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46. A Long Wave in a Rectangular Bay [I]. —Case of Normal Incidence—

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Abstract

In this paper, the long waves in the rectangular bay are elucidated for the normal invasion of a train of periodic waves. The principle of the analysis is based on the method of the buffer domain. Among various results obtained, more interesting facts are that (i) the mouth correction depends upon only the phase of the reflexion coefficient at the estuary for the arrival of the incident waves from the inside of the canal, the values of which are arranged in the paper for several specified values and (ii) the lateral oscillations are most significant for $kl \approx m\pi + \pi/2$ (k: wave number, l: length of the bay and m: non-negative integers), while they vanish completely at $kl = m\pi$.

1. Introduction

We have treated long waves around a breakwater with infinitesimal or finite thickness (*Momoi*, 1969 and 1970a) and in the vicinity of an estuary (*Momoi*, 1970b) based on an exact method, i.e., the buffer domain method. In this paper a long wave in a rectangular bay is elucidated for normal incidence of a wave by the rigorous method employed already previously.

2. Exact Theory

In this section, the exact theory for the wave in the rectangular bay is derived.

2,1. Model used and Nomenclature.

The Configuration of the bay is shown in Fig. 1. The entire domain of waters is assumed to have uniform depth. The length and width of the bay are l and $2d$. The x-axis runs along the coast facing the open sea, the y-axis being taken normally on the axis of symmetry of the bay with the positive off-shore. r and θ in Fig. 1 are polar coordinates.

A train of periodic waves is then propagated normally against the

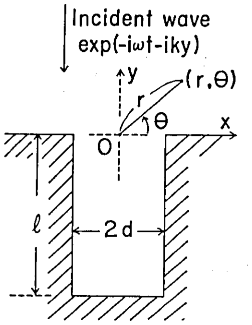


Fig. 1. Model and nemenclature.

coast, which is stated by

$$\exp(-i\omega t - iky), \tag{1}$$

where

ω : the angular frequency,
k : the wave number,
t : the time variable.

2,2. Equation and Boundary Conditions.

For the case of periodic waves, the equation for the long wave is expressed as

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+k^2\zeta=0. \tag{2}$$

The conditions at the rigid boundary are given by

$$\frac{\partial\zeta}{\partial y}=0 \qquad (y=0 \text{ for } |x|>d), \tag{3}$$

$$\frac{\partial\zeta}{\partial x}=0 \qquad (|x|=d \text{ for } y=0 \text{ to } -l), \tag{4}$$

and

$$\frac{\partial\zeta}{\partial y}=0 \qquad (y=-l \text{ for } |x|<d). \tag{5}$$

2,3. Formal Solutions.

Referring to Fig. 2, the entire domain is separated into three parts, i.e.,

domain D_1 : $0<\theta<\pi$ and $r>d$,
domain B : $0<\theta<\pi$ and $r<d$,
domain D_2 : $|x|<d$ and $0>y>-l$.

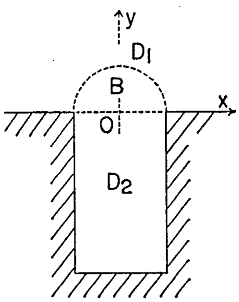


Fig. 2. Geometry of the domains defined.

In the above three domains, domain B is the buffer domain in our method.

Let ζ_j $(j=1, B, 2)$ be the wave heights in domains D_1, B and D_2. The formal solutions are then expressed as

$$\zeta_1=2\cos ky+\sum_{m=0}^{\infty}\zeta_1^{(2m)}H_{2m}^{(1)}(kr)\cos 2m\theta \tag{6}$$

in domain D_1,

$$\zeta_B=\sum_{m=0}^{\infty}\left\{\zeta_B^{(2m)}J_{2m}(kr)\cos 2m\theta+\zeta_B^{(2m+1)}J_{2m+1}(kr)\sin(2m+1)\theta\right\} \tag{7}$$

in domain B,

$$\zeta_2=\sum_{m=0}^{\infty}\zeta_2^{(m)}\cos k_m(l+y)\cos\frac{m\pi}{d}x \tag{8}$$

in domain D_2,

where $\zeta_1^{(2m)}$, $\zeta_B^{(m)}$, $\zeta_2^{(m)}$ are the unknown factors and

$$k_m=\sqrt{k^2-\left(\frac{m\pi}{d}\right)^2} \tag{9}$$

In expression (8), the term $\sin k_m(l+y)$ is excluded for ζ_2 to satisfy condition (5).

2,4. Infinite Simultaneous Equations.

Infinite simultaneous equations are derived from the formal solutions in the foregoing section and from the conditions between the adjacent domains, i.e.,

$$\left.\begin{aligned}\zeta_1&=\zeta_B\\ \frac{\partial\zeta_1}{\partial r}&=\frac{\partial\zeta_B}{\partial r}\end{aligned}\right\}\quad\text{at}\quad r=d\quad\text{and}\quad 0<\theta<\pi, \tag{10}$$

$$\left.\begin{aligned}\zeta_B&=\zeta_2\\ \frac{\partial\zeta_B}{\partial y}&=\frac{\partial\zeta_2}{\partial y}\end{aligned}\right\}\quad\text{at}\quad y=0\quad\text{and}\quad |x|<d. \tag{11}$$

Applying the operator

$$\int_0^{\pi}\cos 2m\theta d\theta\qquad(m=0,1,2,\cdots)$$

after substitution of (6) and (7) into (10), we have

$$\begin{aligned}&\left\{\begin{matrix}J_{2m}\\ J'_{2m}\end{matrix}\right\}\zeta_B^{(2m)}+\frac{1}{\varepsilon_m}\cdot\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{2n+1}{(2n+1)^2-(2m)^2}\cdot\left\{\begin{matrix}J_{2n+1}\\ J'_{2n+1}\end{matrix}\right\}\zeta_B^{(2n+1)}-\left\{\begin{matrix}H_{2m}^{(1)}\\ H_{2m}^{(1)\prime}\end{matrix}\right\}\zeta_1^{(2m)}\\ &\quad=\frac{2}{\varepsilon_m}\left\{\begin{matrix}J_{2m}\\ J'_{2m}\end{matrix}\right\}\qquad(m=0,1,2,\cdots).\end{aligned} \tag{12}$$

In the above,

$$\left.\begin{aligned}\varepsilon_m&=1\qquad(m=0)\\ &=1/2\qquad(m>0)\end{aligned}\right\},$$

and J_p, J'_p, $H_p^{(1)}$ and $H_p^{(1)\prime}$ are the abbreviations of $J_p(kd)$, $J'_p(kd)$, $H_p^{(1)}(kd)$ and $H_p^{(1)\prime}(kd)$ (p : non-negative integers). The expressions in the wavy bracket are taken in the same order. The conventions are followed in the subsequent reductions, unless otherwise stated.

Likewise, substituting (7) and (8) into (11) and applying the operator

$$\int_0^{d}\cos\frac{m\pi}{d}x\,dx\qquad(m=0,1,2,\cdots),$$

we have

$$\left.\begin{aligned}&\sum_{n=0}^{\infty}K_{n,m}\zeta_B^{(2n)}=\varepsilon_m kd\cos k_m l\cdot\zeta_2^{(m)}\\&\sum_{n=0}^{\infty}L_{n,m}\zeta_B^{(2n+1)}=-\varepsilon_m k_m d\sin k_m l\cdot\zeta_2^{(m)}\end{aligned}\right\} \tag{13}$$

$$(m=0,\ 1,\ 2,\ \cdots),$$

where

$$\left.\begin{aligned}K_{n,m}&=\int_0^{kd}J_{2n}(z)\cos\frac{m\pi}{kd}zdz\\L_{n,m}&=(2n+1)\int_0^{kd}\frac{J_{2n+1}(z)}{z}\cos\frac{m\pi}{kd}zdz\end{aligned}\right\}. \tag{14}$$

Eliminations of $\zeta_1^{(2m)}$ from (12) and $\zeta_2^{(m)}$ from (13) yield

$$\frac{i}{kd}\zeta_B^{(2m)}+\frac{1}{\varepsilon_m}\sum_{n=0}^{\infty}\frac{2n+1}{(2n+1)^2-(2m)^2}\cdot(J_{2n+1}H_{2m}^{(1)\prime}-J'_{2n+1}H_{2m}^{(1)})\zeta_B^{(2n+1)}$$

$$=\frac{2}{\varepsilon_m}\cdot\frac{i}{kd}\qquad(m=0,\ 1,\ 2,\ \cdots) \tag{15}$$

and

$$k_m d\sin k_m l\sum_{n=0}^{\infty}K_{n,m}\zeta_B^{(2n)}+kd\cos k_m l\sum_{n=0}^{\infty}L_{n,m}\zeta_B^{(2n+1)}=0 \tag{16}$$

$$(m=0,\ 1,\ 2,\ \cdots).$$

Equations (15) and (16) are now the infinite simultaneous equations to be obtained.

