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41. *Solution of Elastic Waves through a Heterogeneous Medium with Regular Variation in Elasticity.*

By Isao ONDA,

Earthquake Research Institute.

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Summary

As a continuation of the study of waves passing through a medium with periodic structure, the wave equation in a medium where elasticity varies regularly is treated as one of Hill's equation. The same form of the equation appears in the long wave passing through a canal with undulatory bed. It is shown that, the solution in terms of Fourier series given in early investigations is inadequate in this problem, although that solution is very simple in solving process, treatment, etc.

1. Introduction

In early papers (Yoshiyama, 1960; Onda, 1964 and 1966), wave propagation in a medium in which variation of a velocity is regular was discussed, in connection with the stability of progressive waves. It is the purpose of this paper to obtain the periodic solutions of the wave equation in a medium where variation of an elasticity is regular. An analogous equation appears in a problem on long waves passing through a canal with undulatory bed (Hidaka, 1936; Yoshida, 1947). In those calculations a solution expressed in terms of Fourier series was obtained. Such an expression is very easy to solve the equation and convenient to be employed. However, it is doubtful that such an expression forms a fundamental system of solution, since none of the unstable regions can be represented by such a solution.

In section 2, the solution and the stability chart of the wave equation are obtained. The procedure to solve the equation is similar to that outlined in the previous papers. In section 3, the solutions obtained by Prof. Hidaka (1936) and Prof. Yoshida (1947) are examined from a viewpoint of the theory of differential equations.

2. Solutions of wave equation for a structure periodic in elasticity

For the sake of simplicity, it is assumed that a wave is propagated along the x-axis, and an elasticity varies regularly in the same direction. If the method described by Prof. Yoshiyama (1960) is applied, the wave equation is written as

$$\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial^2\varphi}{\partial\tau^2}-\alpha^2\varphi\,, \tag{1}$$

where

$$\varphi=(\rho E)^{1/4}U\,,$$
$$\tau=\int dx/c(x)\,, \tag{2}$$
$$c(x)=\sqrt{E(x)/\rho(x)}\,,$$

$$\alpha^2=-\sqrt{\frac{c}{\rho}}\frac{d}{dx}\left(\rho\, c^2\frac{d}{dx}\frac{1}{\sqrt{\rho c}}\right)=\frac{d}{d\tau}\left\{\frac{1}{2\rho c}\frac{d(\rho c)}{d\tau}\right\}+\left\{\frac{1}{2\rho c}\frac{d(\rho c)}{d\tau}\right\}^2, \tag{3}$$

and ρ is density, E being elasticity, and U being displacement.

Now, let density be uniform throughout the medium and elasticity be denoted by

$$E=E_0(1+\varepsilon\cos\gamma x)\,,\qquad \varepsilon>0 \tag{4}$$

From equation (2), a travel time τ and a velocity distribution are written, respectively, as

$$\tau=\int_0^x\frac{dx}{c_0\sqrt{1+\varepsilon\cos\gamma x}}=\frac{2}{\gamma c_0\sqrt{1+\varepsilon}}\int_0^{\gamma x/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

therefore

$$\operatorname{sn}(a\tau/2)=\sin(\gamma x/2)\,, \tag{5}$$

and

$$c=c_0\sqrt{\frac{1+\operatorname{dn}a\tau+\varepsilon(-1+2\operatorname{cn}a\tau+\operatorname{dn}a\tau)}{1+\operatorname{dn}a\tau}}\,, \tag{6}$$

where $\operatorname{sn}(u)$, $\operatorname{cn}(u)$ and $\operatorname{dn}(u)$ are the Jacobi's elliptic functions,

$$k^2=2\varepsilon/(1+\varepsilon)\,,$$
$$c_0=\sqrt{E_0/\rho}\,,$$

and

$$a=\gamma c_0\sqrt{1+\varepsilon}\,.$$

The wave equation for a harmonic wave is

$$\frac{d^2\varphi}{d\tau^2}+\left\{\omega^2-\frac{\varepsilon a^2}{2(1+\varepsilon)}f(a\tau)\right\}\varphi=0\,, \tag{7}$$

