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1. *Waves Generated from a Linear Horizontal Traction with Finite Source Length on the Surface of a Semi-Infinite Elastic Medium, with Special Remarks on the Theory of Shear Wave Generator.*

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1. Introduction

Waves generated by a traction acting on a free surface of a semi-infinite elastic solid are important, and since Lamb (1904) a great many investigations have been carried out on them. Application of a horizontal traction was first studied by Dr. Nakano (1930). By using expressions of his solution for the elastic wave equations in cylindrical coordinates, the case in which the given horizontal traction is expressed by the n-th order Bessel function was solved. Some numerical analyses were performed by Dr. Hirono (1948) who studied mathematically the mechanism of shallow earthquakes. In those papers, it was pointed out that the longitudinal, the vertically and the horizontally polarized shear and Rayleigh waves as well as two kinds of certain diffracted waves are generated. In those two investigations, the solutions for the radial, the horizontal and the vertical tractions have been closely discussed. A similar work on a horizontal traction was given attention by Cherry (1962).

On the other hand, since 1957, the Seismic Exploration Group of Japan has taken observations by using some transversely sensitive geophones and by hitting with a hammer the end of a weighted slender wooden plate laid down on the ground surface, or by detonation in a gun firmly fixed on a slender plate. In those experiments, it seems that a horizontally polarized shear (so-called SH) wave is generated, by a horizontal traction acting between the plate and the ground surface. The wave observed near the plate is certainly the shear wave from an infinitely linear source, and the wave far from the plate is one from

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an empirical point source (Kobayashi, 1959; Komaki, 1959; Shima and Ohta, 1967).

This paper concerns wave propagation from the shear wave generator developed by the Group, and makes clear the relations existing among the length of the generator, the distance from it and the wave length of the generated waves. Moreover, the condition necessary to make the effective observation of the shear wave is offered. A serious purpose of the theoretical study is not only an interpretation of the phenomena, but also some proposal for the design of the apparatus and the experimental techniques.

In section 2 the outline of the mathematical process carried out in this study is described. In the next section the solution is obtained by the displacement potentials and later is rewritten into the displacement components. The solution is expressed in some integral representations. The detail of the mathematical treatment takes place in Appendix III. In section 4 the integral representations are estimated by the method of steepest descent, and the radiation of the body waves is discussed. A special reference is associated with the transverse displacement. In section 5 the radiation of Rayleigh wave is dealt with. The subsequent section concerns an example of a field experiment. In the last section the result is summarized and the comments for such experiments are stated.

2. Procedure of this study

The shear wave generator developed by the Seismic Exploration Group of Japan utilizes principally the friction acting between its bottom and the ground surface. The in-situ shear wave velocity near the surface in experiments by the Group is about 100 m/sec or less, whereas the wave velocity propagated in iron materials of which the generator is made is certainly of the order of 1 km/sec. Therefore, the generator will be regarded as a rigid source laid on the surface. It is known that the stress due to loading is distributed so as to be a maximum beneath the vicinity of the rim in muddy soil, or beneath the vicinity of the centre in sandy soil. Although it would not be distributed uniformly, the stress distribution is assumed as constant, as the first step of this study. In addition, the underground structure is also assumed as uniform. In the result, a linear source with a finite length laid upon a semi-infinite elastic medium is treated.

In general, the problems of a slender source are solved by any one

of the following treatments mathematically :
1) to use an elliptical coordinate system,
2) to express the source's form by a Bessel-Fourier series, or
3) to superpose the solution from a point source.

In the first treatment, Mathieu and associated Mathieu functions are adopted, and the result is derived in terms of Bessel functions or trigonometrical functions after expanding them into some series. The second treatment seems to be more complicated for the calculation involved. In this paper, the last treatment is followed.

Let us take a force system (F_x, F_y, F_z) applied to a point parallel to the (x, y, z) axes and $(F_r, F_{\theta'}, F_z)$ to the (r, θ', z) axes, respectively. Here, components F_r and $F_{\theta'}$ are related with F_x and F_y by relations

$$\left.\begin{aligned} F_r &= F_x \cos\theta' + F_y \sin\theta', \\ F_{\theta'} &= -F_x \sin\theta' + F_y \cos\theta', \end{aligned}\right\} \tag{1}$$

and F_z is common between two coordinate systems.

Now, we assume that a force acting on the plane $z=0$ is directed towards the y-axis. This condition is equivalent to considering the boundary condition by means of stresses

$$\left.\begin{aligned} \widehat{rz} &= Y(r, \theta') \sin\theta', \\ \widehat{\theta' z} &= Y(r, \theta') \cos\theta', \\ \widehat{zz} &= 0, \end{aligned}\right\} \text{at} \quad z=0, \tag{2}$$

where $Y(r, \theta')$ is the y-component of the force acting on the plane $z=0$. The Hankel transform of $Y(r, \theta')$ with respect to r is written as

$$\overline{Y}(\xi, \theta') = \sum_n \int_0^\infty \varpi \cdot Y(\varpi, \theta') J_n(\xi\varpi) d\varpi \tag{3}$$

where n is the number of nodes in the azimuthal direction of the force. It is cumbersome to express the stress in terms of a sum of functions related with n, especially when the source is of an elongated form. So in this paper, we proceed with the following analysis. Firstly, a solution for a concentrated stress acting at the origin is obtained, that is, the solution gives rise to one from a point force applied at the origin; secondly, this solution is rewritten by means of the new coordinate system in which the new origin is shifted by some distance from the old origin; and lastly, the resulting disturbance is integrated over the extent of the source.

3. The boundary condition and the solution by integral representations

From the preceding remarks of this study the number n is zero, because there are no azimuthal characteristics of the source, and the force $Y(r,\theta')$ is the delta function, $\delta(r)$, because of the concentrated force acting at the origin. Then, the boundary condition is given by

$$\left.\begin{aligned} \widehat{rz}&=-F_0\int_0^\infty \xi\cdot J_0(\xi r)d\xi\cdot \sin\theta', \\ \widehat{\theta' z}&=-F_0\int_0^\infty \xi\cdot J_0(\xi r)d\xi\cdot \cos\theta', \\ \widehat{zz}&=0, \end{aligned}\right\} \text{at} \quad z=0, \tag{4}$$

where F_0 denotes a source intensity acting in the direction $\theta'=\pi/2$ at the origin. These expressions agree with ones approaching an infinitesimal source area in the limit, in Cherry's condition (1962).

The displacement potentials are obtained on account of the convenience in the analysis of the coordinate translation.

3.1. *The solution due to a point source (the first step)*

Substitution of (4) into (AI-6, 7, 8) determines each coefficient, and the resultant forms of the potentials are

$$\left.\begin{aligned} \Phi&=\frac{F_0}{\mu}\int_0^\infty \frac{2\xi^2\nu'}{F(\xi)}J_1(\xi r)e^{-\nu z}\sin\theta' d\xi, \\ \Psi_1&=\frac{F_0}{\mu}\int_0^\infty \frac{1}{\nu'}J_1(\xi r)e^{-\nu' z}\cos\theta' d\xi, \\ \Psi_2&=\frac{F_0}{\mu}\int_0^\infty \frac{(2\xi^2-k^2)}{F(\xi)}J_1(\xi r)e^{-\nu' z}\sin\theta' d\xi, \end{aligned}\right\} \tag{5}$$

where

$$F(\xi)=(2\xi^2-k^2)^2-4\xi^2\nu\nu'. \tag{6}$$

3.2. *Transform of solutions due to the coordinate translation (the second step)*

Before integration over the source elongation, the solutions obtained in the preceding paragraph are transformed into the expressions in a new coordinate system where the origin is shifted in the direction $\theta'=\pi/2$ on the surface $z=0$ by a distance $-r_1$. In Figure 1, the potentials observed at a point Q, which are expressed by (5), have been written by means of the coordinate system where the origin is at point

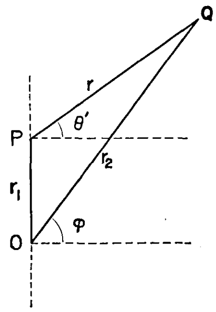

Fig. 1. Translations of coordinate systems.

