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11. *Theoretical Study on Earth Tides by Assuming the Earth as a Visco-elastic Body.*

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Abstract

Forced oscillations of the earth by tide generating potential are investigated by assuming the earth as a visco-elastic body.

First, the Kelvin-Voigt type visco-elastic model is adopted for the earth and equations of motion are deduced. By solving the equations numerically, phase retardations of earth tides are calculated where viscosity within the mantle is assumed to be uniform. As a result, it is noticed that the retardations become marked when the viscosity is over 10^{15} poises.

Second, the retardations are calculated for real earth models where radial distribution of viscosity is converted from Q model by ANDERSON and HART (1978). Obtained retardations are too small to be detected by actual observations. It is concluded that the effect of mantle viscosity to the phase retardations is negligibly small. Through the calculations, Love's numbers, diminishing factor, gravimetric factor, etc. are also calculated for several principal components of tide potential. The results are nearly equal to the values obtained from the solutions of static deformation of the earth. There is, however, slight frequency dependence of Love's numbers.

Finally, relations between the visco-elastic model of the Kelvin-Voigt type and the Maxwell type are studied. From the results, it may be said that even if any kind of vibrating motion of the Kelvin-Voigt type earth is considered, the same vibration is possible for the Maxwell type earth only by taking new appropriate values of viscosity. For example, 2×10^{10} poises of viscosity for the Kelvin-Voigt model corresponds to about 10^{22} poises for the Maxwell model in the frequencies of earth tides.

1. Introduction

The tidal deformation of the earth has been observed by various methods on earth's surface all over the world. The results obtained by these observations have been compared with theoretical values in ordinary cases. Here, the theoretical values are based on the assumptions that the earth to be rigid or perfectly elastic. Some elastic properties of the earth can be clarified from the results.

By comparing the observed results with the theoretical ones, the following quantities are usually determined for a harmonic component in tidal potential. One is the amplitude ratio and the other is phase difference. The amplitude ratio is the direct reflex of the earth's elasticity and then, Love's numbers are calculated as representative values from them. Numerous papers have contributed to this kind of research.

The observed retardation of the phase to the theoretical one seems an index of earth's plasticity. As for the phase retardation, however, only the observed results have been reported and in most cases left unanalyzed afterwards. In considering the phase retardations in the earth tides, it is necessary to establish some theories which connect the retardations with earth's plasticity as a first step. Accordingly, it is required to have information on viscosity within the earth. Nevertheless, the knowledge about earth's viscosity has remained poor till quite recently.

In recent years, the data of the earth's plasticity have gradually been accumulated. They are obtained mainly from the post glacial uplift movement of crust, the attenuation of seismic waves and time decay of earth's free oscillations. For example, HASKEL (1935, 1936) first estimated the viscosity in the upper mantle to be 10^{22} poises from the uplift of Fennoscandian area. MCCONNELL (1965, 1968a, 1968b) also treated the data and obtained the variation in viscosity with depth in the upper mantle. CRITTENDEN (1963) gave the value to be 10^{21} poises from the isostatic rebound of Pleistocene Lake Bonneville.

In another approach, the quality factor Q obtained from the attenuation of vibrating motion in the earth was investigated by a lot of authors. ASADA and TAKANO (1963) obtained the mean Q_p to be 2500–5000 for longitudinal waves for the mantle in the frequency region from 1 to 10 Hz. From another standpoint, KANAMORI (1967) showed Q_p to be 180–240 for the upper mantle and 1600–6000 for the lower mantle. KOVACH and ANDERSON (1964) also obtained the mean Q_s for transverse wave to be 200 for the upper mantle and 2200 for the lower mantle from the deep focus earthquakes which occurred in Argentina. ANDERSON, BEN-MENAHEM and ARCHAMBEAU (1965) constructed a model MM8 in which the distribution of Q_s was given from the earth's surface to 1000 km in depth by using the attenuation of surface waves.

Q in the core is investigated by QAMAR and EISENBERG (1974), SUZUKI and SATO (1970) and BUCHBINDER (1974). On the other hand, NOWROOZI (1968) calculated the mean Q to be 251 ± 42 for spheroidal fundamental mode of vibration from the free oscillation of the earth excited by the Alaskan earthquake of March 1964 and the Aleutian Island earthquake of May 1965.

Although the above results do not necessarily coincide with each other, it is understood that the distribution of the viscosity is about to be revealed and theoretical considerations can begin to explain phenomena related to earth's viscosity.

In 1978, ANDERSON and HART (1978) presented new Q models SL7 and SL8. In these models, radial distributions of Q_p and Q_s are given for the whole earth and they are very convenient to various calculations concerning the earth.

In this paper, first, equations of motion of spheroidal oscillation for the Kelvin-Voigt type visco-elastic earth are derived. From the solution of the equations, phase retardations, Love's numbers and some other factors are calculated for cases of various viscosities and several frequencies. In the calculations, Q model SL8 is applied as the measure of viscosity in the real earth model.

Following the calculations, strain-stress relations for the Maxwell type visco-elastic body are deduced. By comparing them with that of the Kelvin-Voigt body, viscosity relation between both bodies is considered. This relation explains the apparent difference of viscosities obtained from vibrational motion and from creep motion of the body.

2. Fundamental Equations

Equations of motion for an elastic, self-gravitating earth with spherical symmetry may be expressed in terms of polar coordinates (r, θ, ϕ). In the case of spheroidal oscillation, the displacement components u_r, u_θ and u_ϕ in the r-, θ- and ϕ-directions and the additional potential ψ may be expressed in the following forms as

$$\left.\begin{aligned} u_r &= U_n(r)\,Y_n(\theta,\phi), \\ u_\theta &= V_n(r)\{\partial Y_n(\theta,\phi)/\partial\theta\}, \\ u_\phi &= \{V_n(r)/\sin\theta\}\{\partial Y_n(\theta,\phi)/\partial\phi\}, \\ \psi &= P_n(r)\,Y_n(\theta,\phi), \end{aligned}\right\} \tag{2.1}$$

where $Y_n(\theta, \phi)$ is a spherical harmonic function of order n and $U_n(r)$, $V_n(r)$ and $P_n(r)$ are functions of r only. By these relations, the equations of motion are reduced to four equations (ALTERMAN, JAROSCH and PEKERIS, 1959). These expressions are

$$\begin{aligned} &\frac{d}{dr}(\lambda X_n+2\mu\dot{U}_n)+\frac{\mu}{r^2}\{4r\dot{U}_n-4U_n+n(n+1)(-U_n-r\dot{V}_n+3V_n)\} \\ &\quad+\rho P_n+\rho g X_n-\rho\cdot\frac{d}{dr}(gU_n)+\omega^2\rho U_n=0, \end{aligned}$$

$$\left.\begin{aligned}&\frac{d}{dr}\left\{\lambda\left(\dot{V}_n-\frac{V_n}{r}+\frac{U_n}{r}\right)\right\}+\frac{\mu}{r^2}\{5U_n+3r\dot{V}_n-V_n-2n(n+1)V_n\}\\&\quad+\frac{\lambda}{r}X_n+\frac{\rho}{r}P_n-\frac{\rho g}{r}U_n+\omega^2\rho V_n=0,\end{aligned}\right\} \quad (2.2)$$

and

$$\ddot{P}_n+\frac{2}{r}\dot{P}_n-\frac{n(n+1)}{r^2}P_n=4\pi G(\dot{\rho}U_n+\rho X_n),$$

where

$$X_n=\dot{U}_n+\frac{2}{r}U_n-\frac{n(n+1)}{r}V_n,$$

λ and μ: Lamé's elastic moduli,
ρ: density in the undisturbed state,
g: gravity in the undisturbed state,
G: universal gravitational constant

and

ω: circular frequency of the oscillation.

A dot over a quantity implies differentiation with respect to radius r.

In order to investigate the phase retardation in the earth tides theoretically, the earth must be treated as a visco-elastic body. So, some modifications become necessary for equations (2.2) considering the viscous effect of the earth. In this paper, the earth is assumed as a Kelvin-Voigt type visco-elastic body in the first place. Here, the stress components caused by the elastic deformation and by the viscous flow operate on the body as the sum of them at a time. In the case of one dimension, strain-stress relation is given as

$$X=\mu e+\eta\frac{\partial e}{\partial t}, \quad (2.3)$$

where

X: stress,
e: strain,
η: the coefficient of viscosity

and

t: elapsed time.

