BULLETIN OF THE EARTHQUAKE
RESEARCH INSTITUTE
Vol. 45 (1967), pp. 559-566

30. *Notes on Propagation of Elastic Waves through a Heterogeneous Medium with Periodic Structures**⁾.

By Isao ONDA,

Earthquake Research Institute.

(Read May 23, 1967.—Received June 30, 1967.)

1. Introduction

In many of the investigations of a seismic wave propagated through the earth, the heterogeneous medium is approximated by means of an appropriate layering of homogeneous media, and the effect of the heterogeneity on the wave propagation is calculated by using ray theory, which reduces the study to a boundary value problem. However, it is quite laborious to calculate the effect of complexity of a crustal structure or topography on seismic wave propagation by using that treatment. To avoid that laboriousness, we consider instead that the propagation velocity of seismic waves fluctuates about an average velocity spatially and that some structural periods are introduced as elements of the spatial variation. This enables us to reduce the calculation of that effect to the treatment of wave propagation through a heterogeneous medium with a periodic structure. It is well known that wave propagation through such a medium can be solved by means of Hill's equation (*cf.* Humbert, 1926; Strutt, 1932; McLachlan, 1951; Brillouin, 1953).

In previous papers (Yoshiyama, 1960; Yoshiyama and Onda, 1962; Onda, 1964 *a*, *b*; Onda, 1966 *a*, *b*), we have obtained the nature of waves propagated through periodic structures, not by means of direct solutions of the boundary value problem but by means of solutions of Hill's equation. Using a corresponding travel time with distance as a variable, we can reduce the one-dimensional wave equation to one from which a quantitative discussion on wave behaviour for wave frequency is made very easily. Solutions of the equation are obtained by adopting the modified Whittaker's sigma method (Onda, 1964 *b* and 1966 *a*).

---

*⁾ This paper, together with the Author's previous investigations (Onda, 1964 *a*, *b*; 1965: 1966 *a*, *b*), forms part of a thesis, submitted in partial fulfillment of D. Sc. requirements at the University of Tokyo, 1966.

The solutions from this method are very convenient to discuss their convergence and nature.

Generally speaking, two independent solutions of Hill's equation are bounded or unbounded for the whole range of the variable, accordingly as the parameters of each term of the equation give a stable or an unstable region of solutions. In our problem, the stable and unstable regions exist alternately with increasing frequency. However, seismological interest lies usually in the first unstable region which is associated with the lowest frequency (Onda, 1966 a). Moreover, the solution of the equation of waves propagated through a periodic structure cannot be, straightforwardly, connected with the study of progressive waves which play an important role in seismological applications (Rayleigh, 1887; Yoshiyama, 1962). Accordingly, to study the effect on progressive waves, we must make some necessary modifications to these solutions. They are investigated by using the transmission coefficient for a structure in which a periodic one is intervened between two homogeneous media.

The transmission coefficient is expressed by complicated functions of the wave frequency, the velocity undulation and the thickness of the heterogeneous medium. Though only one example was evaluated in a previous paper (Onda, 1966 a), some interesting results are deduced from the calculation of several examples. The first part of this paper concerns them. Similar characteristics are obtained from direct solutions of the boundary value problem, and are connected with the resonant phenomena of waves in periodic structures (sections 3 and 4). The last part deals with a supplement to the wave equation in a medium where variation in velocity is expressed by a sum of many periodicities.

2. Transmission coefficient of waves through a periodic structure

It is important to study the effect of a periodic structure on a progressive wave. However, the direct solution of a wave equation in that medium cannot express the progressive wave explicitly. So, we consider a structure in which a periodic medium is intervened between two homogeneous media and discuss that effect in terms of a transmission coefficient of waves passing through this structure. To eliminate the boundary reflection of waves in this structure, it is assumed that the velocity and its gradient are continuous together at both interfaces.