P. The potentials are transformed by means of another coordinate system where the origin is at point O, by a distance of r_1 from the point P along the y-axis. From equation (5), we find that the r- and θ'-dependences of the potentials are $J_1(\xi r)\sin\theta'$ or $J_1(\xi r)\cos\theta'$. According to the addition theorem (see Appendix III), the potentials which are written in the coordinate system (r_2, φ, z) are obtained as

$$\left.\begin{aligned}\Phi&=\frac{F_0}{\mu}\sum_{m=1}^{\infty}[\Phi_{sm}\sin(2m-1)\varphi+\Phi_{cm}\cos 2m\varphi+\Phi_{co}],\\ \Psi_1&=\frac{F_0}{\mu}\sum_{m=1}^{\infty}[\Psi_{1cm}\cos(2m-1)\varphi+\Psi_{1sm}\sin 2m\varphi],\\ \Psi_2&=\frac{F_0}{\mu}\sum_{m=1}^{\infty}[\Psi_{2sm}\sin(2m-1)\varphi+\Psi_{2cm}\cos 2m\varphi+\Psi_{2co}],\end{aligned}\right\}\tag{7}$$

where

$$\left.\begin{aligned}\Phi_{sm}&=(-1)^m\int_0^\infty\frac{2\xi^2\nu'}{F(\xi)}e^{-\nu z}[J_{2m+1}(\xi r_2)J_{2m}(\xi r_1)-J_{2m-3}(\xi r_2)J_{2m-2}(\xi r_1)]d\xi\\ \Phi_{cm}&=(-1)^m\int_0^\infty\frac{2\xi^2\nu'}{F(\xi)}e^{-\nu z}[J_{2m+2}(\xi r_2)J_{2m+1}(\xi r_1)+J_{2m-2}(\xi r_2)J_{2m-1}(\xi r_1)]d\xi,\\ \Phi_{co}&=\int_0^\infty\frac{2\xi^2\nu'}{F(\xi)}e^{-\nu z}J_2(\xi r_2)J_1(\xi r_1)d\xi;\end{aligned}\right\}\tag{8}$$

$$\left.\begin{aligned}\Psi_{1cm}&=(-1)^m\int_0^\infty\frac{e^{-\nu' z}}{\nu'}[J_{2m+1}(\xi r_2)J_{2m}(\xi r_1)+J_{2m-3}(\xi r_2)J_{2m-2}(\xi r_1)]d\xi,\\ \Psi_{1sm}&=(-1)^m\int_0^\infty\frac{e^{-\nu' z}}{\nu'}[J_{2m+2}(\xi r_2)J_{2m+1}(\xi r_1)-J_{2m-2}(\xi r_2)J_{2m-1}(\xi r_1)]d\xi,\end{aligned}\right\}\tag{9}$$

and

$$\left.\begin{aligned}\Psi_{2sm}&=(-1)^m\int_0^\infty\frac{(2\xi^2-k^2)}{F(\xi)}e^{-\nu' z}[J_{2m+1}(\xi r_2)J_{2m}(\xi r_1)-J_{2m-3}(\xi r_2)J_{2m-2}(\xi r_1)]d\xi,\\ \Psi_{2cm}&=(-1)^m\int_0^\infty\frac{(2\xi^2-k^2)}{F(\xi)}e^{-\nu' z}[J_{2m+2}(\xi r_2)J_{2m+1}(\xi r_1)+J_{2m-2}(\xi r_2)J_{2m-1}(\xi r_1)]d\xi,\\ \Psi_{2co}&=\int_0^\infty\frac{(2\xi^2-k^2)}{F(\xi)}e^{-\nu' z}J_2(\xi r_2)J_1(\xi r_1)d\xi.\end{aligned}\right\}\tag{10}$$

Putting $r_1\to 0$, $r_2\to r$ and $\varphi\to\theta'$, we can easily justify that these expressions tend to ones for the point source (5), by using the relation (AII-6).

3.3. Disturbance from a linear horizontal source with a finite length (the final step)

Lastly, the solution from the point source is superposed over the source elongation. If the source intensity is uniform and the source length is $2a$, the solutions are obtained by integration from $-a$ to a with respect to r_1. As the potentials (8), (9), (10) involve the forms $J_{2m}(\xi r_1)$ or $J_{2m+1}(\xi r_1)$, it is easily found that by using the relation (AII-4) we get

$$\frac{1}{2a}\int_{-a}^{a} J_{2m+1}(\xi r_1)dr_1=0, \tag{11}$$

and

$$\frac{1}{2a}\int_{-a}^{a} J_{2m}(\xi r_1)dr_1=\frac{2}{\xi a}\sum_{n=m}^{\infty} J_{2n+1}(\xi a).$$

Whence

$$\Phi_{cm}=\Psi_{1sm}=\Psi_{2cm}=0, \qquad m=0, 1, 2, \ldots\ldots, \tag{12}$$

and the resulting expressions for the potentials are

$$\left.\begin{aligned}
&\Phi=\frac{F_0}{\mu}\sum_{m=1}^{\infty}\Phi_{sm}\sin(2m-1)\varphi,\\
&\quad\Phi_{sm}=\frac{(-1)^{m+1}\cdot 4}{a}\int_0^{\infty}\frac{\xi\nu'}{F(\xi)}e^{-\nu z}[J_{2m+1}(\xi r_2)\sum_{n=m}^{\infty}J_{2n+1}(\xi a)\\
&\qquad\qquad -J_{2m-3}(\xi r_2)\sum_{n=m}^{\infty}J_{2n-1}(\xi a)]d\xi;\\
&\Psi_1=\frac{F_0}{\mu}\sum_{m=1}^{\infty}\Psi_{1cm}\cos(2m-1)\varphi,\\
&\quad\Psi_{1cm}=\frac{(-1)^{m}\cdot 2}{a}\int_0^{\infty}\frac{e^{-\nu' z}}{\xi\nu'}[J_{2m}(\xi r_2)\sum_{n=m}^{\infty}J_{2n+1}(\xi a)\\
&\qquad\qquad +J_{2m-3}(\xi r_2)\sum_{n=m}^{\infty}J_{2n-1}(\xi a)]d\xi;\\
&\text{and}\\
&\Psi_2=\frac{F_0}{\mu}\sum_{m=1}^{\infty}\Psi_{2sm}\sin(2m-1)\varphi\\
&\quad\Psi_{2sm}=\frac{(-1)^{m+1}\cdot 2}{a}\int_0^{\infty}\frac{(2\xi^2-k^2)}{\xi F(\xi)}e^{-\nu' z}[J_{2m+1}(\xi r_2)\sum_{n=m}^{\infty}J_{2n+1}(\xi a)\\
&\qquad\qquad -J_{2m-3}(\xi r_2)\sum_{n=m}^{\infty}J_{2n-1}(\xi a)]d\xi.
\end{aligned}\right\} \tag{13}$$