Strain-stress relations in an elastic body of three dimensions are generally given for spherical coordinates in the following forms

$$\left.\begin{aligned}R_r&=\lambda\Delta+2\mu e_{rr},\\\Theta_\theta&=\lambda\Delta+2\mu e_{\theta\theta},\\\Phi_\phi&=\lambda\Delta+2\mu e_{\phi\phi},\end{aligned}\right|$$

$$\left.\begin{aligned}\Theta_\phi&=\mu e_{\theta\phi},\\ \Phi_r&=\mu e_{\phi r},\\ R_\theta&=\mu e_{r\theta}:\end{aligned}\right\} \tag{2.4}$$

where

R_r, Θ_θ, Φ_ϕ, Θ_ϕ, Φ_r and R_θ: stress components,

e_{rr}, $e_{\theta\theta}$, $e_{\phi\phi}$, $e_{\theta\phi}$, $e_{\phi r}$ and $e_{r\theta}$: strain components

and

Δ: cubical dilatation; $\Delta=e_{rr}+e_{\theta\theta}+e_{\phi\phi}$.

By referring (2.3), the strain-stress relations (2.4) are expanded for a Kelvin-Voigt type visco-elastic body. They are as follows:

$$\left.\begin{aligned}R_r&=\lambda\Delta+2\mu e_{rr}+\kappa\frac{\partial\Delta}{\partial t}+2\eta\frac{\partial e_{rr}}{\partial t},\\ \Theta_\theta&=\lambda\Delta+2\mu e_{\theta\theta}+\kappa\frac{\partial\Delta}{\partial t}+2\eta\frac{\partial e_{\theta\theta}}{\partial t},\\ \Phi_\phi&=\lambda\Delta+2\mu e_{\phi\phi}+\kappa\frac{\partial\Delta}{\partial t}+2\eta\frac{\partial e_{\phi\phi}}{\partial t},\\ \Theta_\phi&=\mu e_{\theta\phi}+\eta\frac{\partial e_{\theta\phi}}{\partial t},\\ \Phi_r&=\mu e_{\phi r}+\eta\frac{\partial e_{\phi r}}{\partial t},\\ R_\theta&=\mu e_{r\theta}+\eta\frac{\partial e_{r\theta}}{\partial t},\end{aligned}\right\} \tag{2.5}$$

where κ is the second coefficient of viscosity. It may be also denoted by η and the coefficient of volume viscosity η_v as

$$\kappa=\eta_v+\frac{2}{3}\eta. \tag{2.6}$$

In order to obtain the relations (2.5) from (2.4) for an oscillating motion of the body of which the circular frequency is ω, it is sufficient to introduce complex moduli $\hat{\lambda}$ and $\hat{\mu}$ instead of λ and μ. In this case, complex moduli $\hat{\lambda}$ and $\hat{\mu}$ are written as

$$\left.\begin{aligned}\hat{\lambda}&=\lambda+i\omega\kappa,\\ \hat{\mu}&=\mu+i\omega\eta.\end{aligned}\right\} \tag{2.7}$$

According to these replacements, displacement components u_r, u_θ, u_ϕ and additional potential ψ are also taken as complex quantities. In this case, the functions $U_n(r)$, $V_n(r)$ and $P_n(r)$ in equations (2.2) may be expressed in the forms

$$\left.\begin{aligned} U_n(r)&=y_{n1}+i\omega z_{n1},\\ V_n(r)&=y_{n3}+i\omega z_{n3},\\ P_n(r)&=y_{n5}+i\omega z_{n5},\end{aligned}\right\} \tag{2.8}$$

where y_{n1}, y_{n3}, y_{n5}, z_{n1}, z_{n3} and z_{n5} are real functions of r only. For simplicity, a subscript "n" is omitted in later discussions and the above functions are denoted as y_1, y_3, y_5, z_1, z_3 and z_5.

By using complex displacement components, the phase retardation of a displacement component (or additional potential) may be obtained from the ratio of the real part and the imaginary part of the amplitude of the considered component. This is evident from the general relation;

$$(A+iB)e^{i\omega t}=\sqrt{A^2+B^2}\cdot e^{i(\omega t-\varepsilon)}, \tag{2.9}$$

where

$$\tan\varepsilon=-\frac{B}{A}.$$

Then, the phase retardations ε_r, ε_θ, ε_ϕ and ε_ψ with respect to oscillations of r-, θ-, ϕ- components and the potential ψ are given as function of r in the forms

$$\left.\begin{aligned} \tan\varepsilon_r&=-\frac{\omega z_1}{y_1},\\ \tan\varepsilon_\theta&=\tan\varepsilon_\phi=-\frac{\omega z_3}{y_3},\\ \tan\varepsilon_\psi&=-\frac{\omega z_5}{y_5}.\end{aligned}\right\} \tag{2.10}$$

In short, there are three kinds of phase retardations in the earth tides. They are retardations ε_r in radial displacement, ε_θ or ε_ϕ in horizontal displacement and ε_ψ in potential variation. The calculation of them is essentially equal to obtain the solution of differential equations with respect to functions y_1, y_3, y_5, z_1, z_3 and z_5 for given circular frequency ω.

From the above discussions, necessary differential equations are obtained by substituting the relations (2.7) and (2.8) into (2.2) after replacing λ and μ by $\hat{\lambda}$ and $\hat{\mu}$. In deriving the equations, new dependent variables y_2, y_4, y_6, z_2, z_4 and z_6 are introduced in order to simplify later calculations. This is a similar procedure taken by ALTERMAN *et al.* (1959). In our case, they are defined as follows:

$$\begin{aligned} y_2=(\lambda+2\mu)\dot{y}_1+\left(\frac{2\lambda}{r}\right)y_1-\left\{\frac{n(n+1)\lambda}{r}\right\}y_3-\omega^2\Big[(\kappa+2\eta)\dot{z}_1\\ +\left(\frac{2\kappa}{r}\right)z_1-\left\{\frac{n(n+1)\kappa}{r}\right\}z_3\Big],\end{aligned}$$

$$\left.\begin{aligned}
y_4&=\mu\left\{\dot{y}_3-\left(\frac{y_3}{r}\right)+\left(\frac{y_1}{r}\right)\right\}-\omega^2\eta\left\{\dot{z}_3-\left(\frac{z_3}{r}\right)+\left(\frac{z_1}{r}\right)\right\},\\
y_6&=\dot{y}_5-4\pi G\rho y_1,\\
z_2&=(\kappa+2\eta)\dot{y}_1+\left(\frac{2\kappa}{r}\right)y_1-\left\{\frac{n(n+1)\kappa}{r}\right\}y_3+(\lambda+2\mu)\dot{z}_1\\
&\quad+\left(\frac{2\lambda}{r}\right)z_1-\left\{\frac{n(n+1)\lambda}{r}\right\}z_3,\\
z_4&=\eta\left\{\dot{y}_3-\left(\frac{y_3}{r}\right)+\left(\frac{y_1}{r}\right)\right\}+\mu\left\{\dot{z}_3-\left(\frac{z_3}{r}\right)+\left(\frac{z_1}{r}\right)\right\},\\
z_6&=\dot{z}_5-4\pi G\rho z_1.
\end{aligned}\right\}\tag{2.11}$$

Then, stress components R_r, Θ_r and Φ_r are written in complex forms as

$$\left.\begin{aligned}
R_r&=(y_2+i\omega z_2)Y_n(\theta,\phi),\\
\Theta_r&=(y_4+i\omega z_4)\left\{\frac{\partial Y_n(\theta,\phi)}{\partial\theta}\right\},\\
\Phi_r&=(y_4+i\omega z_4)\left(\frac{1}{\sin\theta}\right)\left\{\frac{\partial Y_n(\theta,\phi)}{\partial\phi}\right\}.
\end{aligned}\right\}\tag{2.12}$$

After some elementary but troublesome calculations by using relations (2.7), (2.8) and (2.11), equations (2.2) are finally transformed into twelve simultaneous first-order differential equations. They are expressed in matrix representations as follows:

$$\begin{pmatrix}\frac{\partial y_1}{\partial r}\\ \frac{\partial y_2}{\partial r}\\ \frac{\partial y_3}{\partial r}\\ \frac{\partial y_4}{\partial r}\\ \frac{\partial y_5}{\partial r}\\ \frac{\partial y_6}{\partial r}\end{pmatrix}=A\begin{pmatrix}y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\end{pmatrix}-\omega^2B\begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\end{pmatrix},\tag{2.13}$$

and

$$\begin{pmatrix} \frac{\partial z_1}{\partial r} \\ \frac{\partial z_2}{\partial r} \\ \frac{\partial z_3}{\partial r} \\ \frac{\partial z_4}{\partial r} \\ \frac{\partial z_5}{\partial r} \\ \frac{\partial z_6}{\partial r} \end{pmatrix} = B \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} + A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix}, \tag{2.14}$$

where A and B are 6×6 matrices. By denoting them as

$$A=\begin{pmatrix} a_{11} & \cdots & a_{16} \\ \vdots & & \vdots \\ a_{61} & \cdots & a_{66} \end{pmatrix} \tag{2.15}$$

and

$$B=\begin{pmatrix} b_{11} & \cdots & b_{16} \\ \vdots & & \vdots \\ b_{61} & \cdots & b_{66} \end{pmatrix}, \tag{2.16}$$

the elements in the matrices are given in following forms.