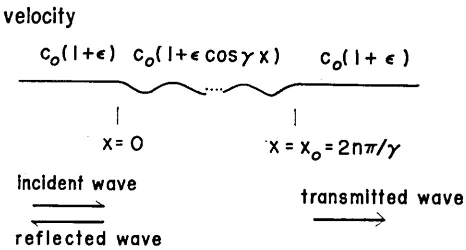

Fig. 1. Schematic illustration of the calculation when $\varepsilon>0$.

The transmission coefficient T of the wave through the medium, where the velocity varies as (see Fig. 1)

$$\left.\begin{aligned} c(x)&=c_0(1+\varepsilon) && x<0 \\ &=c_0(1+\varepsilon\cos\gamma x) && 0<x<x_0=2n\pi/\gamma \\ &=c_0(1+\varepsilon) && x_0<x\ , \end{aligned}\right\} \tag{1}$$

has been calculated in a previous paper (Onda, 1966 *a*, eqs. (12 a; b)). In the unstable region,

$$T=\frac{\exp(-iz_0-i\psi)}{\cosh(\mu z_0\sin 2\sigma)}\cos\psi\ , \tag{2}$$

where

$$z_0=\gamma x_0/2=n\pi\ ,\qquad \mu=\varepsilon/2\ ,$$

$$\tan\psi=\tanh(\mu z_0\sin 2\sigma)(\cos 2\sigma-2\delta-2\delta^2\cos 2\sigma)/\sin 2\sigma\ ,$$

$$\delta=\frac{\varepsilon}{8}+\varepsilon^2\left(\frac{1}{6}-\frac{\cos 2\sigma}{32}\right),$$

$$\frac{2\omega}{\gamma c_0}=1+\frac{\varepsilon}{2}\cos 2\sigma-\frac{7}{16}\varepsilon^2\ .$$

In the stable region,

$$T=\cos\psi'\cdot\exp(-i\nu z_0-i\psi')\ , \tag{3}$$

where

$$\tan\psi'=\frac{\varepsilon}{2}\frac{1}{\nu^2-1}\sin 2\nu z_0\ ,$$

and where ν is determined by

$$\frac{2\omega}{\gamma c_0}=(1-\varepsilon^2)\nu^2\left(1+\frac{\varepsilon^2}{2}\frac{1}{\nu^2-1}\right).$$

Several examples for various undulations ε and thicknesses of the periodic structure have been evaluated, and these results are shown in Fig. 2. From this figure, we obtain that the effect of the periodic structure on progressive waves corresponding to the stable solution of the wave equation apparently occurs rarely, while near the frequency range giving the unstable solution a characteristic attenuation appears. This apparent attenuation has the following characteristics: The larger the velocity undulation ε, the greater becomes the apparent attenuation, and the wider is the associated frequency band; the thicker the heterogeneous medium, the greater becomes the attenuation, and the narrower is the frequency band.

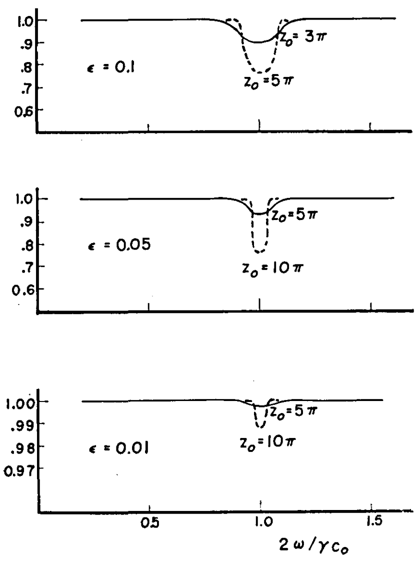

Fig. 2. Modulus of the transmission coefficient for a periodic structure. The velocity distribution is $c(x)=c_0(1+\varepsilon\cos\gamma x)$ for $0<x<x_0$, and $c_0(1+\varepsilon)$ for $x<0$ and $x_0<x$. $z_0/\pi=n$ is the number of velocity maximum ($\varepsilon<0$) or minimum ($\varepsilon>0$) in the intervening medium so that $x_0=2n\pi/\gamma$.