The displacement components are easily calculated by applying suitable operators to these potentials (13). Waves derived from Φ, Ψ_1, Ψ_2 are asssociated with the longitudinal, the horizontally polarized shear and the vertically polarized shear waves, respectively:

$$U_r = u = \frac{F_0}{\mu}\sum_{m=1}^{\infty}[u_m^{(P)} + u_m^{(SH)} + u_m^{(SV)}]\sin(2m-1)\varphi, \tag{14}$$

$$u_m^{(P)} = (-1)^m 2\frac{\partial}{\partial r_2}\int_0^\infty \frac{\xi^2\nu'}{F(\xi)} e^{-\nu z} G_m^{(-)}(\xi r_2, \xi a)d\xi,$$

$$u_m^{(SH)} = (-1)^m \frac{(2m-1)}{r_2}\int_0^\infty \frac{e^{-\nu' z}}{\nu'} G_m^{(+)}(\xi r_2, \xi a)d\xi,$$

$$u_m^{(SV)} = (-1)^{m+1}\frac{\partial}{\partial r_2}\int_0^\infty \frac{\nu'(2\xi^2-k^2)}{F(\xi)} e^{-\nu' z} G_m^{(-)}(\xi r_2, \xi a)d\xi;$$

$$U_\varphi = v = \frac{F_0}{\mu}\sum_{m=1}^{\infty}[v_m^{(P)} + v_m^{(SH)} + v_m^{(SV)}]\cos(2m-1)\varphi, \tag{15}$$

$$v_m^{(P)} = (-1)^m \frac{2(2m-1)}{r_2}\int_0^\infty \frac{\xi^2\nu'}{F(\xi)} e^{-\nu z} G_m^{(-)}(\xi r_2, \xi a)d\xi,$$

$$v_m^{(SH)} = (-1)^m \frac{\partial}{\partial r_2}\int_0^\infty \frac{e^{-\nu' z}}{\nu'} G_m^{(+)}(\xi r_2, \xi a)d\xi,$$

$$v_m^{(SV)} = (-1)^{m+1}\frac{(2m-1)}{r_2}\int_0^\infty \frac{\nu'(2\xi^2-k^2)}{F(\xi)} e^{-\nu' z} G_m^{(-)}(\xi r_2, \xi a)d\xi;$$

and

$$U_z = w = \frac{F_0}{\mu}\sum_{m=1}^{\infty}[w_m^{(P)} + w_m^{(SH)} + w_m^{(SV)}]\sin(2m-1)\varphi, \tag{16}$$

$$w^{(P)} = (-1)^{m+1}\cdot 2\int_0^\infty \frac{\xi^2\nu\nu'}{F(\xi)} e^{-\nu z} G_m^{(-)}(\xi r_2, \xi a)d\xi,$$

$$w_m^{(SH)} = 0,$$

$$w_m^{(SV)} = (-1)^m \int_0^\infty \frac{\xi^2(2\xi^2-k^2)}{F(\xi)} e^{-\nu' z} G_m^{(-)}(\xi r_2, \xi a)d\xi;$$

where

$$G_m^{(\pm)}(\xi r_2, \xi a) = \frac{2}{\xi a}\left[J_{2m+1}(\xi r_2)\sum_{n=m}^{\infty} J_{2n+1}(\xi a) \pm J_{2m-3}(\xi r_2)\sum_{n=m}^{\infty} J_{2n-1}(\xi a)\right]$$

$$= \frac{1}{a}\int_0^a [J_{2m+1}(\xi r_2)J_{2m}(\xi x) \pm J_{2m-3}(\xi r_2)J_{2m-2}(\xi x)]dx. \tag{17}$$

These agree with Cherry's solution (1962, eqs. 17-19) in the limit of a infinitesimal. In the plane $\varphi=0$ and π, both radial and vertical displacements vanish and only the transverse displacement exists, while in the plane $\varphi = \pm\pi/2$, the transverse displacement vanishes and the other two displacements exist. In distant observations, it follows from $J_\nu(x) \propto x^{-1/2}\cos\{x-(2\nu+1)\pi/4\}$ that the components $u^{(P)}$, $u^{(SV)}$; $v^{(SH)}$; $w^{(P)}$, $w^{(SV)}$ constitute the main part, while $u^{(SH)}$; $v^{(P)}$, $v^{(SV)}$; $w^{(SH)}$ are slight or null.

4. Radiation of body waves

The type of the integral involved in expressions (14), (15), (16) has already been treated by Dr. Hirono (1948). In that investigation, the integral was estimated by means of the method of steepest descent in terms of Weyl's integral identity, according to Prof. Sakai (1934). Referring to those results, the waves observed at stations distant compared with the wave length are approximately expressed as:

$$u \doteqdot \frac{F_0}{\mu}\left[\frac{2h^4\sqrt{(\alpha/\beta)^2-\sin^2\theta}\cos\theta\sin^2\theta}{F(-h\sin\theta)}\frac{e^{-ihR}}{R}S_1(ha\sin\theta,\varphi)\right.$$
$$\left.+\frac{k^4\cos 2\theta\cos^2\theta}{F(-k\sin\theta)}\frac{e^{-ikR}}{R}S_1(ka\sin\theta,\varphi)\right], \tag{18}$$

$$v \doteqdot \frac{F_0}{\mu}\frac{e^{-ikR}}{R}S_2(ka\sin\theta,\varphi), \tag{19}$$

and

$$w \doteqdot \frac{F_0}{\mu}\left[\frac{2h^4\sqrt{(\alpha/\beta)^2-\sin^2\theta}\sin\theta\cos^2\theta}{F(-h\sin\theta)}\frac{e^{-ihR}}{R}S_1(ha\sin\theta,\varphi)\right.$$
$$\left.+\frac{k^4\cos 2\theta\sin\theta\cos\theta}{F(-k\sin\theta)}\frac{e^{-ikR}}{R}S_1(ka\sin\theta,\varphi)\right], \tag{20}$$

where $R=\sqrt{r_2^2+z^2}$, θ is measured from the z-axis, and

$$\left.\begin{aligned} S_1(\eta) &= \frac{2}{\eta}\sum_{m=1}^{\infty} J_{2m-1}(\eta)\sin(2m-1)\varphi, \\ S_2(\eta) &= \frac{2}{\eta}\sum_{m=1}^{\infty}\left[J_{2m-1}(\eta)+2\sum_{n=m}^{\infty}J_{2n+1}(\eta)\right]\cos(2m-1)\varphi, \end{aligned}\right\} \tag{21}$$
$$\eta = (h \text{ or } k)a\cdot\sin\theta.$$

In these expressions, the order R^{-2} is neglected, and the integrals are carried out for the relation $R > a\sin\theta$.