2,5. Reduction to Finite Simultaneous Equations.

The infinite simultaneous equations are reduced to finite simultaneous equations in this section.

Approximation

$$\left.\begin{aligned}&J_m(z)\fallingdotseq 0\qquad(m\leqq 2p+1)\\&J_m(z)\equiv 0\qquad(m>2p+1)\end{aligned}\right\}\quad(p\text{: positive integers}) \tag{17}$$

for $z<kd$ is applied upon expression (7) in the buffer domain B.

Using approximation (17), equations (15) and (16) are reduced to

$$\frac{i}{kd}\cdot\frac{1}{J_{2m}H_{2m}^{(1)\prime}}X_{2m+1}+\frac{1}{\varepsilon_m}\sum_{n=0}^{p}\frac{2n+1}{(2n+1)^2-(2m)^2}\cdot\left(1-\frac{J'_{2n+1}H_{2m}^{(1)}}{J_{2n+1}H_{2m}^{(1)\prime}}\right)X_{2n+2}$$

$$=\frac{2}{\varepsilon_m}\cdot\frac{i}{kd}\cdot\frac{1}{H_{2m}^{(1)\prime}}\qquad(m=0,\ 1,\ 2,\ \cdots,\ p) \tag{18}$$

and

$$k_m d \exp(ik_m l) \sin k_m l \sum_{n=0}^{p} \frac{K_{n,m}}{J_{2n}} X_{2n+1}$$

$$+kd \exp(ik_m l) \cos k_m l \sum_{n=0}^{p} \frac{L_{n,m}}{J_{2n+1}} X_{2n+2}=0 \tag{19}$$

$$(m=0, 1, 2, \cdots, p)$$

where

$$X_{m+1}=\zeta_B^{(m)} J_m.$$

Equations (18) and (19) are normalized in order to avoid truncation errors in the numerical calculation by the computer. The normalization factors are $J_{2m}H_{2m}^{(1)\prime}$ or $J_{2n+1}H_{2m}^{(1)\prime}$ in (18) and $\exp(ik_m l)$ in (19). Using $2(p+1)$ equations (18) and (19), the unknown factors

$$\zeta_B^{(m)} \qquad (m=0, 1, 2, \cdots, 2p+1) \tag{20}$$

are calculated through the use of the electronic computer. If one takes the degree of the approximation $2p+1$ high enough, an exact discussion of waves begins to be possible. In the calculation of (18), integrals $K_{n,m}$ and $L_{n,m}$ which are expressed by (14) are computed by the same procedure as that described in Section 2 of the fifth work concerning long waves around an estuary (*Momoi,* 1968).

2,6. Computations of ζ_1, ζ_2 and ζ_3.

With a view to avoiding truncation errors in the computer, expression (8) in the bay is reduced to

$$\zeta_2=\sum_{m=0}^{\infty} C_m F_m(y) \cos \frac{m\pi}{d} x, \tag{21}$$

where

$$C_m=\zeta_2^{(m)} \exp(-ik_m l),$$

$$F_m(y)=\frac{1}{2} \exp(-ik_m y)\Big[\exp\{i\cdot 2k_m(l+y)\}+1\Big].$$

Factor C_m in the above expression is calculated from expression (13) with a slight modification, i.e.,

$$\left.\begin{aligned} C_m&=\frac{1}{\varepsilon_m kdP_m}\sum_{n=0}^{p} K_{n,m}\,\zeta_B^{(2n)} \\ \text{or} \qquad & \\ C_m&=\frac{-1}{\varepsilon_m k_m d\,Q_m}\sum_{n=0}^{p} L_{n,m}\,\zeta_B^{(2n+1)} \end{aligned}\right\}, \tag{22}$$

where

$$P_m=\frac{1}{2}\left\{\exp(i\cdot 2k_m l)+1\right\}$$

or

$$Q_m=\frac{1}{2i}\left\{\exp(i\cdot 2k_m l)-1\right\}.$$

The calculation of C_m is made by the first expression of (22) except for $P_m\approx 0$, while C_m for $P_m\approx 0$ is obtained from the second expression. Wave height ζ_2 in the bay is now computed by substitution of unknown factors $\zeta_B^{(m)}$ obtained in the foregoing section into (22).

For wave height ζ_1, substitution of (20) into the first expression of (12) makes possible the calculation of $\zeta_1^{(2m)}$ in the formal solution (6), which leads to the elucidation of waves in the open sea.

The direct substitution of (20) into (7) finally yields wave height ζ_B in the domain B.

3. Discussions

3,1. Application Range of Ippen-Goda's Theory.

Ippen and Goda (1963) derived an approximate theory for wave induced oscillations in rectangular harbours with the limitation $d\ll L$ (d: a half width of the harbour, L: the wavelength). Their result is presented in the following.

$$F_1(x, y)=2\cos ky+(2A/\pi)\sin kl\cdot f(x, y) \tag{23}$$

in the open sea,

$$F_2(x, y)=A\cos k(l+y) \tag{24}$$

in the habour or bay,

where

$$\frac{2}{A}=\cos kl-\frac{2\sin kl}{\pi}I,$$

$$I=kd\int_0^\infty \frac{\sin^2\xi}{\xi^2\gamma}d\xi,$$

$$f(x, y)=kd\int_0^\infty \frac{\sin\xi}{\xi\gamma}e^{-\gamma y/d}\cos(\xi x/d)d\xi,$$

$$\gamma=\sqrt{\xi^2-(kd)^2}.$$

Factor $|A|$ of expression (24) is plotted in Figs. 3-1 to 3-9 for kd and kl together with the value of $|\zeta_2^{(0)}|$ in the expression (8). These figures reveal that Ippen-Goda's approximate theory suits to discuss the qualitative behavior of *longitudinal* oscillation up to $kd\approx 3.0$.