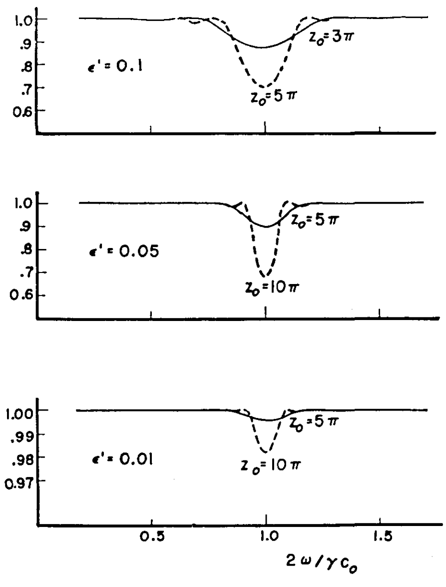

Fig. 3. Modulus of the transmission coefficient for an alternation structure. The velocity distribution is $c_0(1+\varepsilon')$ for $\gamma x<0$, $(2m-2)\pi<\gamma x<(2m-1)\pi$ and $2n\pi<\gamma x$; and is $c_0(1-\varepsilon')$ for $(2m-1)\pi<\gamma x<2m\pi$, where $m=1, 2, \cdots, n(=z_0/\pi)$.

3. Comparison with the transmission coefficient of waves through an alternation of strata

An alternation of two homogeneous layers in which the velocity is $c_0(1\pm\varepsilon')$, respectively, and each thickness is $L/2$ is also a periodic structure. The transmission coefficient in this case is computed for three values of ε' and two values of the thickness of the alternation by means of the matrix method, and these results are shown in Fig. 3. We find that the nature of the attenuation over the frequency range of interest shown in Fig. 3 is similar to one in Fig. 2. There is a slight difference between their micro-structures, which will be elucidated in the near future.

When this alternation covers a very wide extent, the velocity distribution is expressed by

$$c(x)=c_0\left[1-4\varepsilon'\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos 2(2n-1)\pi\frac{x}{L}\right]. \qquad (4)$$

Hence, the transmission coefficient for the n-th maximum attenuation is obtained (Onda, 1964 *b*, eq. (22)) as

$$T_m=\operatorname{sech}\left(\frac{\varepsilon_n}{2}\frac{x_0}{c_0}\omega_n\right), \qquad \omega_n=\frac{n\gamma_0 c_0}{2}. \qquad (5)$$

In the present case, as $n=1$, $\varepsilon_n=4\varepsilon'/\pi$ and $\omega_n=\pi c_0/L$, we have

$$T_m=\operatorname{sech}(2\varepsilon' x_0/L).$$

When ε' is large, the difference between this factor T_m and the coefficient obtained from Fig. 3 may not be neglected, whereas if ε' is small, the difference is negligibly small, *e.g.*, if $\varepsilon'=0.01$, it is smaller than 10^{-3}.

4. Resonance of waves in a periodic structure

In the paper (Onda, 1966 *a*, p. 10), it has been noted that the characteristic attenuation is caused by a type of resonance in the periodic structure. From the preceding section of this paper, the nature of the

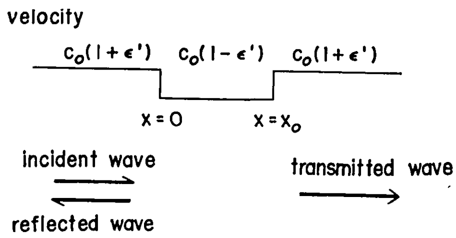

Fig. 4. Schematic illustration of a three layered structure which is the simplest alternation.

transmission coefficient of waves propagated through the alternation of homogeneous layers is in agreement with the one with periodic structure. We consider the simplest case of one cycle of the periodic structure and calculate the amplitude distribution of the displacement in the intermediate medium, $|u_2|$. This structure is schematically shown in Fig. 4. In the result we obtain

$$|u_2|=A\cdot\sqrt{1+2\varepsilon' \cos \pi\frac{x}{x_0}+\varepsilon'^2}\,, \qquad \text{for } k_2x_0=\pi/2\,, \tag{7}$$

where A is a constant and x_0 is the thickness of the intermediate medium, which equals π/γ. It follows from this expression that the loop of the envelope of the displacement amplitude occurs at $x=0$ and x_0 respectively. Consequently, since the condition $k_2x_0=\pi/2$ turns out to be $\omega=\gamma c_2/2$, a resonating oscillation at the frequency giving the unstable region in the periodic structure is verified.