$$\left.\begin{aligned} a_{11} &= -\frac{2p}{qr}, \\ a_{12} &= \frac{\lambda+2\mu}{q}, \\ a_{13} &= \frac{n(n+1)p}{qr}, \\ a_{21} &= -\omega^2\rho - \frac{4\rho g}{r} + \frac{4c_1}{r^2}, \\ a_{22} &= -\frac{4s}{qr}, \\ a_{23} &= -n(n+1)\left(-\frac{\rho g}{r} + \frac{2c_1}{r^2}\right), \\ a_{24} &= \frac{n(n+1)}{r} \\ a_{26} &= -\rho, \end{aligned}\right\}$$

$$\left.\begin{aligned}
a_{31}&=-\frac{1}{r}\\
a_{33}&=\frac{1}{r},\\
a_{34}&=\frac{\mu}{\mu^2+\omega^2\eta^2},\\
a_{41}&=\frac{\rho g}{r}-\frac{2c_1}{r^2},\\
a_{42}&=-\frac{p}{qr},\\
a_{43}&=-\omega^2\rho+\frac{d_1}{r^2},\\
a_{44}&=-\frac{3}{r},\\
a_{45}&=-\frac{\rho}{r},\\
a_{51}&=3\gamma,\\
a_{56}&=1,\\
a_{63}&=-\frac{3n(n+1)\gamma}{r},\\
a_{65}&=\frac{n(n+1)}{r^2},\\
a_{66}&=-\frac{2}{r},
\end{aligned}\right\} \tag{2.17}$$

$$\left.\begin{aligned}
b_{11}&=-\frac{4w}{qr},\\
b_{12}&=-\frac{\kappa+2\eta}{q},\\
b_{13}&=\frac{2n(n+1)w}{qr},\\
b_{21}&=\frac{4c_2}{r^2},\\
b_{22}&=\frac{4w}{qr},\\
b_{23}&=-\frac{2n(n+1)c_2}{r^2},
\end{aligned}\right\} \tag{2.18}$$

$$\left.\begin{aligned} b_{34}&=-\frac{\eta}{\mu^2+\omega^2\eta^2},\\ b_{41}&=-\frac{2c_2}{r^2},\\ b_{42}&=-\frac{2w}{qr},\\ b_{43}&=\frac{d_2}{r^2},\end{aligned}\right\}$$

and all the other elements are zero, where

$$\left.\begin{aligned} p&=\lambda(\lambda+2\mu)+\omega^2\kappa(\kappa+2\eta),\\ q&=(\lambda+2\mu)^2+\omega^2(\kappa+2\eta)^2,\\ s&=\mu(\lambda+2\mu)+\omega^2\eta(\kappa+2\eta),\\ w&=\kappa\mu-\lambda\eta,\\ \gamma&=(4/3)\pi G\rho,\\ c_1&=\{s(3\lambda+2\mu)+\omega^2 w(3\kappa+2\eta)\}/q,\\ c_2&=\{s(3\kappa+2\eta)-w(3\lambda+2\mu)\}/q,\\ d_1&=\{2(2n^2+2n-1)(p\mu-2\omega^2 w\eta)+4(n^2+n-1)(s\mu+\omega^2 w\eta)\}/q,\\ d_2&=\{2(2n^2+2n-1)(2w\mu+p\eta)+4(n^2+n-1)(s\eta-w\mu)\}/q.\end{aligned}\right\} \tag{2.19}$$

As it will be mentioned later, above differential equations are solved for earth's core under the conditions of

$$\mu=\kappa=\eta=0\,.$$

In this case, the relations with respect to "z"s vanish and (2.13) and (2.14) are reduced to four equations. They are already given by ALTERMAN *et al.* (1959). In matrix representations, they are

$$\begin{pmatrix}\dfrac{\partial y_1}{\partial r}\\[2mm] \dfrac{\partial y_2}{\partial r}\\[2mm] \dfrac{\partial y_5}{\partial r}\\[2mm] \dfrac{\partial y_6}{\partial r}\end{pmatrix}=\begin{pmatrix} c_{11} & c_{12} & c_{15} & c_{16}\\ c_{21} & c_{22} & c_{25} & c_{26}\\ c_{51} & c_{52} & c_{55} & c_{56}\\ c_{61} & c_{62} & c_{65} & c_{66}\end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ y_5\\ y_6\end{pmatrix}, \tag{2.20}$$

where

$$\left.\begin{aligned}
c_{11}&=\frac{n(n+1)g}{\omega^2 r^2}-\frac{2}{r},\\
c_{12}&=-\frac{n(n+1)}{\omega^2 r^2 \rho}+\frac{1}{\lambda},\\
c_{15}&=-\frac{n(n+1)}{\omega^2 r^2},\\
c_{21}&=\frac{n(n+1)\rho g^2}{\omega^2 r^2}-\frac{4\rho g}{r}-\omega^2\rho,\\
c_{22}&=-\frac{n(n+1)g}{\omega^2 r^2},\\
c_{25}&=-\frac{n(n+1)\rho g}{\omega^2 r^2},\\
c_{26}&=-\rho,\\
c_{51}&=3\gamma,\\
c_{56}&=1,\\
c_{61}&=-\frac{3n(n+1)\gamma}{\omega^2 r^2},\\
c_{62}&=\frac{3n(n+1)\gamma}{\omega^2 r^2 \rho},\\
c_{65}&=n(n+1)\left\{\frac{3\gamma}{\omega^2 r^2}+\frac{1}{r^2}\right\},\\
c_{66}&=-\frac{2}{r},\\
c_{16}&=c_{52}=c_{55}=0.
\end{aligned}\right\}\qquad(2.21)$$

In an earth model, there are internal surface discontinuities where abrupt change occurs in the values of elastic constants, coefficients of viscosity or density with respect to r-direction. In these cases, boundary conditions must be considered in addition to conditions for the center and for the surface of the earth.

For the elastic earth, the conditions are considered by TAKEUCHI (1950), ALTERMAN *et al.* (1959) and others. In our case, these boundary conditions are summarized as follows:

(1) At the center of the earth: Regularity of the functions y_1, y_2, y_5 and y_6 are necessary. Here, two sets of initial values are taken for numerical integration of differential equations (2.20) for each component of tidal potential.

(2) For the surface of discontinuity in the core: It is sufficient to consider the continuity of functions y_1, y_2, y_5 and y_6.

(3) For the boundary surface between the core and the mantle:

Continuity of the functions y_1, y_2, y_5 and y_6 from the core to the mantle must be considered. On the boundary surface in the mantle, y_4, z_2, z_4 and z_6 are zero. Here, y_3, z_1, z_3 and z_5 can be taken as independent values.

(4) For the surface of discontinuity in the mantle: It is sufficient to consider the continuity of all the functions $y_1, y_2, y_3, y_4, y_5, y_6, z_1 z_2, z_3, z_4, z_5$ and z_6.

(5) For the surface of the earth: From the condition of vanishing the stress components $R_r, \Theta_r(=R_\theta)$ and Φ_r, it is derived that y_2, y_4, z_2 and z_4 must be zero. On the other hand, from the condition of continuity of potential gradient, two conditions

$$y_6+\frac{n+1}{a}y_5=(2n+1)a^{n-1}Gm/d^{n+1}$$

and

$$z_6+\frac{n+1}{a}z_5=0$$

are obtained, where

a: radius of the earth,

m: mass of a disturbing body; for example, the moon or the sun, and

d: distance between the center of the earth and disturbing body.

Under the above conditions for boundary surfaces, numerical integration of differential equations (2.13), (2.14) and (2.20) can be executed for a given earth model.

3. Q value and coefficient of viscosity in the Kelvin-Voigt type visco-elastic body

As the value related to the viscosity, the elastic quality factor Q is often treated in seismology. It is usually estimated from the attenuation of seismic waves or time decay of free oscillations of the earth. Therefore, the relation between Q and coefficients of viscosity is first considered for the Kelvin-Voigt type visco-elastic body. By the relation, Q will be converted into coefficients of viscosity.

Let us now consider a plane wave which moves in the x-direction through the visco-elastic medium. The plane wave may be expressed in the following forms as

$$\begin{aligned}\phi&=Ae^{-\alpha x}e^{i(\omega t-qx)}\\&=Ae^{i(\omega t-\hat{q}x)}\end{aligned}\qquad(3.1)$$

where

ϕ: scalar potential for the wave equation,

A: initial amplitude of the wave,
α: attenuation constant,
q: wave number,

and

$\hat{q}$: complex wave number; $\hat{q}=q-i\alpha$.