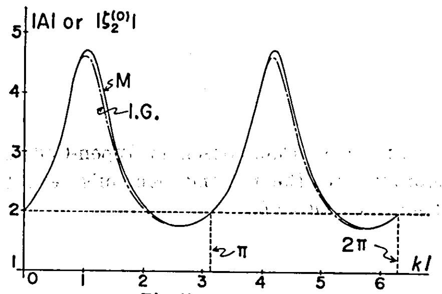


Fig. 3-1. kd=0.5

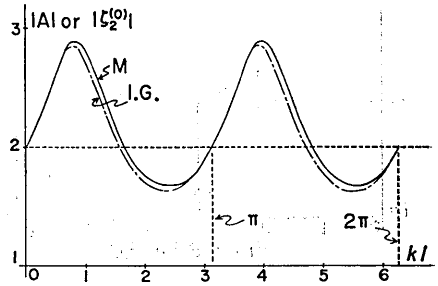


Fig. 3-2. kd=1.0

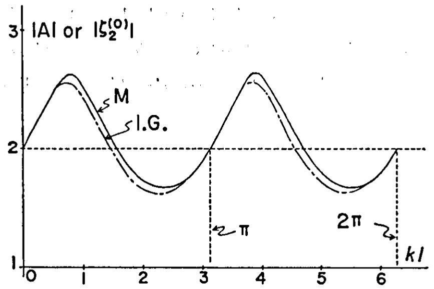


Fig. 3-3. kd=1.2

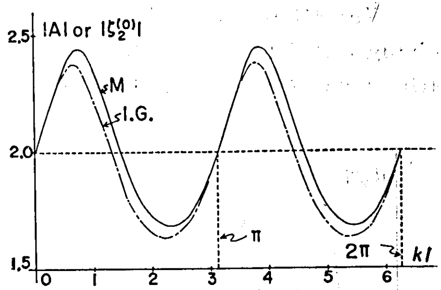


Fig. 3-4. kd=1.4

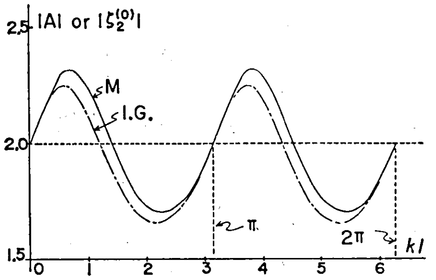


Fig. 3-5. kd=1.6

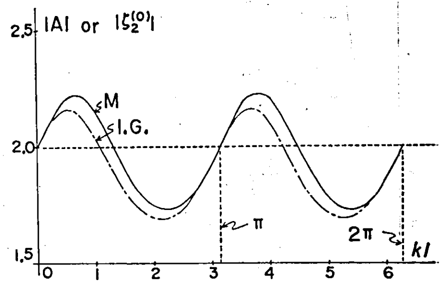


Fig. 3-6. kd=1.8

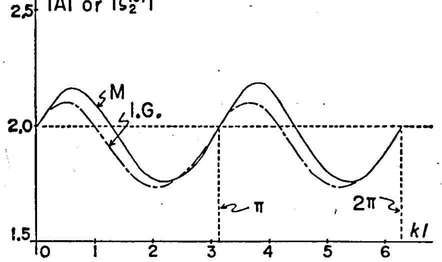


Fig. 3-7. kd=2.0

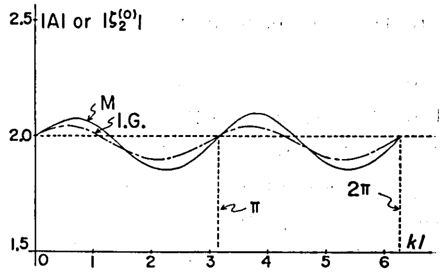


Fig. 3-8. kd=2.6

Figs. 3-1 to 3-8. Comparisons of Ippen-Goda's approximate theory and Momoi's exact theory for $kd=0.5$ to 2.6. The curves designated as **M** and **I.G.** are based on Momoi's and Ippen-Goda's theories respectively. This convention is followed in Fig. 3-9.

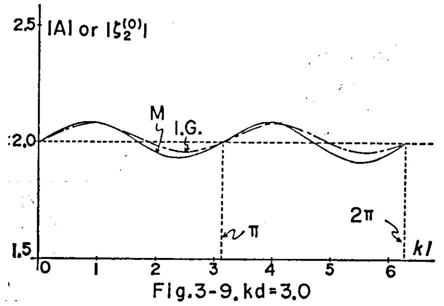
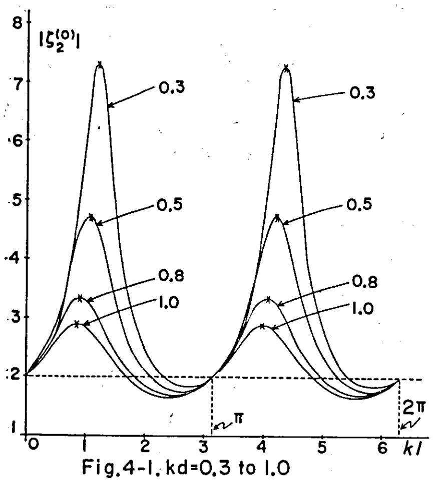
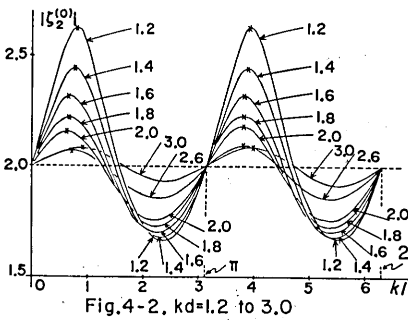


Fig. 3-9. Comparison of Ippen-Goda's approximate theory and Momoi's exact theory for $kd=3.0$.

3,2. Longitudinal Oscillation.

Variations of $|\zeta_2^{(0)}|$ versus kl are given in Figs. 4-1 and 4-2 for the change of kd. These figures expose the following facts. For small kd (the case of Fig. 4-1), the range of kl for $|\zeta_2^{(0)}|>2.0$ exceeds that below





Figs. 4-1 and 4-2. Variations of $|\zeta_2^{(0)}|$ for the change of kd and kl. The first figure is relevant to the variation of kd from 0.3 to 1.0, the second to that from 1.2 to 3.0. The numeral stated on the curve denotes the value of kd.

2.0. The maximum value, which refers to the state of resonance of longitudinal oscillation, becomes larger with the decrease of kd. If one traces the cross points (×) on the curves, it is found that the value of kl, corresponding to the resonance, decreases gradually from $kl=(n+1/2)\pi$ (n: non-negative integer) with kd. The above value attains minimum value $kl\approx n\pi+6.3$ at $kd\approx2.0$, and increases with further increase of kd.

Longitudinal oscillations in the bay are discussed by multiple reflections of two kinds of waves in the problem of an estuary. One of them is the wave transmitted to the canal for normal invasion of the incident wave and the other one the reflected wave for the incoming wave from