where
$$f(u)=\frac{(\mathrm{cn}^2 u-\mathrm{sn}^2 u)\,\mathrm{dn}\,u}{1+\mathrm{dn}\,u}+\frac{5\,\varepsilon\,\mathrm{sn}^2 u\,\mathrm{cn}^2 u}{2(1+\varepsilon)(1+\mathrm{dn}\,u)^2}$$
$$-\frac{\varepsilon\,\mathrm{sn}^2 u\,\mathrm{cn}\,u\,(\mathrm{cn}\,u+\mathrm{dn}\,u)}{(1+\mathrm{dn}\,u)\{1+\mathrm{dn}\,u+\varepsilon(-1+2\,\mathrm{cn}\,u+\mathrm{dn}\,u)\}}$$
$$-\frac{(1+\varepsilon)\{(\mathrm{cn}^2 u-\mathrm{sn}^2 u)\,\mathrm{dn}\,u+(\mathrm{dn}^2 u-k^2\,\mathrm{sn}^2 u)\,\mathrm{cn}\,u\}}{1+\mathrm{dn}\,u+\varepsilon(-1+2\,\mathrm{cn}\,u+\mathrm{dn}\,u)}$$
$$-\frac{3\varepsilon(1+\varepsilon)\,\mathrm{sn}^2 u(\mathrm{cn}\,u+\mathrm{dn}\,u)^2}{2\{1+\mathrm{dn}\,u+\varepsilon(-1+2\,\mathrm{cn}\,u+\mathrm{dn}\,u)\}^2}\,. \tag{8}$$

If a function $f(u)$ be simply $\mathrm{sn}^2 u$, such an equation is called Lamé's equation. So, the most formal solution of equation (7) will be written as some series of Lamé's functions which are a compact solution of Lamé's equation. The most applicable solution in many problems on wave propagations, however, is the periodic one, and then it is convenient to express relation (8) in terms of some trigonometric functions. Equation (7), consequently, becomes Hill's equation.

If it is noted that arguments of these elliptic functions are real, and that a modulus k is restricted between zero and unity, the approximate expressions obtained in the Appendix of this paper are applicable.

Substitution of a variable $2z$ for $a\tau/\vartheta_3^2$, where ϑ_3 is a Theta-function, yields that

$$\frac{d^2\varphi}{dz^2}+\frac{\vartheta_3{}^4}{1+\varepsilon}\left\{4\frac{\omega^2}{\gamma^2 c_0{}^2}-\frac{\varepsilon^2}{8}+\varepsilon\cos 2z+\frac{\varepsilon^2}{8}\cos 4z+O(\varepsilon^3)\right\}\varphi=0\,. \tag{9}$$

The solution of equation (9) can be easily obtained by using the method developed in early papers (Onda, 1964, appendix; 1966, appendix), as follows:

The conditions of unstable regions and solutions in these regions are expressed as

$(n=1)$,

$$\left(\frac{2\omega}{\gamma c_0}\right)^2=1+\frac{\varepsilon}{2}\cos 2\sigma-\frac{\varepsilon^2}{32}(10-\cos 4\sigma)+O(\varepsilon^3)\,, \tag{10}$$

$$\mu=\frac{\varepsilon}{4}\sin 2\sigma+O(\varepsilon^3)\,, \tag{11}$$

$$y(z,\,\sigma)=\sin(z-\sigma)+\frac{\varepsilon}{16}\sin(3z-\sigma)$$
$$+\frac{\varepsilon^2}{256}\left\{\frac{1}{3}\sin(5z-\sigma)+3\sin 2\sigma\cos(3z-\sigma)\right.$$
$$\left.+\cos 2\sigma\sin(3z-\sigma)\right\}+O(\varepsilon^3)\,. \tag{12}$$

$(n=2)$,

$$\left(\frac{2\omega}{\gamma c_0}\right)^2=4-\frac{4}{3}\varepsilon^2+O(\varepsilon^4)\,,$$
$$\mu=O(\varepsilon^4)\,,$$

where n is the limit value of $2\omega/(\gamma c_0)$, when vanishing of ε. In these regions, even if all terms of the order ε^3 are not neglected, the real part of measure of stability μ for other unstable regions expect for the region near $n=1$ can be neglected numerically. In other words, all the solutions expect for the first unstable region alone can be fairly accurately represented by ones in the stable region.