When the velocity $\hat{v}$ of the wave is taken as a complex quantity, it is written as

$$\hat{v}=v+iv^*, \tag{3.2}$$

It may also be given as

$$\hat{v}=\frac{\omega}{\hat{q}}=\frac{\omega(q+i\alpha)}{q^2+\alpha^2}, \tag{3.3}$$

because the velocity of a wave is generally given as the ratio of the circular frequency ω to the wave number $\hat{q}$. Then, the relations

$$\left.\begin{aligned} v&=\frac{\omega q}{q^2+\alpha^2} \\ v^*&=\frac{\omega\alpha}{q^2+\alpha^2} \end{aligned}\right\} \tag{3.4}$$

are obtained. By dividing v^* by v, the attenuation constant α is expressed as

$$\alpha=\frac{v^*}{v}q. \tag{3.5}$$

The values v and v^* in complex velocity may also be given through complex moduli $\hat{\lambda}$ and $\hat{\mu}$ for the Kelvin-Voigt type visco-elastic body. For the longitudinal wave (for which the letter "p" is appended to corresponding quantities), the complex velocity $\hat{v}_p$ is expressed as

$$\begin{aligned} \hat{v}_p&=v_p+iv_p^*=\sqrt{(\hat{\lambda}+2\hat{\mu})/\rho} \\ &=\sqrt{(\lambda+2\mu)/\rho}\cdot\sqrt{1+iu_p} \\ &=\sqrt{(\lambda+2\mu)/\rho}\cdot\{\sqrt{(1+\sqrt{1+u_p^2})/2}+i\sqrt{(-1+\sqrt{1+u_p^2})/2}\}. \end{aligned} \tag{3.6}$$

where

$$u_p=\frac{\omega(\kappa+2\eta)}{\lambda+2\mu}.$$

Then, the relation

$$\frac{v_p^*}{v_p}=\frac{u_p}{1+\sqrt{1+u_p^2}} \tag{3.7}$$

is obtained. Substituting (3.7) into (3.5), the attenuation constant α may be written as

$$\alpha = \frac{u_p}{1+\sqrt{1+u_p^2}}\, q. \tag{3.8}$$

It is generally recognized that the condition

$$u_p \ll 1 \tag{3.9}$$

is always satisfied in the earth. This is equivalent to the following conditions that

$$\left.\begin{array}{l} \lambda+2\mu \gg \omega(\kappa+2\eta), \\ q \gg \alpha \\ v_p \gg v_p^*. \end{array}\right\} \tag{3.10}$$

or

In this case, equation (3.8) is approximated by the relation

$$\alpha = \frac{u_p}{2} q. \tag{3.11}$$

On the other hand, the quality factor Q_p for a longitudinal wave is defined by

$$\frac{1}{Q_p} = \frac{1-e^{-4\pi\alpha/q}}{2\pi}. \tag{3.12}$$

This is approximated under the condition given (3.9) or (3.10) by

$$\frac{1}{Q_p} = \frac{2\alpha}{q} = u_p = \frac{\omega(\kappa+2\eta)}{\lambda+2\mu}. \tag{3.13}$$

This is the relation between Q_p and coefficients of viscosity. By the relation, the value $\kappa+2\eta$ can be calculated from the Q_p for a given circular frequency ω because the elastic moduli λ and μ are known in the earth.

In a similar way, the relation for the transverse wave (for which letter "s" is appended to corresponding quantities)

$$\frac{1}{Q_s} = u_s = \frac{\omega\eta}{\mu} \tag{3.14}$$

is obtained. From the relations (3.13) and (3.14), κ and η can be calculated separatly if Q_p and Q_s were known, for example, from seismic data.

4. Adopted viscosity within the earth

In order to solve the differential equations (2.13) and (2.14), it is necessary to give radial distributions of density, elastic moduli and coefficients of viscosity within the earth beforehand. Although the distributions of density and elastic moduli were well determined from seismic data, coefficients of viscosity have still been left uncertain. Therefore, the coefficients of voscosity must first be estimated.

Before the estimation, the following bold assumptions are taken for the viscosity within the earth.

First, we assume the viscosity to be zero within the earth's core throughout our calculations. Although this is not true, the assumption seems to be permissible considering the fact that the outer core is liquid and the effect of high viscosity value in the inner core is considerably reduced by the outer core.

Table 1. Adopted viscosity values for Kelvin-Voigt model in mantle and related values as assuming the frequency of transverse wave to be 0.04 Hz.

depth	Q_s in SL8	Q_s smoothed	rigidity μ (dyne/cm^2)	$\omega\eta$ (poise/s)	viscosity η (poise)
km			$\times 10^{12}$	$\times 10^{10}$	$\times 10^{10}$
2878	100	100	2.954	2.954	11.754
2800	200	193	2.934	1.520	6.049
2600	500	505	2.804	0.555	2.209
2400	515	518	2.678	0.517	2.057
2200	510	510	2.555	0.501	1.993
2000	495	495	2.439	0.493	1.960
1800	465	470	2.324	0.494	1.967
1600	440	440	2.209	0.502	1.998
1400	410	405	2.090	0.516	2.053
1200	370	365	1.967	0.539	2.144
1000	310	318	1.837	0.578	2.298
800	260	258	1.656	0.642	2.554
650	215	208	1.462	0.703	2.797
650	215	208	1.396	0.671	2.670
500	150	157	1.041	0.663	2.638
350	105	116	0.749	0.646	2.569
350	105	116	0.697	0.601	2.391
300	105	106	0.693	0.654	2.601
200	105	94	0.686	0.730	2.904
100	90	91	0.678	0.745	2.964
60	110	90	0.675	0.750	2.984
60	110	90	0.713	0.792	3.152
45	110	110	0.712	0.647	2.575
45	500	500	0.712	0.142	0.567
15	500	500	0.709	0.142	0.564
15	500	500	0.358	0.072	0.285
0	500	500	0.358	0.072	0.285

Second, we take the assumption that the second coefficient of viscosity κ is zero within the mantle. This fact is supported by the Q model given by ANDERSON *et al.* (1978). Moreover, it is also confirmed that the vibrating motion of the earth is affected little by the value of κ through some trial calculations.

By these simplifications, it becomes sufficient to give the distribution of viscosity η only in the mantle for the calculations. As mentioned before, the coefficient of viscosity η can be converted from the quality factor Q_s. In the present study, Q_s values are adopted from the Q model SL8 given by ANDERSON *et al.* (1978). After smoothing the values in model SL8, they are converted into coefficients of viscosity η according to the relation (3.14). Here, the circular frequency ω is assumed to be 0.04 Hz for transverse seismic waves. The results are presented in Table 1. These are used in later calculations.

5. Phase retardations in the earth tides

In this section, the differential equations (2.13) and (2.14) are numerically solved and phase retardations are calculated.

As for the distributions of density and elastic constants, there is no serious problem. We use here the values given by HADDON and BULLEN (1969) after a slight modification. The modification is trivial and is taken only for the convenience in calculations.

For the distribution of viscosity, two cases are treated in the present study. One is the case where the coefficient of viscosity η is constant all over the mantle (Case I). Here, the variation of phase retardations versus viscosity is considered. The other is the case where the distributions of the coefficient of viscosity η is adopted from the results shown in Table 1 (Case II). Here, phase retardations are calculated for several principal harmonic components in tidal potential.

The numerical integration was executed from the center towards the surface of the earth under the boundary conditions mentioned in Section 2. In the calculation, the Runge-Kutta-Gill method was employed. For the spherical harmonic function, $Y_2(\theta, \phi)$ only was used because the tidal potential is overwhelmingly predominated by the harmonic function of order 2. Two sets of initial values were adopted from analytical solutions of spheroidal oscillation for a uniform earth given by TAKEUCHI and SAITO (1972).

For Case I, two circular frequencies of forced oscillation were considered. One was $\omega_6(=28°984104/\text{hour})$ for semi-diurnal wave M_2 and the other was $\omega_2(=13°943036/\text{hour})$ for diurnal wave O_1. As the coefficient of viscosity in mantle, the following values were successively taken. They were 0, 10^{15}, 3×10^{15}, 10^{16}, 3×10^{16}, 10^{17} and 10^{18} (poises).

Obtained phase retardations are given in Table 2 for M_2 wave and in Fig. 1 for O_1 wave. From the figure, it is clear that phase retardations ε_r and $\varepsilon_\theta(=\varepsilon_\phi)$ rapidly increase as the coefficient of viscosity η increases in the range between 10^{15} and 10^{18} poises. The retardations are small when η is under 10^{15} poises and are nearly 90° when η is over 10^{18} poises.

Table 2. Phase retardations of M_2 wave for various values of viscosity in mantle.

coefficient of viscosity η in poises	retardation in degrees		
	ε	ε_θ, ε_ϕ	ε_ψ
0	0°.00	0°.00	0°.00
10^{15}	4.32	1.64	0.68
3×10^{15}	12.62	5.07	2.00
10^{16}	34.36	19.46	5.25
3×10^{16}	62.40	48.72	6.65
10^{17}	80.72	74.85	3.15
10^{18}	89.06	88.44	0.34

As for ε_ψ in potential variation, it varies quite differently. It takes the maximum value ($\sim$6°) when η is nearly 3×10^{16} poises and then, it decreases as η increases. It seems that ε_ψ approaches to zero as η approaches infinity. This fact is explained in the following way. The additional potential ψ can be divided into two parts. One is the tide generating potential of a disturbing body and the other is the potential due to earth's deformation. When η approaches infinity, the latter closes to zero because the earth's deformation becomes zero and only tide generating potential remains as the additional potential. This is the reason why the retardation ε_ψ becomes zero.