Table 1. Degree of Agreement of $|A^{(\infty)}|$ and $|\zeta_2^{(0)}|$*

KD	KL	A(INF)
0.50000	0.0	2.00000
0.50000	0.15708	2.18394
0.50000	0.31416	2.44987
0.50000	0.47124	2.82758
0.50000	0.62832	3.35002
0.50000	0.78540	4.00870
0.50000	0.94248	4.60892
0.50000	1.09956	4.70724
0.50000	1.25664	4.20400
0.50000	1.41372	3.52873
0.50000	1.57080	2.96187
0.50000	1.72788	2.54552
0.50000	1.88496	2.25089
0.50000	2.04204	2.04587
0.50000	2.19911	1.90732
0.50000	2.35619	1.82041
0.50000	2.51327	1.77635
0.50000	2.67035	1.77092
0.50000	2.82743	1.80361
0.50000	2.98451	1.87752
0.50000	3.14159	2.00000
0.50000	3.29867	2.18394
0.50000	3.45575	2.44987
0.50000	3.61283	2.82758
0.50000	3.76991	3.35002
0.50000	3.92699	4.00870
0.50000	4.08407	4.60892
0.50000	4.24115	4.70724
0.50000	4.39823	4.20400
0.50000	4.55531	3.52873
0.50000	4.71239	2.96187
0.50000	4.86947	2.54552
0.50000	5.02655	2.25089
0.50000	5.18363	2.04587
0.50000	5.34071	1.90732
0.50000	5.49779	1.82041
0.50000	5.65487	1.77635
0.50000	5.81195	1.77092
0.50000	5.96903	1.80361
0.50000	6.12611	1.87752
0.50000	6.28319	2.00000
2.00000	0.0	2.00000
2.00000	0.15708	2.06696
2.00000	0.31416	2.12550
2.00000	0.47124	2.16626
2.00000	0.62832	2.18156
2.00000	0.78540	2.16825
2.00000	0.94248	2.12909
2.00000	1.09956	2.07151
2.00000	1.25664	2.00484
2.00000	1.41372	1.93772
2.00000	1.57080	1.87672
2.00000	1.72788	1.82617
2.00000	1.88496	1.78864
2.00000	2.04204	1.76549
2.00000	2.19911	1.75735
2.00000	2.35619	1.76441
2.00000	2.51327	1.78651
2.00000	2.67035	1.82305
2.00000	2.82743	1.87274
2.00000	2.98451	1.93313
2.00000	3.14159	2.00000
2.00000	3.29867	2.06696
2.00000	3.45575	2.12550
2.00000	3.61283	2.16626
2.00000	3.76991	2.18156
2.00000	3.92699	2.16825
2.00000	4.08407	2.12909
2.00000	4.24115	2.07151
2.00000	4.39823	2.00484
2.00000	4.55531	1.93772
2.00000	4.71239	1.87672
2.00000	4.86947	1.82617
2.00000	5.02655	1.78864
2.00000	5.18363	1.76549
2.00000	5.34071	1.75735
2.00000	5.49779	1.76441
2.00000	5.65487	1.78651
2.00000	5.81195	1.82305
2.00000	5.96903	1.87274
2.00000	6.12611	1.93313
2.00000	6.28319	2.00000
1.00000	0.0	2.00000
1.00000	0.15708	2.16720
1.00000	0.31416	2.36880
1.00000	0.47124	2.58802
1.00000	0.62832	2.78417
1.00000	0.78540	2.89508
1.00000	0.94248	2.87281
1.00000	1.09956	2.72778
1.00000	1.25664	2.51820
1.00000	1.41372	2.30149
1.00000	1.57080	2.10989
1.00000	1.72788	1.95469
1.00000	1.88496	1.83673
1.00000	2.04204	1.75321
1.00000	2.19911	1.70096
1.00000	2.35619	1.67759
1.00000	2.51327	1.68199
1.00000	2.67035	1.71437
1.00000	2.82743	1.77628
1.00000	2.98451	1.87040
1.00000	3.14159	2.00000
1.00000	3.29867	2.16720
1.00000	3.45575	2.36880
1.00000	3.61283	2.58802
1.00000	3.76991	2.78417
1.00000	3.92699	2.89508
1.00000	4.08407	2.87281
1.00000	4.24115	2.72778
1.00000	4.39823	2.51820
1.00000	4.55531	2.30149
1.00000	4.71239	2.10989
1.00000	4.86947	1.95469
1.00000	5.02655	1.83673
1.00000	5.18363	1.75321
1.00000	5.34071	1.70096
1.00000	5.49779	1.67759
1.00000	5.65487	1.68199
1.00000	5.81195	1.71437
1.00000	5.96903	1.77628
1.00000	6.12611	1.87040
1.00000	6.28319	2.00000
3.00000	0.0	2.00000
3.00000	0.15708	2.02672
3.00000	0.31416	2.05143
3.00000	0.47124	2.07130
3.00000	0.62832	2.08386
3.00000	0.78540	2.08746
3.00000	0.94248	2.08159
3.00000	1.09956	2.06707
3.00000	1.25664	2.04578
3.00000	1.41372	2.02032
3.00000	1.57080	1.99353
3.00000	1.72788	1.96809
3.00000	1.88496	1.94624
3.00000	2.04204	1.92973
3.00000	2.19911	1.91974
3.00000	2.35619	1.91694
3.00000	2.51327	1.92152
3.00000	2.67035	1.93317
3.00000	2.82743	1.95110
3.00000	2.98451	1.97400
3.00000	3.14159	2.00000
3.00000	3.29867	2.02672
3.00000	3.45575	2.05143
3.00000	3.61283	2.07130
3.00000	3.76991	2.08386
3.00000	3.92699	2.08746
3.00000	4.08407	2.08159
3.00000	4.24115	2.06707
3.00000	4.39823	2.04578
3.00000	4.55531	2.02032
3.00000	4.71239	1.99353
3.00000	4.86947	1.96809
3.00000	5.02655	1.94624
3.00000	5.18363	1.92973
3.00000	5.34071	1.91974
3.00000	5.49779	1.91694
3.00000	5.65487	1.92152
3.00000	5.81195	1.93317
3.00000	5.96903	1.95110
3.00000	6.12611	1.97400
3.00000	6.28319	2.00000

* KD, KL and A(INF) denote, respectively, *kd*, *kl* and $|A^{(\infty)}|$. The line inserted into the numerals of A(INF) denotes the critical order of $|A^{(\infty)}|$ above which the numeral is in complete agreement with that of $|\zeta_2^{(0)}|$.

Table 2. Degree of p for $A^{(c)}_{error}=0.1$*

KD= 0.05000	KL= 0.0	A(INF)= 2.00000	A(P)= 1.82469	P=23	EPS= 0.10
KD= 0.05000	KL= 1.57080	A(INF)= 14.97135	A(P)= 13.65900	P=23	EPS= 0.10
KD= 0.05000	KL= 3.14159	A(INF)= 2.00000	A(P)= 1.82469	P=23	EPS= 0.10
KD= 0.05000	KL= 4.71239	A(INF)= 14.97135	A(P)= 13.65900	P=23	EPS= 0.10
KD= 0.05000	KL= 6.28319	A(INF)= 2.00000	A(P)= 1.82469	P=23	EPS= 0.10
KD= 0.10000	KL= 0.0	A(INF)= 2.00000	A(P)= 1.98043	P=11	EPS= 0.10
KD= 0.10000	KL= 1.57080	A(INF)= 8.82835	A(P)= 8.74194	P=11	EPS= 0.10
KD= 0.10000	KL= 3.14159	A(INF)= 2.00000	A(P)= 1.98043	P=11	EPS= 0.10
KD= 0.10000	KL= 4.71239	A(INF)= 8.82835	A(P)= 8.74194	P=11	EPS= 0.10
KD= 0.10000	KL= 6.28319	A(INF)= 2.00000	A(P)= 1.98043	P=11	EPS= 0.10
KD= 0.20000	KL= 0.0	A(INF)= 2.00021	A(P)= 1.96328	P= 6	EPS= 0.10
KD= 0.20000	KL= 1.57080	A(INF)= 5.35740	A(P)= 5.48685	P= 6	EPS= 0.10
KD= 0.20000	KL= 3.14159	A(INF)= 2.00021	A(P)= 1.96328	P= 6	EPS= 0.10
KD= 0.20000	KL= 4.71239	A(INF)= 5.35740	A(P)= 5.48685	P= 6	EPS= 0.10
KD= 0.20000	KL= 6.28319	A(INF)= 2.00021	A(P)= 1.96328	P= 6	EPS= 0.10
KD= 0.30000	KL= 0.0	A(INF)= 2.00000	A(P)= 1.90645	P= 4	EPS= 0.10
KD= 0.30000	KL= 1.57080	A(INF)= 4.07106	A(P)= 4.27522	P= 4	EPS= 0.10
KD= 0.30000	KL= 3.14159	A(INF)= 2.00000	A(P)= 1.90645	P= 4	EPS= 0.10
KD= 0.30000	KL= 4.71239	A(INF)= 4.07106	A(P)= 4.27522	P= 4	EPS= 0.10
KD= 0.30000	KL= 6.28319	A(INF)= 2.00000	A(P)= 1.90645	P= 4	EPS= 0.10
KD= 0.40000	KL= 0.0	A(INF)= 2.00965	A(P)= 2.13083	P= 3	EPS= 0.10
KD= 0.40000	KL= 1.57080	A(INF)= 3.40176	A(P)= 3.60690	P= 3	EPS= 0.10
KD= 0.40000	KL= 3.14159	A(INF)= 2.00965	A(P)= 2.13083	P= 3	EPS= 0.10
KD= 0.40000	KL= 4.71239	A(INF)= 3.40176	A(P)= 3.60690	P= 3	EPS= 0.10
KD= 0.40000	KL= 6.28319	A(INF)= 2.00965	A(P)= 2.13083	P= 3	EPS= 0.10
KD= 0.50000	KL= 0.0	A(INF)= 2.00000	A(P)= 1.80839	P= 2	EPS= 0.10
KD= 0.50000	KL= 1.57080	A(INF)= 2.96187	A(P)= 3.24565	P= 2	EPS= 0.10
KD= 0.50000	KL= 3.14159	A(INF)= 2.00000	A(P)= 1.80839	P= 2	EPS= 0.10
KD= 0.50000	KL= 4.71239	A(INF)= 2.96187	A(P)= 3.24565	P= 2	EPS= 0.10
KD= 0.50000	KL= 6.28319	A(INF)= 2.00000	A(P)= 1.80839	P= 2	EPS= 0.10
KD= 1.00000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.14061	P= 1	EPS= 0.10
KD= 1.00000	KL= 1.57080	A(INF)= 2.10989	A(P)= 2.25822	P= 1	EPS= 0.10
KD= 1.00000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.14061	P= 1	EPS= 0.10
KD= 1.00000	KL= 4.71239	A(INF)= 2.10989	A(P)= 2.25822	P= 1	EPS= 0.10
KD= 1.00000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.14061	P= 1	EPS= 0.10
KD= 2.00000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.01911	P= 1	EPS= 0.10
KD= 2.00000	KL= 1.57080	A(INF)= 1.87672	A(P)= 1.89465	P= 1	EPS= 0.10
KD= 2.00000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.01911	P= 1	EPS= 0.10
KD= 2.00000	KL= 4.71239	A(INF)= 1.87672	A(P)= 1.89465	P= 1	EPS= 0.10
KD= 2.00000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.01911	P= 1	EPS= 0.10
KD= 3.00000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.00362	P= 1	EPS= 0.10
KD= 3.00000	KL= 1.57080	A(INF)= 1.99353	A(P)= 1.99714	P= 1	EPS= 0.10
KD= 3.00000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.00362	P= 1	EPS= 0.10
KD= 3.00000	KL= 4.71239	A(INF)= 1.99353	A(P)= 1.99714	P= 1	EPS= 0.10
KD= 3.00000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.00362	P= 1	EPS= 0.10