In the stable region

$$\left(\frac{2\omega}{\gamma c_0}\right)^2=\left(1-\frac{\varepsilon^2}{8}\frac{3\nu^2-4}{\nu^2-1}\right)\nu^2+O(\varepsilon^4)\,, \tag{13}$$

$$\varphi_s(z,\,\nu)=\sin\nu z-\frac{\varepsilon}{8}\left\{\frac{\sin(\nu+2)z}{\nu+1}-\frac{\sin(\nu-2)z}{\nu-1}\right\}$$
$$+\frac{\varepsilon^2}{128}\left\{\frac{\sin(\nu+4)z}{\nu+1}-\frac{\sin(\nu-4)z}{\nu-1}\right\}+O(\varepsilon^3)\,, \tag{14}$$

the second solution $\varphi_c(z,\,\nu)$ being obtained by substitution of cosine for sine in $\varphi_s(z,\,\nu)$.

From these relations, the stability chart is shown in Fig. 1. The area of the unstable region is narrow, in comparison with one for a case of velocity fluctuation (cf. Fig. 1 of the previous paper; Onda, 1966). Comparing these figures, the effect of a periodic structure is less remarkable than that for the case of velocity fluctuation, and the nature of these solutions is similar to each other.

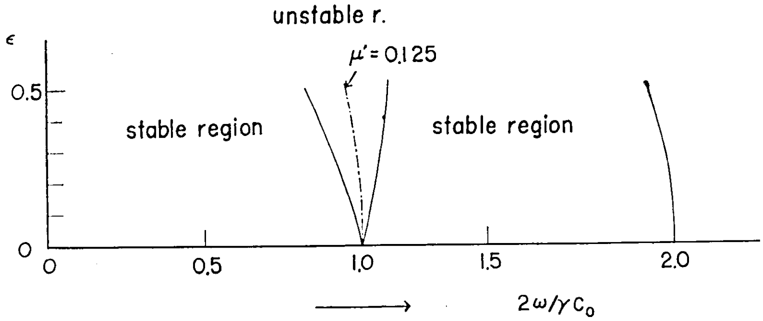

Fig. 1. The stability chart for the solutions of the wave equation in the medium with regular variation in elasticity: $E=E_0(1+\varepsilon\cos\gamma x)$. μ' is the maximum value of μ for $\varepsilon=0.5$.

The resultant expression of displacement in this medium is given by the form

$$U=\frac{1}{(1+\varepsilon\cos\gamma x)^{1/4}}\left[A_1 e^{\mu z}y(z,\sigma)+B_1 e^{-\mu z}y(z,-\sigma)\right], \tag{15}$$

where μ and $y(z,\sigma)$ are given by expressions (11) and (12) respectively, and σ ranges between 0 and $\pi/2$, and is associated with the wave frequency in the relation (10). In another frequency domain,

$$U=\frac{1}{(1+\varepsilon\cos\gamma x)^{1/4}}\left[A_1\varphi_s(z,\nu)+B_1\varphi_c(z,\nu)\right], \tag{16}$$

where $\varphi_s(z,\nu)$ and $\varphi_c(z,\nu)$ are given by expression (14) and ν is associated with the wave frequency in relation (13). The variable z is expressed by

$$z=\frac{\gamma c_0\sqrt{1+\varepsilon}}{2\vartheta_3^2}\cdot\tau=\vartheta_3^{-2}F(\gamma x/2,k)$$
$$=\frac{\gamma x}{2}\left\{1-\frac{\varepsilon}{2}\frac{\sin\gamma x}{\gamma x}+\frac{3}{16}\varepsilon^2\frac{\sin 2\gamma x}{2\gamma x}+O(\varepsilon^3)\right\},$$

where $F(\zeta,k)$ is the elliptic integral of the first kind.

It is to be remarked that these expressions of solution are quite different from the Fourier series.