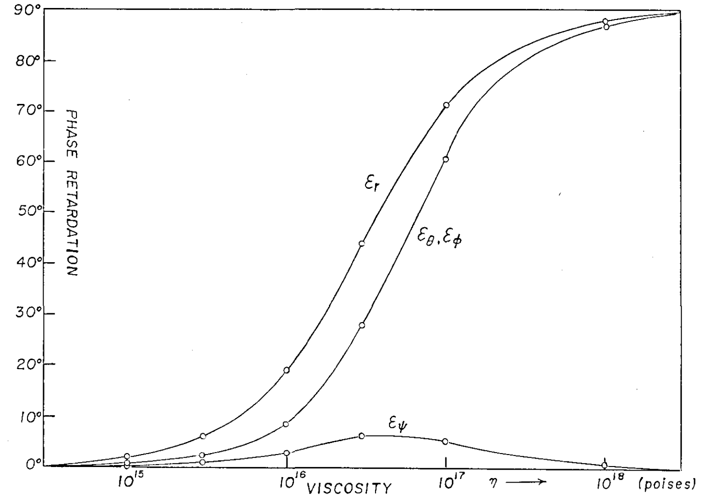

Fig. 1. Phase retardations ε_r in radial displacement, $\varepsilon_\theta(=\varepsilon_\phi)$ in horizontal displacement and ε_ψ in potential variation versus viscosity values in the mantle for O_1 wave.

Table 3. Phase retardations for several principal components in tidal potential.

wave	frequency ω	retardation in degrees			retardation in time for directions		
		ε_r	$\varepsilon_\theta, \varepsilon_\phi$	ε_ψ	r-	θ, ϕ-	ψ-
	/hour	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-6}$	ms	ms	ms
Q_1	13°399	51°8	17°6	8°1	13.93	4.72	2.18
O_1	13.943	54.0	18.3	8.4	〃	4.71	〃
P_1	14.959	57.9	19.6	9.0	〃	〃	〃
K_1	15.041	58.2	19.7	9.1	〃	〃	〃
N_2	28.440	110.7	36.9	17.3	14.01	4.67	2.19
M_2	28.984	112.8	37.6	17.7	〃	〃	〃
S_2	30.000	116.8	38.9	18.3	14.02	4.66	2.20
K_2	30.082	117.1	39.0	18.4	〃	〃	〃
M_3	43.476	170.9	55.4	26.9	14.15	4.59	2.22

For case II, the viscosity distribution given in Table 1 was adopted and phase retardations for following nine principal waves were calculated by similar procedure in Case I. The circular frequencies considered here were;

$$\omega_1 = 13°398661/\text{hour} \quad \text{for } Q_1,$$
$$\omega_2 = 13°943036/\text{hour} \quad \text{for } O_1,$$
$$\omega_3 = 14°958931/\text{hour} \quad \text{for } P_1,$$
$$\omega_4 = 15°041069/\text{hour} \quad \text{for } K_1,$$
$$\omega_5 = 28°439730/\text{hour} \quad \text{for } N_2,$$
$$\omega_6 = 28°984104/\text{hour} \quad \text{for } M_2`$$
$$\omega_7 = 30°000000/\text{hour} \quad \text{for } S_2,$$
$$\omega_8 = 30°082137/\text{hour} \quad \text{for } K_2,$$

and

$$\omega_9 = 43°476156/\text{hour} \quad \text{for } M_3.$$

The results are summarized in Table 3.

As expected from results in Case I, the retardations obtained from the adopted earth model are negligibly small. Each kind of retardation is, however, approximately proportional to the circular frequency of the vibration. Then, it may be said that the retardations expressed in time are nearly equal for each kind of displacement or variation. This fact is clearly shown in Table 3 although there remains slight dependence of the retardations on frequency.

6. Viscosity relations between the Kelvin-Voigt model and the Maxwell model

As shown in Section 4, adopted values of the viscosity in our calculations are at most 10^{11} poises except for the lowest layer of the mantle. On the other hand, plenty of research shows the viscosity in the mantle as greater than 10^{21} poises as mentioned in Section 1. These high values of viscosity coefficient apparently contradict with our values. This contradiction, however, originates from the definition for the coefficient of viscosity. These points may be explained in what follows.

There are two models which often represent the visco-elastic body. One is the Kelvin-Voigt model as mentioned earlier, and the other is the Maxwell model.

In the case of the Kelvin-Voigt model, The displacement occurs within a limit for an ever-lasting constant stress operating on the body. Here, the creep of the body never occurs. Then, the model is inadequate to interpret the phenomena which involve the creep motion like a post gracial rebound movement. For this case, the Maxwell model is adequate and often adopted. In this model, the creep occurs even by small stress when it works continuously.

In the Maxwell model, strain rate is expressed as the sum of the quantity proportional to stress rate caused by the elastic deformation and the quantity proportional to stress accompanied with the viscous flow. In the case of one dimension, the relation is written as

$$\frac{de}{dt}=\frac{1}{\mu}\frac{dX}{dt}+\frac{1}{\eta}X. \tag{6.1}$$

In an elastic body of three dimensions, strain-stress relations in spherical coordinates are generally given in the following forms:

$$\left.\begin{aligned}
e_{rr}&=\frac{1}{2\mu(3\lambda+2\mu)}\{2(\lambda+\mu)R_r-\lambda(\Theta_\theta+\Phi_\phi)\},\\
e_{\theta\theta}&=\frac{1}{2\mu(3\lambda+2\mu)}\{2(\lambda+\mu)\Theta_\theta-\lambda(\Phi_\phi+R_r)\},\\
e_{\phi\phi}&=\frac{1}{2\mu(3\lambda+2\mu)}\{2(\lambda+\mu)\Phi_\phi-\lambda(R_r+\Theta_\theta)\},\\
e_{\theta\phi}&=\frac{1}{\mu}\cdot\Theta_\phi,\\
e_{\phi r}&=\frac{1}{\mu}\cdot\Phi_r,\\
e_{r\theta}&=\frac{1}{\mu}\cdot R_\theta,
\end{aligned}\right\} \tag{6.2}$$

These equations are merely transpositions of (2.4).

In the Maxwell type visco-elastic body of three dimensions, the strain-stress relations are analogically obtained from (6.2) by referring (6.1). The expressions are

$$\left.\begin{aligned}
\frac{\partial e_{rr}}{\partial t}&=\frac{1}{2\mu(3\lambda+2\mu)}\left\{2(\lambda+\mu)\frac{\partial R_r}{\partial t}-\lambda\left(\frac{\partial \Theta_\theta}{\partial t}+\frac{\partial \Phi_\phi}{\partial t}\right)\right\}\\
&\quad+\frac{1}{2\eta(3\kappa+2\eta)}\{2(\kappa+\eta)R_r-\kappa(\Theta_\theta+\Phi_\phi)\},\\
\frac{\partial e_{\theta\theta}}{\partial t}&=\frac{1}{2\mu(3\lambda+2\mu)}\left\{2(\lambda+\mu)\frac{\partial \Theta_\theta}{\partial t}-\lambda\left(\frac{\partial \Phi_\phi}{\partial t}+\frac{\partial R_r}{\partial t}\right)\right\}\\
&\quad+\frac{1}{2\eta(3\kappa+2\eta)}\{2(\kappa+\eta)\Theta_\theta-\kappa(\Phi_\phi+R_r)\},\\
\frac{\partial e_{\phi\phi}}{\partial t}&=\frac{1}{2\mu(3\lambda+2\mu)}\left\{2(\lambda+\mu)\frac{\partial \Phi_\phi}{\partial t}-\lambda\left(\frac{\partial R_r}{\partial t}+\frac{\partial \Theta_\theta}{\partial t}\right)\right\}\\
&\quad+\frac{1}{2\eta(3\kappa+2\eta)}\{2(\kappa+\eta)\Phi_\phi-\kappa(R_r+\Theta_\theta)\},\\
\frac{\partial e_{\theta\phi}}{\partial t}&=\frac{1}{\mu}\cdot\frac{\partial \Theta_\phi}{\partial t}+\frac{1}{\eta}\cdot\Theta_\phi,\\
\frac{\partial e_{\phi r}}{\partial t}&=\frac{1}{\mu}\cdot\frac{\partial \Phi_r}{\partial t}+\frac{1}{\eta}\cdot\Phi_r,\\
\frac{\partial e_{r\theta}}{\partial t}&=\frac{1}{\mu}\cdot\frac{\partial R_\theta}{\partial t}+\frac{1}{\eta}\cdot R_\theta.
\end{aligned}\right\}\tag{6.3}$$