* The representation KD=kd, KL=kl, A(INF)=$|A^{(\infty)}|$, A(P)=$|A^{(p)}|$, P=p and EPS=$A^{(c)}_{error}$ are employed, of which the conventions are followed in Table 3.

the inside of the canal. These problems have already been discussed in a series of papers concerning long waves in the vicinity of an estuary (*Momoi*, 1968 and 1970b).

Let T and R be the transmission and reflexion coefficients at the mouth of the estuary, respectively, for the invasions of the incoming wave from the open sea and from the inside of the canal. Using the same notation and definition as in Section 2, the transmitted wave from the open sea to the bay is expressed as Te^{-iky}. The transmitted wave produces the reflected wave at the head of the bay ($y=-l$) to be des-

Table 3. Degree of p for $A^{(c)}_{error}=0.01$

KD= 0.05000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.01138	P=46	EPS= 0.01
KD= 0.05000	KL= 1.57080	A(INF)= 14.97135	A(P)= 14.88709	P=46	EPS= 0.01
KD= 0.05000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.01138	P=46	EPS= 0.01
KD= 0.05000	KL= 4.71239	A(INF)= 14.97135	A(P)= 14.88709	P=46	EPS= 0.01
KD= 0.05000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.01138	P=46	EPS= 0.01
KD= 0.10000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.01899	P=23	EPS= 0.01
KD= 0.10000	KL= 1.57080	A(INF)= 8.82835	A(P)= 8.91218	P=23	EPS= 0.01
KD= 0.10000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.01899	P=23	EPS= 0.01
KD= 0.10000	KL= 4.71239	A(INF)= 8.82835	A(P)= 8.91218	P=23	EPS= 0.01
KD= 0.10000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.01899	P=23	EPS= 0.01
KD= 0.20000	KL= 0.0	A(INF)= 2.00021	A(P)= 1.99362	P=12	EPS= 0.01
KD= 0.20000	KL= 1.57080	A(INF)= 5.35740	A(P)= 5.37542	P=12	EPS= 0.01
KD= 0.20000	KL= 3.14159	A(INF)= 2.00021	A(P)= 1.99362	P=12	EPS= 0.01
KD= 0.20000	KL= 4.71239	A(INF)= 5.35740	A(P)= 5.37542	P=12	EPS= 0.01
KD= 0.20000	KL= 6.28319	A(INF)= 2.00021	A(P)= 1.99362	P=12	EPS= 0.01
KD= 0.30000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.01129	P= 8	EPS= 0.01
KD= 0.30000	KL= 1.57080	A(INF)= 4.07106	A(P)= 4.04832	P= 8	EPS= 0.01
KD= 0.30000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.01129	P= 8	EPS= 0.01
KD= 0.30000	KL= 4.71239	A(INF)= 4.07106	A(P)= 4.04832	P= 8	EPS= 0.01
KD= 0.30000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.01129	P= 8	EPS= 0.01
KD= 0.40000	KL= 0.0	A(INF)= 2.00965	A(P)= 2.00618	P= 7	EPS= 0.01
KD= 0.40000	KL= 1.57080	A(INF)= 3.40176	A(P)= 3.39589	P= 7	EPS= 0.01
KD= 0.40000	KL= 3.14159	A(INF)= 2.00965	A(P)= 2.00618	P= 7	EPS= 0.01
KD= 0.40000	KL= 4.71239	A(INF)= 3.40176	A(P)= 3.39589	P= 7	EPS= 0.01
KD= 0.40000	KL= 6.28319	A(INF)= 2.00965	A(P)= 2.00618	P= 7	EPS= 0.01
KD= 0.50000	KL= 0.0	A(INF)= 2.00000	A(P)= 1.98165	P= 5	EPS= 0.01
KD= 0.50000	KL= 1.57080	A(INF)= 2.96187	A(P)= 2.93469	P= 5	EPS= 0.01
KD= 0.50000	KL= 3.14159	A(INF)= 2.00000	A(P)= 1.98165	P= 5	EPS= 0.01
KD= 0.50000	KL= 4.71239	A(INF)= 2.96187	A(P)= 2.93469	P= 5	EPS= 0.01
KD= 0.50000	KL= 6.28319	A(INF)= 2.00000	A(P)= 1.98165	P= 5	EPS= 0.01
KD= 1.00000	KL= 0.0	A(INF)= 2.00000	A(P)= 1.99062	P= 3	EPS= 0.01
KD= 1.00000	KL= 1.57080	A(INF)= 2.10989	A(P)= 2.09999	P= 3	EPS= 0.01
KD= 1.00000	KL= 3.14159	A(INF)= 2.00000	A(P)= 1.99062	P= 3	EPS= 0.01
KD= 1.00000	KL= 4.71239	A(INF)= 2.10989	A(P)= 2.09999	P= 3	EPS= 0.01
KD= 1.00000	KL= 6.28319	A(INF)= 2.00000	A(P)= 1.99062	P= 3	EPS= 0.01
KD= 2.00000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.00197	P= 2	EPS= 0.01
KD= 2.00000	KL= 1.57080	A(INF)= 1.87672	A(P)= 1.87487	P= 2	EPS= 0.01
KD= 2.00000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.00197	P= 2	EPS= 0.01
KD= 2.00000	KL= 4.71239	A(INF)= 1.87672	A(P)= 1.87487	P= 2	EPS= 0.01
KD= 2.00000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.00197	P= 2	EPS= 0.01
KD= 3.00000	KL= 0.0	A(INF)= 2.00000	A(P)= 2.00362	P= 1	EPS= 0.01
KD= 3.00000	KL= 1.57080	A(INF)= 1.99353	A(P)= 1.99714	P= 1	EPS= 0.01
KD= 3.00000	KL= 3.14159	A(INF)= 2.00000	A(P)= 2.00362	P= 1	EPS= 0.01
KD= 3.00000	KL= 4.71239	A(INF)= 1.99353	A(P)= 1.99714	P= 1	EPS= 0.01
KD= 3.00000	KL= 6.28319	A(INF)= 2.00000	A(P)= 2.00362	P= 1	EPS= 0.01

cribed by $Te^{i2kl+iky}$. The wave expressed lastly is reflected at the bay mouth and propagated toward the bay head, which is described as $TRe^{i2kl-iky}$. This wave is again reflected at the bay head to yield the wave $TRe^{i4kl+iky}$.