These results can also be obtained by means of a procedure in the previous paper (Onda, 1964): If, after approximating $2z$ for γx in expression

(4), the term α^2 is expanded in a power series of ε, the wave equation is written as

$$\frac{d^2\varphi}{dz^2}+\left\{4\frac{\omega^2}{\gamma^2 c_0{}^2}+\frac{\varepsilon^2}{8}\left(1+\frac{\varepsilon^2}{4}\right)+\varepsilon\left(1-\frac{\varepsilon^2}{16}\right)\cos 2z-\frac{\varepsilon^2}{8}\cos 4z+O(\varepsilon^3)\right\}\varphi=0\,. \quad (18)$$

Since $$|\cos\gamma x-\cos 2z|=\frac{\varepsilon}{4}|1-\cos 2\gamma x|+O(\varepsilon^2)\,,$$

however, equation (18) should be discussed within the order $\varepsilon^2/4$. The condition of stability derived from this equation is numerically similar to that from the calculation of expressions (9) to (14).

It is notable that this solution is not the expression of progressive waves. The stability of progressive waves through this heterogeneous medium must be discussed by means of calculating the transmission coefficient in a structure where this medium intervenes between two homogeneous media, as shown in the previous papers, in which the transmission coefficient has been calculated and some pulse transmission has been discussed.

3. A note on a solution in a canal with undulatory bed

For the propagation of long waves in a canal, the wave equation is written (Lamb, 1932, p. 274) as

$$\frac{\partial^2 U}{\partial t^2}=\frac{g}{b}\frac{\partial}{\partial x}\left(bh\frac{\partial U}{\partial x}\right), \quad (19)$$

where

U = the free surface elevation,

b = width of a canal,

h = depth of a canal,

and

g = acceleration of gravity.

If U, b/g and bh are taken for the displacement, density and elasticity respectively, equation (19) is in agreement with the wave equation in an elastic body. Therefore, if the width is uniform and the depth varies regularly, that is,

$$h=h_0(1+2\Delta\cos\gamma x)\,, \quad (20)$$

the solutions must be the same as that obtained in the preceding paragraph ; (15) and (16). Prof. Hidaka (1936) and Prof. Yoshida (1947) investigated this problem, the former assuming the form of solution

$$\sum_{n=1}^{\infty} A_n \cos(n\gamma x+\delta)\,, \tag{21}$$

and found that the coefficient sequence A_n was absolutely convergent. This can be transformed into the form

$$=\sum_{n=1}^{\infty}\frac{A_n}{2}\{\exp(in\gamma x+i\delta)+\exp(-in\gamma x-i\delta)\}$$

$$=\sum_{n=-\infty}^{\infty} A'_n \exp(in\gamma x)\,.$$

The latter assumed the form of solution

$$\sum_{n=-\infty}^{\infty} A_n \exp\{i(n+\eta)\gamma x\}\,, \tag{22}$$

where η is a constant.

The expression (21) or (22) is regarded not only as a Fourier series but also as a Laurent series at the point $\exp(i\gamma x)=0$. Substitution

$$\zeta=\exp(i\gamma x) \tag{23}$$

to the wave equation for harmonic oscillations yields

$$\zeta\frac{d}{d\zeta}\left\{(\varDelta\zeta^2+\zeta+\varDelta)\frac{dU}{d\zeta}\right\}-k_0^2U=0\,, \tag{24}$$

where

$$k_0^2=\omega^2/(\gamma^2 g h_0)\,.$$

The positions of the singular points and their exponents are given, if one writes them following the expression of Riemann's P equation, by the form

$$\left\{\begin{matrix}\zeta_1^{-1}, & \zeta_1, & 0, & \infty & \\ 0, & 0, & 0, & 0, & \zeta \\ 0, & 0, & 1, & 1, & \end{matrix}\right\}, \tag{25}$$

where

$$\zeta_1=-\frac{1}{2\varDelta}+\sqrt{\left(\frac{1}{2\varDelta}\right)^2-1}$$

$$=-\Delta\{1+\Delta^2+2\Delta^4+5\Delta^6+O(\Delta^8)\}\ . \qquad (26)$$

The differential equation with four regular points is known as a kind of Lamé's equation.