For later discussions, it is convenient to express the stress components as explicit functions of strain components. By integrating the latter half of relations (6.3), the stress components Θ_ϕ, Φ_r and R_θ may easily be obtained in the following forms;

$$\left.\begin{aligned}
\Theta_\phi&=\mu e_{\theta\phi}-\frac{\mu^2}{\eta}\cdot\exp\left(-\frac{\mu t}{\eta}\right)\int e_{\theta\phi}\cdot\exp\left(\frac{\mu t}{\eta}\right)dt+C_1\exp\left(-\frac{\mu t}{\eta}\right),\\
\Phi_r&=\mu e_{\phi r}-\frac{\mu^2}{\eta}\cdot\exp\left(-\frac{\mu t}{\eta}\right)\int e_{\phi r}\cdot\exp\left(\frac{\mu t}{\eta}\right)dt+C_2\exp\left(-\frac{\mu t}{\eta}\right),\\
R_\theta&=\mu e_{r\theta}-\frac{\mu^2}{\eta}\cdot\exp\left(-\frac{\mu t}{\eta}\right)\int e_{r\theta}\cdot\exp\left(\frac{\mu t}{\eta}\right)dt+C_3\exp\left(-\frac{\mu t}{\eta}\right).
\end{aligned}\right\}\tag{6.4}$$

where C_1, C_2 and C_3 are integral constants. As the stress components may, however, be considered as superposition of periodic variations for tidal deformation, these exponential terms may not exist. Then, they become

$$C_1=C_2=C_3=0. \tag{6.5}$$

For obtaining the stress component R_r, it is convenient to calculate the following quantities

$$\frac{\partial}{\partial t}(e_{rr}+e_{\theta\theta}+e_{\phi\phi}) \quad \text{and} \quad \frac{\partial}{\partial t}(2e_{rr}-e_{\theta\theta}-e_{\phi\phi})$$

as a first step. Then, the linear combinations of stress components $R_r+\Theta_\theta+\Phi_\phi$ and $2R_r-\Theta_\theta-\Phi_\phi$ are obtained similarly in the cases of Θ_ϕ, Φ_r and R_θ. From the sum of them, R_r is written in the form;

$$\begin{aligned} R_r=&2\mu e_{rr}+\lambda\Delta-\frac{1}{3}\cdot\frac{(3\lambda+2\mu)^2}{3\kappa+2\eta}\exp\left(-\frac{3\lambda+2\mu}{3\kappa+2\eta}t\right)\int\Delta\exp\left(\frac{3\lambda+2\mu}{3\kappa+2\eta}t\right)dt \\ &-2\frac{\mu^2}{\eta}\cdot\exp\left(-\frac{\mu t}{\eta}\right)\int e_{rr}\cdot\exp\left(\frac{\mu t}{\eta}\right)dt \\ &+\frac{2\mu^2}{3\eta}\cdot\exp\left(-\frac{\mu t}{\eta}\right)\int\Delta\exp\left(\frac{\mu t}{\eta}\right)dt. \end{aligned} \tag{6.6}$$

Other components Θ_θ and Φ_ϕ are also calculated in similar ways. After all, they are reduced in the following forms as

$$\left.\begin{aligned} R_r&=\lambda\Delta+2\mu e_{rr}-\frac{3\lambda+2\mu}{3}(P_{rr}+P_{\theta\theta}+P_{\phi\phi})-2\mu Q_{rr}+\frac{2\mu}{3}(Q_{rr}+Q_{\theta\theta}+Q_{\phi\phi}), \\ \Theta_\theta&=\lambda\Delta+2\mu e_{\theta\theta}-\frac{3\lambda+2\mu}{3}(P_{rr}+P_{\theta\theta}+P_{\phi\phi})-2\mu Q_{\theta\theta}+\frac{2\mu}{3}(Q_{rr}+Q_{\theta\theta}+Q_{\phi\phi}), \\ \Phi_\phi&=\lambda\Delta+2\mu e_{\phi\phi}-\frac{3\lambda+2\mu}{3}(P_{rr}+P_{\theta\theta}+P_{\phi\phi})-2\mu Q_{\phi\phi}+\frac{2\mu}{3}(Q_{rr}+Q_{\theta\theta}+Q_{\phi\phi}), \\ \Theta_\phi&=\mu e_{\theta\phi}-\mu Q_{\theta\phi}, \\ \Phi_r&=\mu e_{\phi r}-\mu Q_{\phi r}, \\ R_\theta&=\mu e_{r\theta}-\mu Q_{r\theta}. \end{aligned}\right\} \tag{6.7}$$

where P_{ij} and Q_{ij} are expressed in the forms

$$\left.\begin{aligned} P_{ij}&=\frac{3\lambda+2\mu}{3\kappa+2\eta}\cdot\exp\left(-\frac{3\lambda+2\mu}{3\kappa+2\eta}t\right)\int e_{ij}\cdot\exp\left(\frac{3\lambda+2\mu}{3\kappa+2\eta}t\right)dt, \\ Q_{ij}&=\frac{\mu}{\eta}\cdot\exp\left(-\frac{\mu t}{\eta}\right)\int e_{ij}\cdot\exp\left(\frac{\mu t}{\eta}\right)dt. \end{aligned}\right\} \tag{6.8}$$

Then, all the stress components are obtained as explicit functions of strain components for the Maxwell type visco-elastic body of three dimensions. In the relations given in (6.8), exponential terms are also neglected.

Let us now consider to calculate P_{ij} and Q_{ij} for strain components e_{ij} corresponding to a circular frequency ω. In this case, e_{ij} is generally expressed in the form;

$$e_{ij}=F_{ij}(\theta,\phi)\exp(i\omega t), \tag{6.9}$$

where $F_{ij}(\theta,\phi)$ is an appropriate function of θ and ϕ.

By substituting (6.9), integrations in (6.8) are executed and the relations

$$\left.\begin{aligned} P_{ij}&=\frac{\alpha}{\alpha^2+\omega^2}\left\{\alpha e_{ij}-\frac{\partial e_{ij}}{\partial t}\right\},\\ Q_{ij}&=\frac{\beta}{\beta^2+\omega^2}\left\{\beta e_{ij}-\frac{\partial e_{ij}}{\partial t}\right\}, \end{aligned}\right\} \tag{6.10}$$

are obtained, where

$$\alpha=\frac{3\lambda+2\mu}{3\kappa+2\eta},$$

and

$$\beta=\frac{\mu}{\eta}. \tag{6.11}$$

By substituting (6.10) into (6.7), stress components for the deformation of which the circular frequency is ω are finally obtained. They are as follows:

$$\left.\begin{aligned} R_r&=\frac{1}{3}\left\{(3\lambda+2\mu)\frac{\omega^2}{\alpha^2+\omega^2}-2\mu\frac{\omega^2}{\beta^2+\omega^2}\right\}\Delta+2\mu\frac{\omega^2}{\beta^2+\omega^2}e_{rr}\\ &\quad+\frac{1}{3}\left\{(3\kappa+2\eta)\frac{\alpha^2}{\alpha^2+\omega^2}-2\eta\frac{\beta^2}{\beta^2+\omega^2}\right\}\frac{\partial\Delta}{\partial t}+2\eta\frac{\beta^2}{\beta^2+\omega^2}\cdot\frac{\partial e_{rr}}{\partial t},\\ \Theta_\theta&=\frac{1}{3}\left\{(3\lambda+2\mu)\frac{\omega^2}{\alpha^2+\omega^2}-2\mu\frac{\omega^2}{\beta^2+\omega^2}\right\}\Delta+2\mu\frac{\omega^2}{\beta^2+\omega^2}e_{\theta\theta}\\ &\quad+\frac{1}{3}\left\{(3\kappa+2\eta)\frac{\alpha^2}{\alpha^2+\omega^2}-2\eta\frac{\beta^2}{\beta^2+\omega^2}\right\}\frac{\partial\Delta}{\partial t}+2\eta\frac{\beta^2}{\beta^2+\omega^2}\cdot\frac{\partial e_{\theta\theta}}{\partial t},\\ \Phi_\phi&=\frac{1}{3}\left\{(3\lambda+2\mu)\frac{\omega^2}{\alpha^2+\omega^2}-2\mu\frac{\omega^2}{\beta^2+\omega^2}\right\}\Delta+2\mu\frac{\omega^2}{\beta^2+\omega^2}e_{\phi\phi}\\ &\quad+\frac{1}{3}\left\{(3\kappa+2\eta)\frac{\alpha^2}{\alpha^2+\omega^2}-2\eta\frac{\beta^2}{\beta^2+\omega^2}\right\}\frac{\partial\Delta}{\partial t}+2\eta\frac{\beta^2}{\beta^2+\omega^2}\cdot\frac{\partial e_{\phi\phi}}{\partial t},\\ \Theta_\phi&=\mu\frac{\omega^2}{\beta^2+\omega^2}e_{\theta\phi}+\eta\frac{\beta^2}{\beta^2+\omega^2}\cdot\frac{\partial e_{\theta\phi}}{\partial t},\\ \Phi_r&=\mu\frac{\omega^2}{\beta^2+\omega^2}e_{\phi r}+\eta\frac{\beta^2}{\beta^2+\omega^2}\cdot\frac{\partial e_{\phi r}}{\partial t},\\ R_\theta&=\mu\frac{\omega^2}{\beta^2+\omega^2}e_{r\theta}+\eta\frac{\beta^2}{\beta^2+\omega^2}\cdot\frac{\partial e_{r\theta}}{\partial t}. \end{aligned}\right\} \tag{6.12}$$