The above procedure is repeated. Let W_p be the wave produced as the result of p time reflections at the bay mouth. W_p is given by

$$W_p=\sum_{n=0}^{p} TR^n e^{i2nkl-iky}+\sum_{n=0}^{p} TR^n e^{i2(n+1)kl+iky}.$$

After a simple algebraic reduction, the above expression is reduced to

$$W_p = A^{(p)} \cos k(l+y), \tag{25}$$

where

$$A^{(p)} = 2Te^{ikl} \frac{1 - R^{p+1} e^{i2(p+1)kl}}{1 - Re^{i2kl}}. \tag{26}$$

If p tends to infinity, the relation

$$\begin{aligned} \zeta_2^{(0)} &\approx \lim_{p \to \infty} A^{(p)} \\ &= A^{(\infty)} \\ &= \frac{2Te^{ikl}}{1 - Re^{i2kl}} \end{aligned} \tag{27}$$

must hold.

The exact theory in Section 2 is ascertained numerically through the use of relation (27) in order to avoid unexpected results arising from truncation errors in the numerical calculations. The calculated results are shown in Table 1. Inspection of this table reveals that $|A^{(\infty)}|$ is in very good agreement with $|\zeta_2^{(0)}|$.

Let $A_{error}^{(p)}$ be the relative error of $A^{(p)}$ from $A^{(\infty)}$ to be defined as

$$A_{error}^{(p)} = \left| \frac{A^{(\infty)} - A^{(p)}}{A^{(\infty)}} \right|. \tag{28}$$

If one sets a critical value $A_{error}^{(c)}$ of $A_{error}^{(p)}$, the smallest p making $A_{error}^{(p)} \leqq A_{error}^{(c)}$ is evaluated from (28). The evaluations are carried out for $A_{error}^{(c)} = 0.1$ and 0.01. The results are arranged in Tables 2 and 3.

Inspection of these tables exposes the following. The degree of p decreases rapidly with increase of kd, which denotes strong trapping of the wave in the bay for very long wavelength. The value of p for the one percentage error (Table 3) is nearly twice of that for the ten percentage error (Table 2). When $kd = 1.0$, the error becomes less than 10 (or 1) percent after one (or three) time reflection at the bay mouth for $A_{error}^{(c)} = 0.1$ (or 0.01).

3,3. Mouth Correction.

The mouth correction for free oscillations in the rectangular bay is discussed in this section. The uncorrected period T_u of the free oscillation of the bay, having the node at its mouth and the loop at its end, is given by the formula

$$T_u = \frac{2\pi l}{V(\pi/2 + n\pi)} \tag{29}$$

(V: the velocity of the long wave and n: the non-negative integers), provided the correction due to the mouth be neglected. The corrected period T_c may readily be found in the following way.

The corrected period of the free oscillation corresponds to that of the resonance of the wave in the bay discussed in the foregoing section. In order to derive the expression of the resonance period, equation (27) is employed, which is obtained on the basis of the method of multiple reflection. The validity of the use of expression (27) in place of the rigorous expression $\zeta_2^{(0)}$ in section 2 is well ascertained by Table 1. Using (27), the period T_c is derived as follows.

The absolute value of (27) becomes

$$|A^{(\infty)}|=\frac{2|T|}{B}, \tag{30}$$

where

$$\begin{aligned}|B|^2&=B\cdot\overline{B} \qquad (\overline{B}: \text{ the conjugate value of } B)\\ &=1-A_R e^{i\alpha+i2kl}-A_R e^{-i\alpha-i2kl}+A_R^2\end{aligned} \tag{31}$$

and

$$R=A_R e^{i\alpha} \qquad (\text{the vector expression of complex } R).$$

The procedure making $|A^{(\infty)}|$ a maximum for variable kl in (30) can be replaced by that making $|B|^2$ a minimum. Differentiation of $|B|^2$ of (31) yields a minimum point of kl and the maximum value of $|A^{(\infty)}|$, i.e.,

$$kl=-\frac{\alpha}{2}+n\pi \qquad (n=0, 1, 2, \cdots) \tag{32}$$

and

$$|A^{(\infty)}|_{\max}=\frac{2|T|}{1-A_R}. \tag{33}$$

Setting down $-\alpha=\pi-2\alpha_M$ in (32), the corrected period T_c becomes

$$T_c=\frac{2\pi l}{V(\pi/2+n\pi-\alpha_M)} \qquad (n=0, 1, 2, \cdots), \tag{34}$$

where the relation $k=2\pi/VT_c$ is employed. As found in (32), the mouth correction is dependent only upon α (the phase of the reflexion coefficient for invasion of the incoming wave from the inside of the canal).

The ratio of the corrected period to the uncorrected is given, from (29) and (34), by

$$\frac{T_c}{T_u}=\frac{\pi/2+n\pi}{\pi/2+n\pi-\alpha_M} \qquad (n=0, 1, 2, \cdots). \tag{35}$$

Letting n tend to infinity, the expression (35) is reduced to

$$\frac{T_c}{T_u} \rightarrow 1 \qquad (n \rightarrow \infty).$$

The above relation denotes that higher modes of free oscillations are less sensitive to the mouth correction than the fundamental mode.

The calculated results for the fundamental oscillation ($n=0$) are arranged in Table 4. According to this table, the ratio of the corrected

Table 4. Mouth correction in the rectangular bay*

KD=	0.05	AL=	0.12356	KL=	1.44724	L/B=	14.472	TC/TU=	1.08538	AMAX=	40.33
KD=	0.06	AL=	0.14123	KL=	1.42957	L/B=	11.913	TC/TU=	1.09879	AMAX=	33.69
KD=	0.08	AL=	0.17349	KL=	1.39731	L/B=	8.733	TC/TU=	1.12416	AMAX=	25.41
KD=	0.10	AL=	0.20252	KL=	1.36828	L/B=	6.841	TC/TU=	1.14801	AMAX=	20.45
KD=	0.20	AL=	0.31690	KL=	1.25390	L/B=	3.135	TC/TU=	1.25273	AMAX=	10.59
KD=	0.30	AL=	0.40159	KL=	1.16921	L/B=	1.949	TC/TU=	1.34347	AMAX=	7.33
KD=	0.40	AL=	0.47012	KL=	1.10068	L/B=	1.376	TC/TU=	1.42712	AMAX=	5.74
KD=	0.50	AL=	0.52719	KL=	1.04361	L/B=	1.044	TC/TU=	1.50516	AMAX=	4.75

* The representations KD=kd, AL=α_M, KL=kl, L/B=l/b, TC/TU=T_c/T_u and AMAX $=|A^{(\infty)}|_{\max}$ are employed.

period to the uncorrected begins to be gradually larger than 1.0 with the decrease of l/b from infinity (l: length of the bay and b: breadth of the bay) to amount finally to about 1.5 for $l/b \approx 1.0$. The values in Table 4 are plotted in Fig. 5 together with those obtained by Honda et al's (1908) on the basis of approximated method. Fig. 5 reveals that Momoi's result is in very good agreement with Honda's in the range over $l/b \approx 7$, while the difference becomes more significant with the decrease of l/b below 7. This departure might be ascribed to the approximation employed in Honda et al's theory. Deriving their theory, they employed approximations that the potential energy in the open sea might be neglected as compared with that in the bay, that there exist no lateral modes in the bay and further that $l \gg b$. These approximations may be acceptable only for a relatively small kd in which the wave height in the bay is great compared with that in the open sea as ascertained from the values of $|A^{(\infty)}|_{\max}$ in Table 4, and in which the lateral modes are of negligible order. Since small values of kd for resonance, as shown in Table 4, refer to large values of l/b, the agreement of

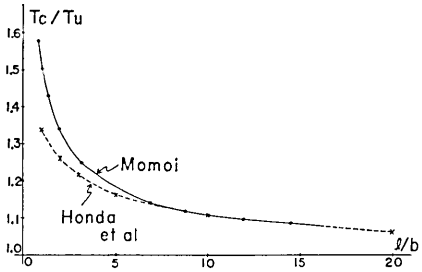


Fig. 5. Comparison of the mouth corrections calculated by Momoi's and Honda et al's theories.