5. Wave equation in a heterogeneous medium with some periodicities

We assume that the velocity varies as

$$c(x)=c_0(1+\varepsilon_1 \cos \gamma_2 x+\varepsilon_2 \cos \gamma_2 x)\,, \tag{8}$$

where the ratio γ_1/γ_2 is rational, and that the product $\varepsilon_2\gamma_2$ is of the order of magnitude of the product $\varepsilon_1\gamma_1$. If $\gamma_1/\gamma_2=m_1/m_2$, m_1 and m_2 being any mutually prime integers, we can define γ_0 satisfying $\gamma_j=m_j\gamma_0$, $(j=1, 2)$. For simplicity, let each product $\varepsilon_j m_j$ be of the order δ. As a result, the wave equation valid within the order δ^2 is expressed by

$$
\begin{aligned}
\frac{d^2\varphi}{d\xi^2}+\Bigg\{\left(\frac{2\omega}{\gamma_0 c_0}\right)^2 & \frac{1}{1-(\varepsilon_1+\varepsilon_2)^2}-\frac{\varepsilon_1^2 m_1^2+\varepsilon_2^2 m_2^2}{2}+2\varepsilon_1 m_1^2\cos 2m_1\xi \\
& +2\varepsilon_2 m_2^2\cos 2m_2\xi+\frac{5}{2}\varepsilon_1^2 m_1^2\cos 4m_1\xi+\frac{5}{2}\varepsilon_2^2 m_2^2\cos 4m_2\xi \\
& +\varepsilon_1\varepsilon_2\left(m_1^2+m_1m_2+m_2^2+\frac{m_2^3}{m_1}+\frac{m_1^3}{m_2}\right)\cos 2(m_1+m_2)\xi \\
& +\varepsilon_1\varepsilon_2\left(m_1^2-m_1m_2+m_2^2-\frac{m_2^3}{m_1}-\frac{m_1^3}{m_2}\right)\cos 2(m_1-m_2)\xi \\
& +O(\delta^3)\Bigg\}\varphi=0\ ,
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
\varphi &= (\text{displacement})\cdot\sqrt{(\rho c)}\ , \\
\xi &= \frac{\gamma_0 c_0}{2}\sqrt{1-(\varepsilon_1+\varepsilon_2)^2}\int\frac{dx}{c(x)} \\
&= \frac{\gamma_0 c_0}{2}\Bigg[1-\varepsilon_1\frac{\sin m_1\gamma_0 x}{m_1\gamma_0 x}-\varepsilon_2\frac{\sin m_2\gamma_0 x}{m_2\gamma_0 x} \\
&\quad +\frac{\varepsilon_1^2}{2}\frac{\sin 2m_1\gamma_0 x}{2m_1\gamma_0 x}+\frac{\varepsilon_2^2}{2}\frac{\sin 2m_2\gamma_0 x}{2m_2\gamma_0 x} \\
&\quad +\varepsilon_1\varepsilon_2\left\{\frac{\sin(m_1+m_2)\gamma_0 x}{(m_1+m_2)\gamma_0 x}+\frac{\sin(m_1-m_2)\gamma_0 x}{(m_1-m_2)\gamma_0 x}-1\right\}+O(\delta^3)\Bigg]\ .
\end{aligned}
\tag{10}
$$

If $m_1=m_2=1$ and $\varepsilon_1=\varepsilon_2=\varepsilon/2$, they agree with the equation and the variable of periodic variation in velocity, respectively, obtained in the previous paper (Onda, 1966 *a*, eqs. (3) and (4)). The solution of this equation is obtained easily by means of so-called Whittaker's sigma method (Onda, 1964 *b*, Appendix; 1966 *a*, Appendix).