By comparing these expressions of stress components with that of (2.5), it is easily found that relations (6.12) are obtained by replacing λ, μ, κ and η in (2.5) with the values given as follows:

$$\left.\begin{aligned}
&\lambda\rightarrow\frac{1}{3}\left\{(3\lambda+2\mu)\frac{\omega^2}{\alpha^2+\omega^2}-2\mu\frac{\omega^2}{\beta^2+\omega^2}\right\},\\
&\mu\rightarrow\mu\frac{\omega^2}{\beta^2+\omega^2},\\
&\kappa\rightarrow\frac{1}{3}\left\{(3\kappa+2\eta)\frac{\alpha^2}{\alpha^2+\omega^2}-2\eta\frac{\beta^2}{\beta^2+\omega^2}\right\},\\
&\eta\rightarrow\eta\frac{\beta^2}{\beta^2+\omega^2}.
\end{aligned}\right\} \tag{6.13}$$

They are rewritten in simpler forms as

$$\left.\begin{aligned}
3\lambda+2\mu&\rightarrow(3\lambda+2\mu)\frac{\omega^2}{\alpha^2+\omega^2},\\
\mu&\rightarrow\mu\frac{\omega^2}{\beta^2+\omega^2},\\
3\kappa+2\eta&\rightarrow(3\kappa+2\eta)\frac{\alpha^2}{\alpha^2+\omega^2},\\
\eta&\rightarrow\eta\frac{\beta^2}{\beta^2+\omega^2}.
\end{aligned}\right\} \tag{6.14}$$

The above relations connect elastic constants and coefficients of viscosity of the Kelvin-Voigt body with that of the Maxwell body. Accordingly, for the case where a Kelvin-Voigt body and a Maxwell body which are similar in shape and which oscillate in the same way under an external force, the following relations hold for elastic constants and coefficients of viscosity of both bodies.

$$\left.\begin{aligned}
3\lambda_K+2\mu_K&=(3\lambda_M+2\mu_M)\frac{\omega^2}{\alpha_M{}^2+\omega^2},\\
\mu_K&=\mu_M\frac{\omega^2}{\beta_M{}^2+\omega^2},\\
3\kappa_K+2\eta_K&=(3\kappa_M+2\eta_M)\frac{\alpha_M{}^2}{\alpha_M{}^2+\omega^2},\\
\eta_K&=\eta_M\frac{\beta_M{}^2}{\beta_M{}^2+\omega^2}.
\end{aligned}\right\} \tag{6.15}$$

where suffixes "K" and "M" are appended to the quantities corresponding to the Kelvin-Voigt body and the Maxwell body, respectively.

Then, when a kind of viscosity of the earth is known as a Maxwell body, corresponding viscosity for a Kelvin-Voigt body can be calculated according to relations (6.15).

For example, when it is assumed that the mantle is a Maxwell body of which elastic constants and coefficients of viscosity are

$$\lambda_M = 2\times 10^{12} \text{ dynes/cm}^2,$$
$$\mu_M = 2\times 10^{12} \text{ dynes/cm}^2,$$

and

$$\kappa_M = \eta_M = 10^{22} \text{ poises},$$

the following values are calculated for the Kelvin-Voigt body that

$$\lambda_K = 2\times 10^{12} \text{ dynes/cm}^2,$$
$$\mu_K = 2\times 10^{12} \text{ dynes/cm}^2,$$

and

$$\kappa_K = \eta_K = 2\times 10^{10} \text{ poises}.$$

For simplicity, $\omega^2 = 2\times 10^{-8}/\text{sec}^2$ is adopted as the circular frequency in this case. From the results, it is evident that coefficients of viscosity diminish of the order of 10^{12} in spite of the fact that elastic moduli keep unchanged. This is the main reason why apparent contradiction occurs in considering the viscosity in the mantle as mentioned before. Greater values of viscosity correspond to the case where the mantle is assumed as a Maxwell body, on the contrary, smaller values correspond to the case of a Kelvin-Voigt body.

7. Tidal Love's numbers

Tidal Love's numbers are calculated through the solutions of differential equations (2.13), (2.14) and (2.20). They are given by y_1, y_3 and y_5 for the earth's surface in following forms as

Table 4. Love's numbers and factors to rigid earth for several principal components in tidal potential.

wave	frequency /hour	Love's numbers			factors to rigid earth		
		h	l	k	D	L	G
Q_1	13°.399	0.60177	0.08520	0.30155	0.69077	1.21635	1.15846
O_1	13.943	0.61083	〃	0.30157	0.69074	1.21637	1.15847
P_1	14.959	0.61094	0.08521	0.30163	0.69069	1.21642	1.15850
K_1	15.041	0.61095	〃	〃	0.69068	〃	〃
N_2	28.440	0.61307	0.08531	0.30265	0.68958	1.21734	1.15910
M_2	28.984	0.61319	〃	0.30270	0.68952	1.21739	1.15913
S_2	30.000	0.61341	0.08532	0.30281	0.68940	1.21748	1.15919
K_2	30.084	0.61342	0.08533	0.30282	0.68939	1.21749	1.15920
M_3	43.476	0.61705	0.08550	0.30456	0.68751	1.21906	1.16021

Table 5. The variation of Love's numbers and factors to rigid earth versus values of coefficient of viscosity for M_2 wave.

coefficient of viscosity η in poises	Love's numbers			factors to rigid earth		
	h	l	k	D	L	G
0	0.6132	0.0853	0.3027	0.6895	1.2174	1.1591
10^{15}	0.6115	0.0852	0.3018	0.6903	1.2166	1.1588
3×10^{15}	0.5986	0.0847	0.2947	0.6961	1.2100	1.1566
10^{16}	0.4850	0.0748	0.2380	0.7530	1.1632	1.1280
3×10^{16}	0.1910	0.0336	0.0984	0.9074	1.0648	1.0434
10^{17}	0.0252	0.0049	0.0156	0.9904	1.0107	1.0018
10^{18}	0.0003	0.0001	0.0002	0.9999	1.0001	1.0000

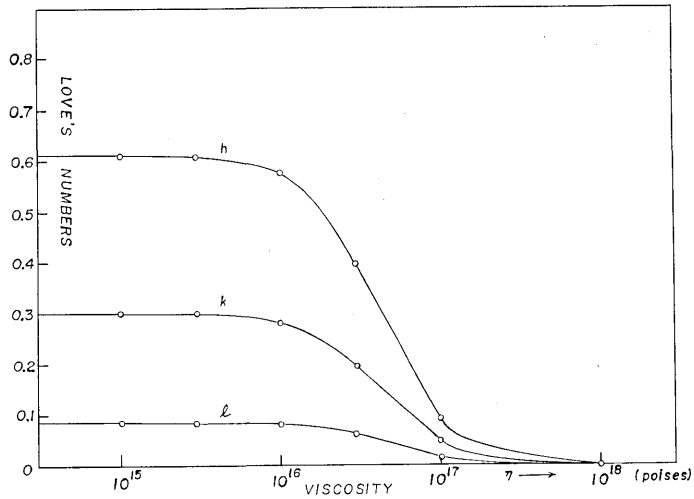

Fig. 2. Love's numbers h, l and k versus viscosity values in the mantle for O_1 wave.

$$\left.\begin{aligned} h &= y_1(a)d^3g(a)/Gma^2, \\ l &= y_3(a)d^3g(a)/Gma^2, \\ k &= y_5(a)d^3/Gma-1. \end{aligned}\right\} \qquad (7.1)$$

The meaning of the symbols are already given in Section 2. The results are shown in Tables 4, 5 and Fig. 2.