Momoi's and Honda et al's results is more favourable in the range of larger l/b than smaller l/b.

3,4. Lateral Oscillation.

The amplitude factor of higher modes is discussed. The first mode $|\zeta_2^{(1)}|$ of expression (8) is plotted as function of kd and kl in Fig. 6. Since the first mode in the bay is described as $\zeta_2^{(1)} \cos k_1(l+y) \cos (\pi x/d)$ in complete form, $|\zeta_2^{(1)}|$ denotes the amplitude of the first mode at the midpoint and the two corners of the head of the bay ($y=-l$, $x=0$ and $\pm d$). In Fig. 6, it is of greatest interest that the first mode disappears for $kl=n\pi$ (n: positive integers). After trying various numerical calculations, all the higher modes seem to vanish for the above critical values $kl=n\pi$, though the computed results are not shown in this paper. An approximate verification for the above fact, though not complete, is given in the following. A thorough verification will be made after the reflexion coefficient at the mouth of the canal has been discussed for the outflow of the wave of higher modes. The amplitude factor for higher modes is derived by the method of multiple reflexions under a certain assumption.

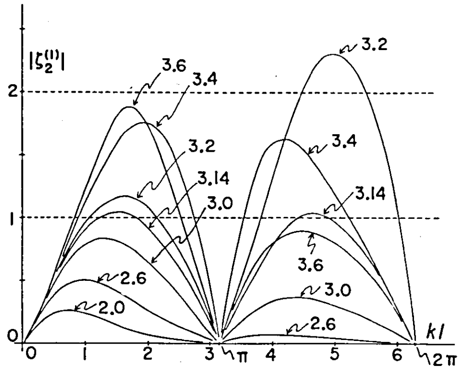


Fig. 6. Variation of $|\zeta_2^{(1)}|$ for the changes of kd and kl. The numerals on the curves denote the values of kd.

For the arrival of the incident wave $e^{-i\omega t-iky}$ from the open sea at the mouth of the estuary, the higher modes generated in the canal are expressed as

$$\zeta_i^{(0)}=\sum_{m=1}^{\infty} T_m e^{-ik_m y} \cos \frac{m\pi}{d} x, \tag{36}$$

where T_m is the amplitude factor of higher mode. The wave in (36) is reflected at the bay head to produce the retrograding wave

$$\zeta_r^{(0)}=\sum_{m=1}^{\infty} T_m e^{ik_m (y+2l)} \cos \frac{m\pi}{d} x. \tag{37}$$

Supposing that the wave reflected from the bay head, expressed by (37), does not reach the bay mouth with a sufficient magnitude, a slight amount of the wave is reflected at the mouth toward the inside of the bay, which may be permitted to be set equal to zero. This supposition might be accepted, if l/d is large enough in the range $kd<\pi$. The above assumption is employed repeatedly for the reflected wave of the above type. The zeroth mode of the reflected wave from the bay head is $Te^{ik(y+2l)}$ as described in Section 3,2, which produces the higher modes of the wave reflected at the bay mouth, i.e.,

$$\zeta_i^{(1)}=\sum_{m=1}^{\infty}R_m Te^{2ikl}e^{-ik_m y}\cos\frac{m\pi}{d}x. \tag{38}$$

In the above, R_m is the amplitude factor of higher mode of the reflected wave ζ_{ref} at the mouth of the estuary for the invasion of the incident wave $e^{-i\omega t+iky}$ from the inside of the canal, as expressed by

$$\text{higher modes of }\ \zeta_{ref}=\sum_{m=1}^{\infty}R_m e^{-ik_m y}\cos\frac{m\pi}{d}x.$$

The reflected wave of (38) at the bay head is given by

$$\zeta_r^{(1)}=\sum_{m=1}^{\infty}R_m Te^{2ikl}e^{ik_m(y+2l)}\cos\frac{m\pi}{d}x. \tag{39}$$

Under the assumption employed in the present analysis, the reflected waves of (39) are no more produced at the bay mouth.

The above process is repeated. After the n-th arrival of the reflected zeroth mode at the bay mouth, the higher modes produced at the bay mouth and head are, respectively,

$$\zeta_i^{(n)}=\sum_{m=1}^{\infty}R_m TR^{n-1}e^{2nikl}e^{-ik_m y}\cos\frac{m\pi}{d}x \tag{40}$$

and

$$\zeta_r^{(n)}=\sum_{m=1}^{\infty}R_m TR^{n-1}e^{2nikl}e^{ik_m(y+2l)}\cos\frac{m\pi}{d}x. \tag{41}$$

Using (36) to (41), the higher mode waves $\zeta_{h.m.}^{(n)}$ in the bay, after the $(n+1)$-th arrival of the zeroth mode at the mouth, are given by

$$\begin{aligned}\zeta_{h.m.}^{(n)}&=\sum_{n=0}^{n+1}(\zeta_i^{(n)}+\zeta_r^{(n)})\\&=\sum_{m=1}^{\infty}A_m^{(n)}\cos k_m(y+l)\cos\frac{m\pi}{d}x,\end{aligned} \tag{42}$$

where

$$A_m^{(n)}=2e^{ik_m l}\left(T_m+R_m Te^{2ikl}\frac{1-R^{n+1}e^{2(n+1)ikl}}{1-Re^{2ikl}}\right) \tag{43}$$

after a few algebraic reductions. If n tends to infinity, (43) is reduced to

$$A_m^{(\infty)}=2e^{ik_ml}\left(T_m+\frac{R_mTe^{2ikl}}{1-Re^{2ikl}}\right). \tag{44}$$

Within the range of the assumption employed in the above development, the relation

$$\zeta_2^{(m)}\approx A_m^{(\infty)} \tag{45}$$

must now hold, where $\zeta_2^{(m)}$ is the amplitude factor of higher mode in the bay for the exact theory developed in Section 2.

Using the relations $T=1-R$ and $T_m=R_m$ $(m\geq 1)$, the verification of which is carried out in the appendix, the amplitude factor (44) is reduced to

$$A_m^{(\infty)}=2e^{ik_ml}T_m\cdot\frac{1-e^{2ikl}}{1-Re^{2ikl}}. \tag{46}$$

Putting $kl=n\pi$ (n: positive integers) in (46), this expression is found to be completely equal to zero. The vanishing of $|\zeta_2^{(1)}|$ at $kl=n\pi$ in Fig. 6 is now ascertained.

As known from (46), $|\zeta_2^{(1)}|$ takes a maximum value for $kl\approx m\pi+\pi/2$ (m: non-negative integers).

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Appendix

In this appendix, the verification of the relation

$$\zeta_{op}^{(m)}+\zeta_{cl}^{(m)}=\varphi_m \tag{a1}$$
$$(\varphi_0=1,\ \varphi_m=0 \text{ for } m\geqq 1)$$

is performed. $\zeta_{op}^{(m)}$ and $\zeta_{cl}^{(m)}$ are the amplitude factors of the waves produced in the canal, respectively, for the invasion of the incoming waves e^{-iky} and e^{iky} from the open sea and the head of the canal (refer to Fig. a1). The wave heights ζ_{op} for the former and ζ_{cl} for the latter are described as

$$\zeta_{op}=\sum_{m=0}^{\infty}\zeta_{op}^{(m)}e^{-ik_m y}\cos\frac{m\pi}{d}x \tag{a2}$$

and

$$\zeta_{cl}=e^{iky}+\sum_{m=0}^{\infty}\zeta_{cl}^{(m)}e^{-ik_m y}\cos\frac{m\pi}{d}x, \tag{a3}$$

where

$$k_m=\sqrt{k^2-\left(\frac{m\pi}{d}\right)^2}.$$

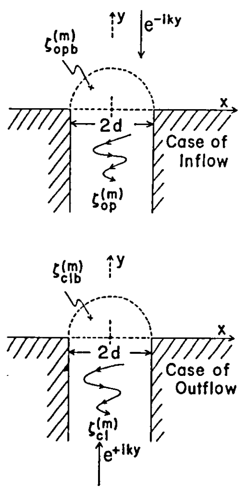

Fig. a1. Nomenclature of the model used.