If the displacement components are transformed from cylindrical coordinates into spherical polar coordinates (R, θ, φ), the expressions (18), (19), (20) yield the following:

$$U_R=\frac{F_0}{\mu}\frac{h^4\sqrt{(\alpha/\beta)^2-\sin^2\theta}\sin 2\theta}{F(-h\sin\theta)}\frac{e^{-ihR}}{R}S_1(ha\sin\theta, \varphi), \tag{22}$$

$$U_\theta=\frac{F_0}{\mu}k^4\frac{\cos 2\theta\cos\theta}{F(-k\sin\theta)}\frac{e^{-ikR}}{R}S_1(ka\sin\theta, \varphi), \tag{23}$$

and

$$U_\varphi=\frac{F_0}{\mu}\frac{e^{-ikR}}{R}S_2(ka\sin\theta, \varphi). \tag{24}$$

It follows that the component U_R is associated with the P wave, U_θ with the shear wave which is coupled with the P wave to supply the boundary condition and corresponds to the SV wave in the cylindrical coordinates, and U_φ with the shear wave of torsion which to the SH wave. The expressions S_1 and S_2 are regarded as the radiation patterns in the azimuthal distribution of the P or SV wave and of the SH wave, respectively. If the source length is negligible compared with the wave length, the azimuthal patterns tend to the expressions

$$S_1(\eta, \varphi)\rightarrow \sin\varphi, \text{ and } S_2(\eta, \varphi)\rightarrow \cos\varphi. \tag{25}$$

Under this situation the azimuthal distribution of the SH wave varies as $\cos\varphi$, while the distribution of both the P and the SV waves as $\sin\varphi$. The vertical radiation patterns were obtained by Cherry (1962). If the source length is not small, the resultant pattern is modified, but its modification diminishes with increasing depth and when approaching the direction of the bisection of the source. The azimuthal pattern of the P and the SV waves is shown in Figure 2. It follows that the effect of the source length is important. However, since this pattern is modified by the vertical pattern, the effect on the observed wave will not be so large as the azimuthal pattern expressed in Fig. 2. For angles such that $\sin\theta>\beta/\alpha$, the observed shear wave has a phase shift and becomes complication, because $F(-k\sin\theta)$ is complex (Hirono, 1948; Cherry, 1962). Although the change of wave forms caused by a phase shift is important for discussions of the wave propagation, our interest lies mainly in the transverse displacement.

The radiation pattern of the transverse component $S_2(ka\sin\theta, \varphi)$ is shown in Fig. 3. If the wave length of a shear wave is L, we have

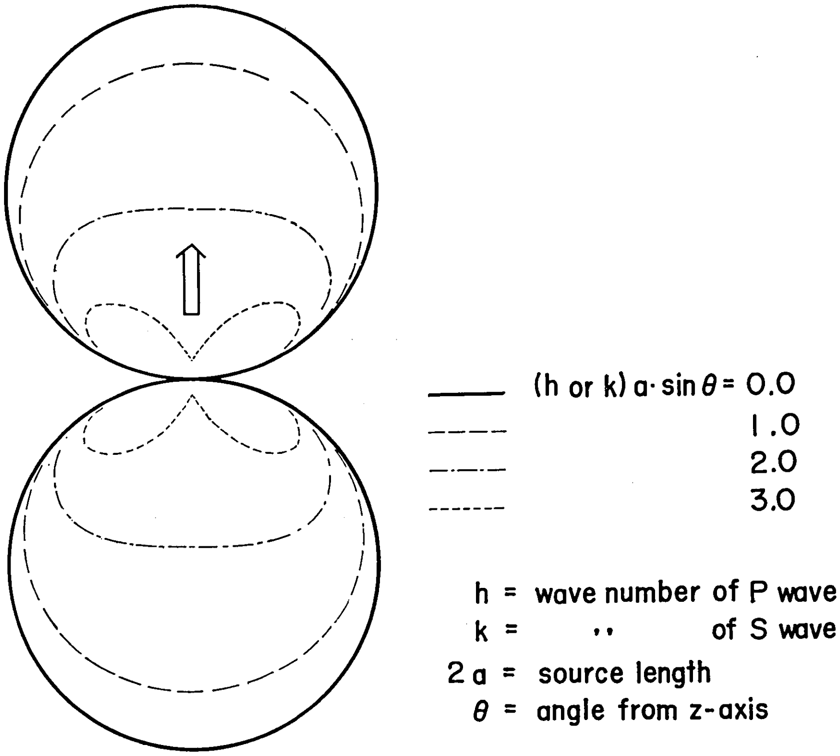

Fig. 2. Azimuthal radiation pattern of P or SV wave.

$$ka = 2\pi a / L = \pi \cdot (2a) / L, \tag{26}$$

and ka is the product of π and the ratio of the source length to the wave length. If the accuracy of amplitude observations is within ten per cent, the effect of the source length cannot be neglected for the azimuth of the observation $|\varphi|$ larger than about ten degrees when the source length is comparable with the wave length. Accordingly, to obtain the SH wave as simply as possible, the observation must take place in the direction bisecting the source, $\varphi=0$. It is easily justified, too, by the formula of Bessel functions (AII-12) that when $\varphi=0$ the radiation pattern S_2 is unity, *i.e.*, the effect of the source length on the wave length vanishes.

Next, we consider the effect of the source length on the distance from the source, in the $\varphi=0$ direction. We derive results directly from the relation (15) but not from (19). In this vertical plane, the transverse component is expressed by the sum of three terms (see Appendix IV) :

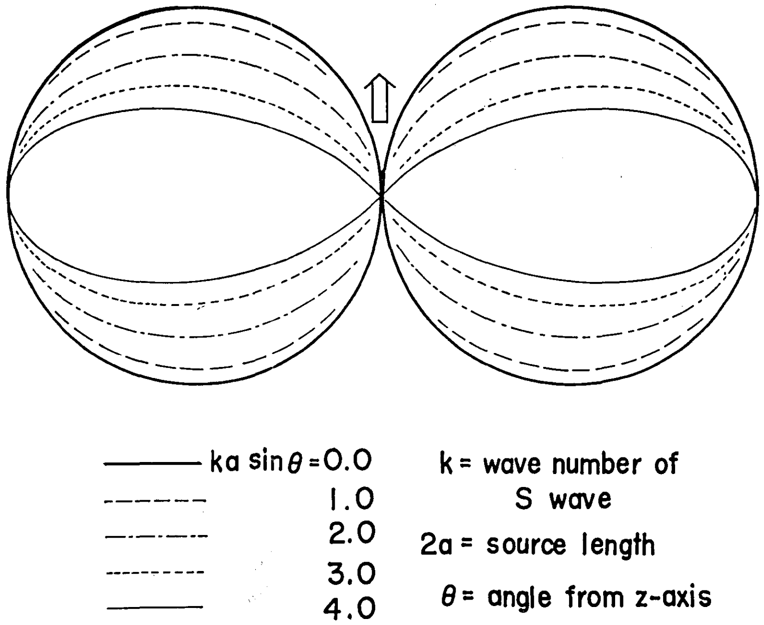

Fig. 3. Radiation pattern of SH wave.

$$
\begin{aligned}
v^{(P)}/(F_0/\mu) = & -\frac{2}{r_2}\int_0^\infty \frac{\xi^2\nu'}{F(\xi)} e^{-\nu z} J_1(\xi r_2)\,d\xi \\
& +\frac{1}{3!}\left(\frac{a}{r_2}\right)^2 \cdot r_2 \frac{\partial}{\partial r_2}\frac{1}{r_2}\int_0^\infty \frac{\xi^2\nu'}{F(\xi)} e^{-\nu z} J_1(\xi r_2)\,d\xi \\
& -\frac{16}{a r_2^2}\int_0^a dr_1 \int_0^\infty \frac{\xi\nu'}{F(\xi)} e^{-\nu z} \sum_{m=1}^{\infty} (-1)^m \cdot m^2 J_{2m}(\xi r_2) J_{2m}(\xi r_1)\,d\xi \\
& +O(a^4/r_2^4),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
v^{(SH)}/(F_0/\mu) = & \frac{\partial}{\partial r_2}\int_0^\infty \frac{e^{-\nu' z}}{\nu'} J_1(\xi r_2)\,d\xi \\
& +\frac{1}{3!}\left(\frac{a}{r_2}\right)^2 \frac{\partial}{\partial r_2} H_0^{(1)}(kR) - \frac{2}{3!}\frac{1}{r_2}\left(\frac{a}{r_2}\right)^2 \{H_0^{(2)}(kR) - H_0^{(2)}(kz)\} \\
& +O(a^4/r_2^4),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
v^{(SV)}/(F_0/\mu) = & \frac{1}{r_2}\int_0^\infty \frac{\nu'(2\xi^2-k^2)}{F(\xi)} J_1(\xi r_2) e^{-\nu' z}\,d\xi \\
& -\frac{1}{3!\cdot 2}\left(\frac{a}{r_2}\right)^2 r_2 \frac{\partial}{\partial r_2}\frac{1}{r_2}\int_0^\infty \frac{\nu'(2\xi^2-k^2)}{F(\xi)} e^{-\nu' z} J_1(\xi r_2)\,d\xi \\
& +\frac{8}{a r_2^2}\int_0^\infty dr_1 \int_0^\infty \frac{\nu'(2\xi^2-k^2)}{\xi F(\xi)} e^{-\nu' z} \sum_{m=1}^{\infty} (-1)^m \cdot m^2 J_{2m}(\xi r_2) J_{2m}(\xi r_1)\,d\xi \\
& +O(a^4/r_2^4).
\end{aligned}
\tag{29}
$$