On the other hand, the following three factors to values for a perfectly rigid earth are frequently treated in the analysis of earth tides. They are the diminishing factor $D(=1+k-h)$ obtained by observations of tidal tilting, the magnified ratio of latitude variation $L(=1+k-l)$ obtained by observations of periodic variation of latitude and the gravi-

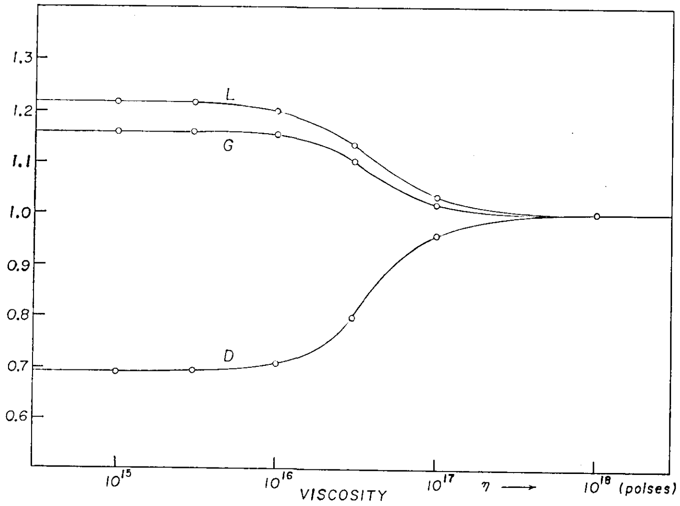

Fig. 3. The diminishing factor D, magnified factor L and gravimetric factor G versus viscosity values in the mantle for O_1 wave.

metric factor $G\{=1+h-(3/2)k\}$ obtained by observations of tidal variation of gravity. These factors are also calculated and given in Tables 4, 5 and Fig. 3.

Table 4 gives the three factors in addition to tidal Love's numbers for each component of tidal potential considered in Section 5. There, a slight dependence of Love's numbers on circular frequency of vibration is recognized.

In Table 5, Love's numbers and the three factors for M_2 wave are given according to the coefficient of viscosity η from calculations of Case I in Section 5, where Love's numbers show sudden decreases as η takes values in the range between 10^{15} and 10^{18} poises. Similar relations for the O_1 wave are shown in Fig. 3. As a matter of course, Love's numbers h, l and k become zero and factors D, L and G approach to unity as η increases because the earth becomes rigid if we let η approach infinity.

Tidal Love's numbers for S_2 wave given here almost coincide with that of SAITO (1974) and one of the results of ALTERMAN *et al.* (1959) obtained from dynamic solution of the earth's oscillation. This also indicates the smallness of the effect of viscosity in the mantle for earth tides.

8. Summary and conclusion

In the present research, the viscous effect of the mantle has been considered for the earth tides. The phase retardations have been calculated for various viscosity models and for several tidal components. In the course of calculations, tidal Love's numbers are also obtained. The results are summarized as follows. In this summary, viscosity for the Kelvin-Voigt type visco-elastic body is denoted by η.

(1) Three kinds of phase retardation may be defined for tidal deformation of the earth. They are retardations ε_r in radial displacement, $\varepsilon_\theta(=\varepsilon_\phi)$ in horizontal displacement and ε_ψ in potential variation. For these retardations, the relation $\varepsilon_r > \varepsilon_\theta > \varepsilon_\psi$ always holds in our results.

(2) As η increases, both ε_r and ε_θ always increase. Such a tendency becomes marked in the case of $\eta > 10^{15}$ poises. When η exceeds 10^{18} poises, they become about 90°. It is supposed that limit values of ε_r and ε_θ may be 90° for the case of $\eta \to \infty$.

(3) For ε_ψ in potential variation, as η increases, ε_ψ reaches the maximum value and afterwards gradually it decreases. The maximum value of ε_ψ is less than 10° and occurs when η is about 3×10^{16} poises for tidal components O_1 and M_2. The limit value is supposed to be 0° for $\eta \to \infty$. The physical meaning of such variation is clear.

(4) Love's numbers h, l and k decrease and approach to zero as η increases. On the other hand, factors D, L and G close to unity. It is quite natural because the earth is considered to be perfectly rigid in the case of $\eta \to \infty$. They undergo rapid changes of the values when η is between 10^{15} and 10^{18} poises as in the cases of changes in ε_r and ε_θ.

(5) From the general relations mentioned above, it seems that large phase retardation may not be expected in the real earth because the viscosity values converted from Q_s within the mantle are only of the order of 10^{10} poises in average. This speculation is confirmed by calculations of Case II in Section 5 where retardations are obtained to be under 10^{-4} degrees. It may be difficult to detect these small retardations actually by observations.

(6) In actual observation of earth tides, a few degrees of phase retardations are sometimes detected. By our analysis, it is concluded that the retardations do not depend on mantle viscosity but on some other causes, for example, load variation of ocean tides, effect of topography around observation sites or effect of temperature variation on measuring apparatus.

(7) Calculated phase retardations in Case II are approximately proportional to circular frequency ω in our considering range of frequency. This fact may imply that each kind of retardation converted to time interval is nearly constant regardless of frequency variation. Strictly speaking, there remains a slight dependency between ω and retardations in time interval.

(8) There is also a slight dependency between Love's numbers and circular frequency ω. As ω increases, Love's numbers h, l and k increase

a little.

(9) As the effect of mantle viscosity is very small, the obtained solutions of differential equations approximately correspond to dynamic solutions for forced oscillation of the elastic earth. These solutions are already obtained by ALTERMAN *et al.* (1959), SAITO (1974) and others. It seems difficult to compare our results directly to their ones because the adopted earth model and the circular frequencies are different from one another. The obtained Love's numbers in our calculations, however, almost coincide with that of Saito (for Wang's model, 1972) and one of the Alterman's results (for Bullen's B model, 1950) to three places of decimals in the case of $\omega=30°$/hour.

(10) The coefficients of viscosity and elastic moduli are generally determined according to the adopted model for a visco-elastic body. In the case of vibrating motion, they are convertible to each other in the Kelvin-Voigt model and in the Maxwell model through the circular frequency. Actually, coefficients of viscosity κ and η in the mantle are greatly affected by the model, although elastic moduli λ and μ undergo practically no change for the circular frequency range in the earth tides. After all, it may be said that only viscosity values depend on the adopted model.

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11. 粘弾性地球に対する地球潮汐の理論的考察

地震研究所 長 沢 工

ここでは，地殻，マントルを粘弾性体と考えた場合，地球潮汐の状態がどのようになるか，特にその場合の位相遅れがどのぐらいになるかを理論的に考察している．地球潮汐の位相遅れは，高さ方向の変位に対するもの，水平方向の変位に対するもの，およびポテンシャルの変化に対するものの3種が定義できる．

まず，地球をケルビン・フォークト型の粘弾性体であると仮定して，地球振動の方程式を弾性体のものから書き直し，それを数値的に解くことによって運動の状態を求めた．はじめにマントルに一様粘性を仮定し，種々の粘性に対する地球の強制振動の計算を行ない，位相遅れの量を求めた．ここで計算に使用した地球内部の密度，弾性定数などは，HADDON and BULLEN (1969) によった．この結果によると，マントルの粘性が 10^{15} ポアズをこえると位相遅れが顕著になり，10^{18} ポアズに達するとほぼ 90° に近づく．ポテンシャル変化の位相遅れだけは少し様子が異るが，これは外力のポテンシャルも付加ポテンシャルに含めて考えているためで，地球の変形によって生ずるポテンシャル変化だけを考えるなら，他の種の位相遅れと同様の結果になるものと思われる．

次に，ANDERSON and HART (1978) の Q モデルによって，Q値をケルビン・フォークト型の粘性値に換算し，粘性の深さによる分布を求め，その分布を使用して，現実の地球になるべく近いものとして地球振動の計算を行なった．Q モデルから得られた粘性値は 10^{10} ポアズの桁であるので，得られる位相遅れの小さいことは予想されたが，果して，角度で 10^{-4} 度に達するかどうかという小さい結果が求められた．このように小さな位相遅れは，通常の地球潮汐の観測では検出が困難と思われる．また，現実に観測されるもっと大きな量の位相遅れは，マントル，地殻の粘性によるものではなく海洋潮汐による荷重，観測点周囲の地形の影響など，他の原因によるものであることが推測される．

この計算の結果から，主要分潮に対する，ラブ数 h, l, k や，減衰定数，重力定数などの D, L, G なども計算された．これらの値は分潮によって多少の差があり，わずかな周波数依存性が認められる．

最後に，粘弾性モデルとして，マクスウェル型を採用したときにどうなるかの考察を行なった．変位と応力の関係をマクスウェル型粘弾性体に対するものに書き直して比較することにより，地球潮汐

のような振動型の運動に対しては，粘性係数の値を変えるだけで，ケルビン・フォークト型の場合と同じ運動に到達することがみちびかれた．マントルの粘性は，ケルビン・フォークト型として，ほぼ 10^{10} ポアズの桁であるが，これをマクスウェル型に換算すると，振動の周波数によって多少の差はあるが，ほぼ 10^{22} ポアズの桁になることがわかる．今までに，氷河が溶け去ったあとのスカンジナビア地方の隆起運動などから，10^{22} ポアズ程度の粘性値が求められているが，これは，マクスウェル型粘弾性体の粘性値に対応しているものである．

いずれにしても，地殻，マントルの粘性の影響は，地球潮汐そのものに対しては非常に小さいことが，この研究によって結論されたといってよい．