In conclusion, the author wishes to express his sincere thanks to Professor Ryoichi Yoshiyama for his valuable encouragement, and also to Professor Yasuo Satô and Dr. Ryosuke Sato for their helpful suggestions. A part of the computation was done through the project UNICON, IBM Japan.

References

BRILLOUIN, L., 1953, *Wave Propagation in Periodic Structures.* Dover Publ., 255pp.
HUMBERT, P., 1926, *Fonctions de Lamé et fonctions de Mathieu.* Fascicule X du Mémorial

des sciences mathématiques. Paris, Gauthier-Villars, 58 pp.
McLachlan, N. W., 1951, *Theory and Application of Mathieu Functions.* Oxford, 401 pp.
Onda, I., 1964 *a*, "Effect of the Intermediate Dissipative Medium on the Transmission of Elastic Waves through a Heterogeneous Medium having Periodic Structures," *Bull. Earthq. Res. Inst.*, **42**, 11-18.
Onda, I., 1964 *b*, "Propagation and Apparent Attenuation of Elastic Waves in a Heterogeneous Medium with Certain Periodic Structures," *Bull. Earthq. Res. Inst.*, **42**, 427-447.
Onda, I., 1965, "Effects of Intervenient Boundaries on the Elastic Propagation," *Bull. Earthq. Res. Inst.*, **43**, 1-14.
Onda, I., 1966 *a*, "Plane Pulse Propagation through a Heterogeneous Medium with a Periodic Structure," *Bull. Earthq. Res. Inst.*, **44**, 1-12.
Onda, I., 1966 *b*, "Solutions of Elastic Waves through a Heterogeneous Medium with Regular Variation in Elasticity," *Bull. Earthq. Res. Inst.*, **44**, 765-777.
Rayleigh, Lord, 1887, "Maintenance of Vibrations by Forces of Double Frequency, and Propagation of Waves through a Medium with a Periodic Structure," *Phil. Mag.*, **24**, 145-159.
Strutt, M. J. O., 1932, *Lamésche- Mathieusche- und verwandte Funktionen in Physik und Technik.* Ergebnisse der Mathematik und ihrer Grenzgebiete, **3**, Springer, 116pp.
Yoshiyama, R., 1960, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction," *Bull. Earthq. Res. Inst.*, **38**, 467-478.
Yoshiyama, R., 1962, see Yoshiyama and Onda, 1962.
Yoshiyama, R., and Onda, I., 1962, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction, Part 2; with Special Remarks on Waves Propagation in a Periodic Structure," *Bull. Earthq. Res. Inst.*, **40**, 391-398.

---

30. 周期構造を伝わる波に関するノート

地震研究所　音　田　　功

前の論文 (Onda, 1966 *a*) で，周期構造を伝わる波の透過率を求めた．この論文では，その表現式を用いて，速度変動量と周期構造の厚さを変えて数値計算を試み，次の結果を得た．方程式の安定解を与える周波数の波は，見掛け上，ほとんどこの周期構造の影響を受けないように見えるけれども，不安定解を与える周波数を中心に，著しい見掛けの減衰が現われ，その減衰の性質は次のようである：速度変化が激しくなるにつれて，関係せる周波数幅は拡がり，減衰量が大きくなる．また，周期構造の厚さが厚くなるにつれて，関係せる周波数幅は狭くなり，減衰量は増加する．そして，この見掛けの減衰は，周期構造に伝わる波の共鳴に起因するものであることを，前の論文とは別の立場から証明した．

最後に，速度が二つの構造上の波長をもつている場合の波動方程式を，前の論文で示したよりも，さらに微小項を含む形で求めた．これは更に，速度変化がフーリエ級数で表わされた不均質媒質中の波動を，より詳しく論ずるときに必要な表現に，容易に拡張されるであろう．