In these expressions, the first term of each component of the transverse displacement is the integral representation corresponding to the solution from a point source, and the second and later terms signify the effect of the source length. Consequently, this effect is of the same order as the square of the ratio of the source length to the distance between the centre and the station. In the SH component, the disturbance from the linear source turns out to be of the order of the square of the ratio, because the second term consists of the solution from the linear source.

5. Radiation of Rayleigh wave

It is well known that Rayleigh wave is estimated by the contribution of some pole on a suitable Riemann sheet. Referring to Dr. Hirono's calculation (1948) again, the effect of the Rayleigh wave at large distance is given by,

$$\left.\begin{aligned} u^{(\mathrm{R})}&=\frac{F_0}{\mu}\sqrt{\frac{2}{\pi}}\frac{k^2\kappa\nu_R'}{F'(-\kappa)}e^{-i\pi/4}\frac{\partial}{\partial(\kappa r_2)}\frac{e^{-i\kappa r_2}}{\sqrt{\kappa r_2}}[1+O(1/\kappa r_2)]S_1(\kappa a,\varphi),\\ v^{(\mathrm{R})}&=\frac{F_0}{\mu}\sqrt{\frac{2}{\pi}}\frac{k^2\kappa\nu_R'}{F'(-\kappa)}e^{-3i\pi/4}\frac{e^{-i\kappa r_2}}{(\kappa r_2)^{3/2}}[1+O(1/\kappa r_2)]S_3(\kappa a,\varphi),\\ w^{(\mathrm{R})}&=\frac{F_0}{\mu}\sqrt{\frac{2}{\pi}}\frac{\kappa^2(2\kappa^2-k^2-\nu_R\nu_R')}{F'(-\kappa)}e^{-3i\pi/4}\frac{e^{-i\kappa r_2}}{\sqrt{\kappa r_2}}[1+O(1/\kappa r_2)]S_1(\kappa a,\varphi),\end{aligned}\right\}\tag{30}$$

where $S_1(\kappa a, \varphi)$ is given by equation (21),

$$S_3(\kappa a,\varphi)=\frac{2}{\kappa a}\sum_{m=1}^{\infty}(2m-1)J_{2m-1}(\kappa a)\cos(2m-1)\varphi,\tag{31}$$

κ is the wave number of Rayleigh wave, and

$$\nu_R=\sqrt{\kappa^2-h^2},\qquad \nu_R'=\sqrt{\kappa^2-k^2},\tag{32}$$

$$F'(-\kappa)=-\frac{2(4\kappa^4-k^4)}{\kappa}+4\kappa^3\left(\frac{\nu_R}{\nu_R'}+\frac{\nu_R'}{\nu_R}\right).\tag{33}$$

Then, the transverse component of the Rayleigh wave does not vanish but is of higher order than the other components. This results from the azimuthal distribution of the source traction, as mentioned by Dr. Nakano (1928). The principal components of the Rayleigh wave are radial and vertical, and its orbital motion is determined uniquely. The radiation pattern is given by the equation

$$\frac{2}{\kappa a}\sum_{m=1}^{\infty} J_{2m-1}(\kappa a)\sin(2m-1)\varphi, \tag{34}$$

which agrees with the azimuthal pattern of the P or the SV wave, and tends to $\sin\varphi$ with decreasing source length. Accordingly, the longer the source length, the less is the radiation of Rayleigh wave.

6. Field experiments

The Seismic Exploration Group of Japan held a joint experiment in the vicinity of Shirane City, Niigata Prefecture, in September 1964. As one of the experimental subjects, the directional properties of the waves which were generated by detonation of a gun firmly fixed on a slender plate were checked. Transversely sensitive geophones were placed at 22.5 degree intervals around a circle 10 meters in radius, and at the centre of the circle the shear wave generator was placed. The generator consisted essentially of a length of thick-walled steel tubing (one meter long) that has coupled closely to the ground by steel pegs. Detectors of electro-magnetic type having natural frequencies of 27 cps were used. The records were obtained by a conventional 24-channel recording system (ETL M-3).

According to a travel time-distance plot of the SH signals from transversely sensitive detectors along a line perpendicular to the generator, the shear wave velocity of the surface and subsurface layers at the field is determined as about 70 m/sec and 130 m/sec respectively. The critical distance is about 14 meters, so that thickness of the surface layer is calculated as about 4 meters.

Because the observed records were of velocity-type, they were turned into ones of displacement-type by numerical integration. The result of the circular spread is shown in Fig. 4. The relation between the maximum amplitude of the displacements and the azimuth is shown in Fig. 5. From Fig. 4, the apparent period of the wave showing the maximum amplitude is about 50 milli-seconds. As the shear velocity is 70 m/sec, the apparent wave length of this wave is estimated as 3.5 meters. By relation (26), we have, with the source length of one meter,

$$ka=\pi/3.5\fallingdotseq 0.9.$$

In Fig. 5, the azimuthal variation for 0.9 of ka, $S_2(0.9,\varphi)$, is drawn, in

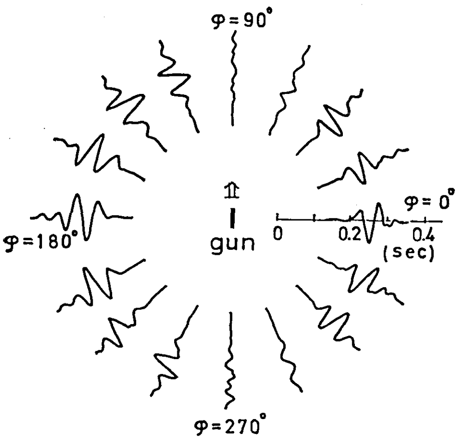

Fig. 4. Test of directional properties of SH wave from a gun-type wave generator. The length of the generator is 1 m, and the distance from the generator's centre is 10 m. The records are of displacement-type.

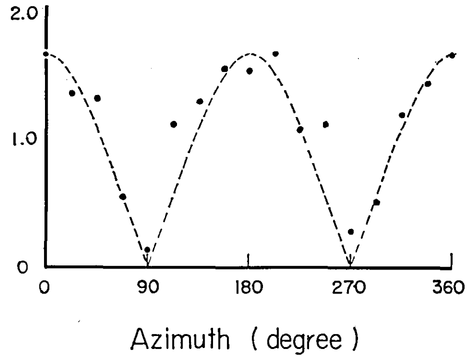

Fig. 5. Azimuthal variation of maximum amplitude. The scale of the ordinate is arbitrary. The dotted line denotes a calculated value of the radiation pattern of SH wave for $ka=0.9$.

which the amplitudes of zero degree in azimuth is taken as a unit. Agreement between the observed and the calculated values is extremely good.