For the geometry of the model used, Fig. a1 should be referred to. The expression (a2) refers to (13) of the fifth work (*Momoi*, 1968) concerning the long wave around the estuary and the expression (a3) to (4) of the seventh paper (*Momoi*, 1970b), the former of which is referred to as paper V and the latter as paper VII in the following discussions.

The amplitude factors $\zeta_{op}^{(m)}$ and $\zeta_{cl}^{(m)}$ are expressed as

$$\zeta_{op}^{(m)}=\frac{1}{\varepsilon_m kd}\sum_{n=0}^{l}K'_{n,m}\zeta_{opb}^{(2n)} \tag{a4}$$

from (9) in paper V

and

$$\zeta_{cl}^{(m)}=\frac{1}{\varepsilon_m kd}\sum_{n=0}^{l}K'_{n,m}\zeta_{clb}^{(2n)}-\varphi_m \tag{a5}$$

from (9) in paper VII

where

$\varepsilon_0=1$, $\varepsilon_m=1/2$ $(m\geqq1)$,

$\zeta_{opb}^{(2n)}=\bar{\zeta}_2^{(2n)}$ of paper V,

$\zeta_{clb}^{(2n)}=\zeta_2^{(2n)}$ of paper VII,

$K'_{n,m}=I(J_{2n}, m)$ of paper V

and m are non-negative integers. The addition of (a4) to (a5) yields

$$\zeta_{op}^{(m)}+\zeta_{cl}^{(m)}=\frac{1}{\varepsilon_m kd}\sum_{n=0}^{l}K'_{n,m}(\zeta_{opb}^{(2n)}+\zeta_{clb}^{(2n)})-\varphi_m \tag{a6}$$

$$(m=0, 1, 2, \cdots).$$

Expression (a6) will be used later.

Finite simultaneous equations for determining $\zeta_{opb}^{(2m)}$ and $\zeta_{opb}^{(2m+1)}$ $(=\zeta_2^{(2m+1)}$ in paper V) are given by (6) and (7) in paper V as follows.

$$i\cdot\frac{k_m d}{kd}\sum_{n=0}^{l}K'_{n,m}\zeta_{opb}^{(2n)}+\sum_{n=0}^{l}L'_{n,m}\zeta_{opb}^{(2n+1)}=0 \tag{a7}$$

and

$$\varepsilon_m\zeta_{opb}^{(2m)}-ikd\sum_{n=0}^{l}P_{m,n}\zeta_{opb}^{(2n+1)}=2, \tag{a8}$$

where the following transfer of the notations is made, i.e.,

$k_m=k_1^{(m)}$ in paper V,

$L'_{n,m}=I\left(\frac{J_{2n+1}}{r}, m\right)$ in paper V,

$$P_{m,n}=\frac{2n+1}{(2n+1)^2-(2m)^2}\left\{J_{2n+1}(kd)H_{2m}^{(1)\prime}(kd)-J'_{2n+1}(kd)H_{2m}^{(1)}(kd)\right\}$$

and $m=0, 1, 2, \cdots, l$.

Likewise, $\zeta_{clb}^{(2m)}$ and $\zeta_{clb}^{(2m+1)}$ $(=\zeta_2^{(2m+1)}$ in paper VII) are given by equations (11) and (12) in paper VII, i.e.,

$$i\cdot\frac{k_m d}{kd}\sum_{n=0}^{l}K'_{n,m}\zeta_{clb}^{(2n)}+\sum_{n=0}^{l}L'_{n,m}\zeta_{clb}^{(2n+1)}=\varphi_m\cdot 2ikd \tag{a9}$$

and

$$\varepsilon_m\zeta_{clb}^{(2m)}-ikd\sum_{n=0}^{l}P_{m,n}\zeta_{clb}^{(2n+1)}=0, \tag{a10}$$

where $m=0, 1, 2, \cdots, l$.

The additions of (a7) to (a9) and (a8) to (a10) yield

$$i\cdot\frac{k_m d}{kd}\sum_{n=0}^{l}K'_{n,m}\xi^{(2n)}+\sum_{n=0}^{l}L'_{n,m}\xi^{(2n+1)}=\varphi_m\cdot 2ikd \tag{a11}$$

and

$$\varepsilon_m\xi^{(2m)}-ikd\sum_{n=0}^{l}P_{m,n}\xi^{(2n+1)}=2, \tag{a12}$$

where

$$\xi^{(m)}=\zeta_{opb}^{(m)}+\zeta_{clb}^{(m)} \tag{a13}$$

and $m=0, 1, 2, \cdots, l$.

Applying the operators

$$\int_0^d \cos\frac{m\pi}{d}z dz \qquad (m=0, 1, 2, \cdots)$$

to the formula relevant to the Bessel function

$$\sum_{n=0}^{\infty}\frac{1}{\varepsilon_n}J_{2n}(kz)=1,$$

we have

$$\sum_{n=0}^{\infty}\frac{K'_{n,m}}{\varepsilon_n}=\varphi_m kd. \qquad \text{(a14)}$$

Using (a14), equations (a11) are reduced to

$$i\cdot\sum_{n=0}^{l}K'_{n,m}\left(\xi^{(2n)}-\frac{2}{\varepsilon_n}\right)+\frac{kd}{k_m d}\sum_{n=0}^{l}L'_{n,m}\xi^{(2n+1)}=0 \qquad \text{(a15)}$$

$$(m=0, 1, 2, \cdots, l).$$

A slight modification of equation (a12) yields

$$\varepsilon_m\left(\xi^{(2m)}-\frac{2}{\varepsilon_m}\right)-ikd\sum_{n=0}^{l}P_{m,n}\xi^{(2n+1)}=0 \qquad \text{(a16)}$$

$$(m=0, 1, 2, \cdots, l).$$

Solving equations (a15) and (a16) with respect to $\xi^{(2n)}-2/\varepsilon_n$ and $\xi^{(2n+1)}$ $(n=0, 1, 2, \cdots, l)$, we have

$$\left.\begin{aligned}\zeta_{opb}^{(2n)}+\zeta_{clb}^{(2n)}&=\frac{2}{\varepsilon_n}\\ \zeta_{opb}^{(2n+1)}+\zeta_{clb}^{(2n+1)}&=0\end{aligned}\right\}\quad (n=0, 1, 2, \cdots, l) \qquad \text{(a17)}$$

through the use of (a13).

Substitution of the first expression of (a17) into (a6) and use of (a14) give the final result, i.e.,

$$\zeta_{op}^{(m)}+\zeta_{cl}^{(m)}=\varphi_m.$$

The verification is now completed.

46. 矩形湾における長波 [I]

—垂直入射の場合—

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本報告においては矩形湾に外海から周期波が垂直に入射する場合の湾水振動が厳密な方法で論じられている．解析には buffer domain の方法が用いられている．得られた結果の中でより興味のある事実は次のごとくである．

(i) 湾口補正は水路の中から周期波が河口に到達したときそこでおこる反射波の反射係数の位相のみに依存する．

(ii) 湾水の横振動は $kl \approx m\pi + \pi/2$ (k: 波数, l: 湾の長さ, m: 負でない整数) のとき最も卓越し，$kl = m\pi$ のとき完全に消失する．