This experiment is not the best example. There are two reasons for this; one is that $ka=0.9$ and the other that $2a/r_2=1/10$. Nevertheless, it follows that the experiment of the directivity for the SH wave radiation has the advantage of checking the efficiency of the shear wave generator.

7. Concluding remarks

A field experiment where a horizontal traction with a finite length is applied on the ground surface has been carried out. In this paper, a mathematical model for the experiment is studied, and the following results are obtained:

1) The azimuthal variation of the radial and the vertical displacements is expressed in terms of a series of $\sin(2m-1)\varphi$, while that of the transverse displacement in terms of a series of $\cos(2m-1)\varphi$. In every component, terms involving any even multiple of azimuth, $2m\varphi$, vanish.

2) At far stations, the longitudinal (P) and the vertically polarized shear (SV) waves predominate in the radial and the vertical components,

while the horizontally polarized shear (SH) wave predominates in the transverse one. Near the surface there is a nodal plane of the P and the SV waves, for these waves at the surface are of the same order and vary as the square of the ratio of the wave length to the distance, at most. On the free surface, the SH and Rayleigh waves are significant. Considering the reflected and the critically refracted waves, we must certainly interpret the radiation pattern into the medium, but the effect of the source length disappears with increasing depth.

3) In the vertical bisecting the source, the SH wave predominates, and the effect of the source length on the wave length is neglected. If the wave length is comparable with the source length, the observed amplitude of the SH wave diminishes by one tenth for directions deflected by ten degrees from this vertical, and the other waves are observed there.

4) In this vertical also, the effect of the source length on the distance from the source is of the same order as the square of the ratio of the source length to the distance. The leading term is one from a point source, while the perturbed term is one from a linear source.

5) The radiation pattern of the Rayleigh wave agrees with the azimuthal one of the P or the SV wave. It becomes less important with increasing source length.

Elastic waves from a horizontal traction with a finite length have the above-mentioned characteristics. It is hoped for the design of the S wave generator and the S wave observations that the following are taken into consideration :

1) In order to make the wave energy concentrate into the vertical bisecting the source as effectively as possible, it is necessary that the source intensity is symmetrical about the centre, because the total energy generated from the source is finite and symmetrical intensity leads to no terms with the even multiple of the azimuth.

2) The observation on the shear wave should take place on this vertical, because the simplest possible wave form is obtained there.

3) The ratio of the source length to the wave length should be large, as then the waves other than the SH wave become less observable.

In the future, investigations will be carried out to complete the subject of this paper. We must consider stratified structures or more complicated structures, because the actual earth is never uniform. The boundary condition assumed in this paper is that the intensity of the source is uniform at every point. This condition in the experiments may not be so. Moreover, in the experiments, transient waves have been observed, so that we must solve the problem by taking a suitable initial condition into account. It is necessary, too, to clarify whether the linear theory of elasticity holds good for small distances from the source. Although the ability to control the wave length or period is suggested for the reason that the effect of the source length on the wave length is neglected in the vertical bisecting of the source, this prediction must be examined by the experiments and its validity must be justified.

Acknowledgements

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Appendix I. Displacements and stresses in cylindrical coordinates

Prof. Sezawa (1928) and Dr. Nakano (1928) have obtained the expressions of displacement and stress components in a circular cylindrical coordinate system. In some problems, however, it is better to use displacement potentials. The displacement vector U is expressed by a sum of the gradient of a scalar potential Φ and the curls of two solenoidal vector potentials A and B;

$$U=U_1+U_2+U_3, \tag{AI-1}$$

where

$$\begin{aligned} U_1&=-\operatorname{grad}\Phi \\ U_2&=-\operatorname{rot}A,\ \operatorname{div}A=0, \\ U_3&=-\operatorname{rot}B,\ \operatorname{div}B=0. \end{aligned} \tag{AI-2}$$

and

In the problems of the cylindrical coordinates, vector A is directed towards the z-axis and B can be regarded simply as the curl of vector A (Morse and Feshbach, 1953, pp. 1764-1767). If a plane specified by $z=0$ is a free surface, vector U_1 is related to a longitudinal (P) wave, U_2 to a horizontally polarized shear (SH) wave and U_3 to a vertically polarized shear (SV) wave.

If the r-, θ- and z-components of vector A are written as (A_r, A_θ, A_z) we have

$$A=(0, 0, \Psi_1), \qquad \text{and} \qquad B=\operatorname{rot}(0, 0, \Psi_2), \tag{AI-3}$$

and three potentials Φ, Ψ_1, Ψ_2 satisfy the following equations, respectively,

$$\left.\begin{aligned} \nabla^2\Phi-\alpha^{-2}\partial^2\Phi/\partial t^2=0, \\ \nabla^2\Psi_{1,2}-\beta^{-2}\partial^2\Psi_{1,2}/\partial t^2=0, \end{aligned}\right\} \tag{AI-4}$$

where α, β are the velocities of P and S waves, respectively.

If any harmonic wave train in time is considered, these equations are Helmholtz's equations. Neglecting the time factor exp $(i\omega t)$, where ω is the circular frequency, we have the solutions of equations (AI-4)

$$\left.\begin{aligned} &\Phi=\sum A_n Z_n(\xi r)e^{-\nu z}{}^{\cos}_{\sin}n\theta, \qquad \nu=\sqrt{\xi^2-h^2},\ h=\omega/\alpha, \\ &\Psi_1=\sum B_n Z_n(\xi r)e^{-\nu' z}{}^{\sin}_{-\cos}n\theta, \ \nu'=\sqrt{\xi^2-k^2},\ k=\omega/\beta, \\ &\Psi_2=\sum C_n Z_n(\xi r)e^{-\nu' z}{}^{\cos}_{\sin}n\theta, \end{aligned}\right\} \tag{AI-5}$$

where Z_n is some cylindrical function. The displacement components are calculated by (AI-2) and (AI-5). The stress components treated in this paper are written as

$$\widehat{zz}/\mu=\left\{\left(1-2\frac{\beta^2}{\alpha^2}\right)k^2-2\frac{\partial^2}{\partial z^2}\right\}\Phi-2\left(k^2+\frac{\partial^2}{\partial z^2}\right)\frac{\partial\Psi_2}{\partial z}$$

$$=\sum\left[A_n(k^2-2\xi^2)e^{-\nu z}+2C_n\nu'\xi^2e^{-\nu' z}\right]Z_n(\xi r){}^{\cos}_{\sin}n\theta, \tag{AI-6}$$

$$\widehat{rz}/\mu=-2\frac{\partial^2\Phi}{\partial z\partial r}-\frac{1}{r}\frac{\partial^2\Psi_1}{\partial\theta\partial z}-\left(k^2+2\frac{\partial^2}{\partial z^2}\right)\frac{\partial\Psi_2}{\partial r}$$

$$=\sum\left[2A_n\nu e^{-\nu z}+C_n(k^2-2\xi^2)e^{-\nu' z}\right]\frac{\partial}{\partial r}Z_n(\xi r){}^{\cos}_{\sin}n\theta$$

$$-\sum B_n n\nu'\frac{Z_n(\xi r)}{r}e^{-\nu' z}{}^{\cos}_{\sin}n\theta, \tag{AI-7}$$

$$\widehat{\theta z}/\mu = -\frac{2}{r}\frac{\partial^2 \Phi}{\partial\theta\partial z} + \frac{1}{r}\frac{\partial^2 \Psi_1}{\partial r\partial z} - \left(k^2 + 2\frac{\partial^2}{\partial z^2}\right)\frac{1}{r}\frac{\partial \Psi_2}{\partial \theta}$$

$$= -\sum n\cdot[2A_n\nu e^{-\nu z} + C_n(k^2-2\xi^2)e^{-\nu' z}]\frac{1}{r}Z_n(\xi r)\begin{matrix}\sin\\ -\cos\end{matrix}n\theta$$

$$+\sum B_n n\nu'\frac{\partial}{\partial r}Z_n(\xi r)e^{-\nu' z}\begin{matrix}\sin\\ -\cos\end{matrix}n\theta. \qquad \text{(AI-8)}$$

Appendix II. Formulae of Bessel functions

We summarize the identities and formulae of the Bessel functions which are frequently used in this paper.

1. *Identities*

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z), \qquad \text{(AII- 1)}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z), \qquad \text{(AII- 2)}$$

$$[J_\nu(z)/z^\nu]' = -J_{\nu+1}(z)/z^\nu, \qquad \text{(AII- 3)}$$

$$J_\nu(e^{m\pi i}z) = e^{\nu m\pi i}J_\nu(z), \qquad \text{(AII- 4)}$$

$$J_{-n}(z) = e^{n\pi i}J_n(z), \qquad \text{(AII- 5)}$$

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m}}{m!\,(m+n)!} \quad (z \neq \text{ negative integer}). \qquad \text{(AII- 6)}$$

2. *Integral representations*

$$J_n(z) = \frac{1}{\pi i^n}\int_0^\pi e^{iz\cos\alpha}\cos n\alpha d\alpha, \qquad \text{(AII- 7)}$$

$$J_0(\sqrt{z^2+y^2}) = \frac{1}{\pi}\int_0^\pi e^{iy\cos\alpha}\cos(z\sin\alpha)d\alpha, \qquad \text{(AII- 8)}$$

$$H_0^{(1)}(k\sqrt{r^2+z^2}) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-\sqrt{\xi^2-k^2}|z|+i\xi r}}{\sqrt{\xi^2-k^2}}d\xi, \qquad \text{(AII- 9)}$$

$$J_0(x)J_0(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}J_0(\sqrt{x^2+y^2-2xy\cos\alpha})d\alpha. \qquad \text{(AII-10)}$$

3. *Series expressions*

$$\int J_m(z)dz = 2\sum_{n=0}^{\infty}J_{m+2n+1}(z), \qquad \text{(AII-11)}$$

$$\sum_{n=0}^{\infty}\varepsilon_n J_{2n}(z) = 1, \qquad \varepsilon_0 = 1,\ \varepsilon_n = 2 \qquad (n=1, 2, 3, \ldots.), \qquad \text{(AII-12)}$$

$$\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z) = (\sin z)/2, \qquad \text{(AII-13)}$$

$$\sum_{n=0}^{\infty}(2n+1)J_{2n+1}(z)=z/2. \tag{AII-14}$$

4. *Addition theorems* (As to the notations, refer to Fig. A-1.)

$$e^{i\nu\psi}J_{\nu}(\varpi)=\sum_{m=-\infty}^{\infty}J_{\nu+m}(Z)J_{m}(z)e^{im\phi}, \tag{AII-15}$$

$$e^{-i\nu\psi}J_{\nu}(\varpi)=\sum_{m=-\infty}^{\infty}J_{\nu+m}(Z)J_{m}(z)e^{-im\phi}. \tag{AII-16}$$

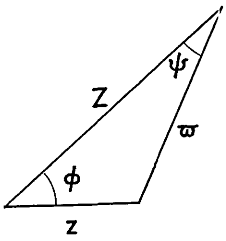

Fig. A-1.

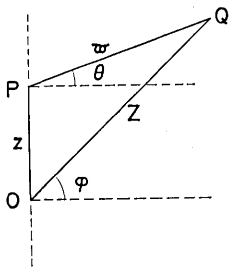

Fig. A-2 Translation of the origin.

Appendix III. Derivation of equations (7)

Let us consider the functions

$$J_s=J_1(\varpi)\sin\theta, \quad \text{and} \quad J_c=J_1(\varpi)\cos\theta. \tag{AIII-1}$$

If the origin P is shifted to the point O by a distance z, functions J_s and J_c, which are expressed by ϖ and θ, yield by means of the addition theorems (AII-15 & 16), in terms of Z and φ, where $\angle QOP=\pi/2-\varphi$, $\angle PQO=\varphi-\theta$ and $z<Z$,

$$\left.\begin{aligned} e^{-i\theta}J_1(\varpi)&=\sum_{m=-\infty}^{\infty}J_{m+1}(Z)J_m(z)e^{-im\pi/2+i(m-1)\varphi},\\ e^{i\theta}J_1(\varpi)&=\sum_{m=-\infty}^{\infty}J_{m+1}(Z)J_m(z)e^{im\pi/2-i(m-1)\varphi},\end{aligned}\right\} \tag{AIII-2}$$

whence

$$\left.\begin{aligned} J_c=&\sum_{m=-\infty}^{\infty}(-1)^m J_{2m+1}(Z)J_{2m}(z)\cos(2m-1)\varphi\\ &+\sum_{m=-\infty}^{\infty}(-1)^m J_{2m+2}(Z)J_{2m+1}(z)\sin 2m\varphi,\\ J_s=&\sum_{m=-\infty}^{\infty}(-1)^m J_{2m+1}(Z)J_{2m}(z)\sin(2m-1)\varphi\\ &+\sum_{m=-\infty}^{\infty}(-1)^m J_{2m+2}(Z)J_{2m+1}(z)\cos 2m\varphi.\end{aligned}\right\} \tag{AIII-3}$$

By using the relation between the negative and positive order Bessel functions (AII-5), the negative order functions are transformed into the positive order, and then we get

$$J_c=\sum_{m=1}^{\infty}(-m)^m[J_{2m+1}(Z)J_{2m}(z)+J_{2m-3}(Z)J_{2m-2}(z)]\cos(2m-1)\varphi$$
$$+\sum_{m=1}^{\infty}(-1)^m[J_{2m+2}(Z)J_{2m+1}(z)-J_{2m-2}(Z)J_{2m-1}(z)]\sin 2m\varphi, \quad \text{(AIII-4)}$$

$$J_s=\sum_{m=1}^{\infty}(-1)^m[J_{2m+1}(Z)J_{2m}(z)-J_{2m-3}(Z)J_{2m-1}(z)]\sin(2m-1)\varphi$$
$$+\sum_{m=1}^{\infty}(-1)^m[J_{2m+2}(Z)J_{2m+1}(z)+J_{2m-2}(Z)J_{2m-1}(z)]\cos 2m\varphi$$
$$+J_2(Z)J_1(z). \quad \text{(AIII-5)}$$

Putting $\varpi=\xi r$, $Z=\xi r_2$ and $z=\xi r_1$ into (AIII-4) and (AIII-5), we can easily obtain the expressions (7).

Appendix IV. Derivation of equations (27), (28), (29)

These equations (27), (28), (29) have similar forms, so that we consider equation (28) only. From (15), with $\varphi=0$,

$$V=v^{(\mathrm{SH})}/(F_0/\mu)=\sum_{m=1}^{\infty}v_m^{(\mathrm{SH})}$$
$$=\frac{1}{2a}\frac{\partial}{\partial r_2}\int_0^a dr_1\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}\sum_{m=1}^{\infty}(-1)^m[J_{2m+1}(\xi r_2)J_{2m}(\xi r_1)$$
$$+J_{2m-3}(\xi r_2)J_{2m-2}(\xi r_1)]d\xi$$
$$=\frac{1}{2a}\frac{\partial^2}{\partial r_2^2}\int_0^a dr_1\int_0^{\infty}\frac{e^{-\nu' z}}{\xi\nu'}\sum_{m=0}^{\infty}\varepsilon_n(-1)^n J_{2n}(\xi r_2)J_{2n}(\xi r_1)d\xi,$$

where $\varepsilon_n=1(n=0),=2(n\geq 1)$, by the addition theorem (AII-15) with $\nu=0$,

$$V=-\frac{1}{2a}\frac{\partial^2}{\partial r_2^2}\int_0^a dr_1\int_0^{\infty}\frac{e^{-\nu' z}}{\xi\nu'}J_0(\xi\sqrt{r_2^2+r_1^2})d\xi.$$

By (AII-8), it is transformed into

$$V=-\frac{i}{2a\pi}\frac{\partial}{\partial r_2}\int_0^a dr_1\int_0^{\pi/2}\cos\alpha d\alpha\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}\cos(\xi r_2\cos\alpha)\cos(\xi r_1\sin\alpha)d\xi$$
$$=\frac{1}{2a\pi i}\frac{\partial}{\partial r_2}\int_{-a}^{a}dr_1\int_0^{\pi/2}\cos\alpha d\alpha\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}\cos(\xi r_2\cos\alpha+\xi r_1\sin\alpha)d\xi$$

and by (AII-9),

$$=-\frac{1}{4a}\frac{\partial}{\partial r_2}\int_{-a}^{a} dr_1\int_0^{\pi/2} H_0^{(1)}(k\sqrt{(r_2\cos\alpha+r_1\sin\alpha)^2+z^2})\cos\alpha d\alpha.$$

The substitution of the variable r_1 by $\eta\cdot r_2$ yields

$$V=-\frac{1}{2}\frac{\partial}{\partial r_2}\frac{r_2}{a}\int_0^{\pi/2}\cos\alpha d\alpha\int_0^{a/r2} H_0^{(1)}(k\sqrt{r_2^2(\cos\alpha+\eta\sin\alpha)^2+z^2})d\eta.$$

Since $a/r_2<1$, Taylor expansion of the integrand is applied, then

$$V=\frac{1}{2}\frac{\partial}{\partial r_2}\int_0^{\pi/2} H_0^{(1)}(k\sqrt{z^2+r_2^2\cos^2\alpha})\cos\alpha d\alpha$$
$$+\frac{1}{3!\cdot 2}\frac{\partial}{\partial r_2}\frac{a^2}{r_2}\frac{\partial}{\partial r_2}\int_0^{\pi/2} H_0^{(1)}(k\sqrt{z^2+r_2^2\cos^2\alpha})\tan\alpha d\alpha+O(a^4/r_2^4).$$

Again by (AII-9),

$$V=\frac{\partial}{\partial r_2}\int_0^{\pi/2}\cos\alpha d\alpha\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}\cos(\xi r_2\cos\alpha)d\xi$$
$$+\frac{a^2}{3!}\frac{\partial}{\partial r_2}\frac{1}{r_2}\frac{\partial}{\partial r_2}\int_0^{\pi/2}\tan\alpha d\alpha\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}\cos(\xi r_2\cos\alpha)d\xi+O\left(\frac{a^4}{r_2^4}\right),$$
$$=\frac{\partial}{\partial r_2}\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}J_1(\xi r_2)d\xi$$
$$+\frac{a^2}{3!}\frac{\partial}{\partial r_2}\frac{1}{r_2}\frac{\partial}{\partial r_2}\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}d\xi\int_0^1\cos(\xi r_2 x)\frac{dx}{x}+O(a^4/r_2^4).$$

The first term is the solution from a point source. The second term is transformed as follows:

$$\frac{a^2}{3!}\frac{\partial}{\partial r_2}\frac{1}{r_2}\int_0^{\infty}\frac{\xi e^{-\nu' z}}{\nu'}d\xi\int_0^1[-\sin(\xi r_2 x)]dx$$
$$=\frac{1}{3!}\frac{a^2}{r_2^2}\frac{\partial}{\partial r_2}\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}\cos(\xi r_2)d\xi-\frac{2}{3!}\frac{a^2}{r_2^3}\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}\cos(\xi r_2)d\xi$$
$$+\frac{2}{3!}\frac{a^2}{r_2^3}\int_0^{\infty}\frac{e^{-\nu' z}}{\nu'}d\xi,$$
$$=\frac{1}{3!}\left(\frac{a}{r_2}\right)^2\frac{\partial}{\partial r_2}H_0^{(1)}(k\sqrt{r_2^2+z^2})-\frac{2}{3!}\left(\frac{a}{r_2}\right)^2\frac{H_0^{(1)}(k\sqrt{r_2^2+z^2})}{r_2}$$
$$+\frac{2}{3!}\left(\frac{a}{r_2}\right)^2\frac{H_0^{(1)}(kz)}{r_2}.$$

These expressions are equation (28). The other expressions also are obtained in a similar way.

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1. 半無限弾性体の表面に置かれた有限長線状水平力源から発生する波動について

— SH 波発生装置の理論的研究 —

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地震探鉱実験グループでは約10年前から，地表面に板を寝かせてその一端を叩いたり，或いはその板の上に大砲を固定させて火薬を爆発させたりして，その接触面に摩擦によるせん断応力を与え，S波を発生させている．この論文では，この問題を理想化して，半無限弾性体の自由表面におかれた有限長の線状せん断応力源を考えて，変位の方位方向成分について，応力源の長さに関する影響を導く．そして，S波発生装置の設計や実験方法について，二・三の注意を与える．

さて，この様な応力源から発生する波は次の性質をもっている．

1) 応力源を二等分する方向の一つを $\varphi=0$ にとると，変位の半径方向 (u) と鉛直方向 (w) の成分は $\sin(2m-1)\varphi$ の級数として，又方位方向の成分 (v) は $\cos(2m-1)\varphi$ の級数として表わされる．

2) 遠方では，u と w では P, SV，レイリー波の各成分が，v では SH 波の成分が卓越し，表面では SH 波とレイリー波のみが観測される.

3) SH 波の射出パターンには $\varphi=0$ 以外の方位で，源の長さの影響は現われるが，$\varphi=0$ の方向でそれは消える．また深くなるにつれてその影響は小さくなる.

4) $\varphi=0$ の鉛直面で，源の長さの影響は (長さ)/(距離) の自乗の大きさである．即ち，変位の主部は点源からのものであって，上記の大きさの線状源の補正項をもつ.

5) P, SV 及びレイリー波の射出パターンは源の長さの影響が著しい.

次に，S 波発生装置を設計する場合には次の点に考慮が払われることが望ましい.

1) SH 波のエネルギーを $\varphi=0$ の方向に効果的に発生させるには，源の強さがその中心に対して対称であること.

2) 観測は $\varphi=0$ の線上で行なうこと.

3) (源の長さ)/(波長) の比が大きいこと.

ところで，実験現場はこの論文で取扱った様に一様なものではないので，成層構造とかその他複雑な構造を考える必要がある．また，源の強さが一定であるという境界条件のもとで解かれているが，この条件は検討を要する．更に，実験は衝撃的な力を利用しているので，適当な初期条件を考えなくてはいけない．今後，これらの点を明らかにして行きたい.