Now, it must be borne in mind that the wave equation does not alter in form with substitution of ζ^{-1} for ζ. Therefore, the relation between the coefficients of the solution around zero is the same as one around infinity, and one around ζ_1 is the same as one around ζ_1^{-1}. From the view-point of the theory of differential equations, the power series solution around a singular point ζ_0 is given by the forms (*cf.* Erdélyi, 1956, p. 61)

$$\sum_{n=0}^{\infty} A_n(\zeta-\zeta_0)^{n+\eta} \text{ when } \zeta_0 \text{ is a regular point,} \qquad (27)$$

and

$$\sum_{n=-\infty}^{\infty} A_n(\zeta-\zeta_0)^{n+\eta} \text{ when } \zeta_0 \text{ is an irregular point,} \qquad (28)$$

where η is an exponent at a point ζ_0. From these considerations, the type of solution (22) consists of two solutions one of which is a solution around zero and the other around infinity respectively, and then it seems that the solution does not form the fundamental system of the general solution.

For the purpose of reference, the reason why the type of solution (22) can be taken in the Mathieu's equation

$$\frac{d^2w}{dz^2}+(a-2q\cos 2z)\,w=0 \qquad (29)$$

is as follows: Substituting

$$\zeta=\exp(iz)\,,$$

one can write the equation as

$$\zeta^2\frac{d^2w}{d\zeta^2}+\zeta\frac{dw}{d\zeta}+(q\zeta^2-a+q\zeta^{-2})w=0\,. \qquad (30)$$

Thus, only two singular points of zero and infinity are both irregular, and then the fundamental system of the general solution is given by the form

$$\sum_{n=-\infty}^{\infty} a_n\,\zeta^{n+\eta}=\sum_{n=-\infty}^{\infty} a_n\,e^{i(n+\eta)z}. \qquad (31)$$

Reverting to the subject, since the difference of the exponents at both

$\zeta=0$ and ∞ is unity, the first solution is expressed in terms of Taylor series at $\zeta=0$, and the second solution involves a logarithm. Therefore the general solution of the differential equation (24), which is analytic around zero, is

$$\left.\begin{aligned}&\sum_{n=1}^{\infty} a_n \zeta^n = \sum_{n=1}^{\infty} a_n e^{in\gamma x},\\ &\text{and}\\ &(\log \zeta)\cdot \sum a_n\zeta^n + \frac{\varDelta a_1}{k_0^2} - \sum_{n=1}^{\infty} b_n \zeta^n = i\gamma x \sum_{n=1}^{\infty} a_n e^{in\gamma x} + \frac{\varDelta a_1}{k_0^2} - \sum_{n=1}^{\infty} b_n e^{in\gamma x},\end{aligned}\right\} \quad (32)$$

where the coefficients a_n are combined with the following relations

$$(1-k_0^2)\, a_1 + 1\cdot 2\cdot \varDelta a_2 = 0$$

$$(n-1)n\, \varDelta a_{n-1} + (n^2-k_0^2)\, a_n + n(n+1)\varDelta a_{n+1} = 0, \text{ for } n \geq 2\,,$$

and

$$(1-k_0^2) b_1 + 1\cdot 2\cdot \varDelta b_2 = -(2a_1 + 3\varDelta a_2)\,, \quad (33)$$

$$\begin{aligned}(n-1)n\varDelta b_{n-1} &+ (n^2-k_0^2)\, b_n + n(n+1)\varDelta b_{n+1}\\ &= -\{(2n-1)\varDelta a_{n-1} + 2n\, a_n + (2n+1)\varDelta a_{n+1})\}, \text{ for } n \geq 2\end{aligned}$$

Here, substitution of ζ^{-1} for ζ does not alter the solution (32) and the relations between the coefficients (33). Prof. Hidaka (1936) showed that the series sequence a_n is convergent for $\varDelta < 1/2$. According to the theory of differential equations, the convergence radius of this solution is given by a distance to the nearest singular point, and then it is suggested that the second series solution would be divergent for a range of almost $\varDelta < 1/2$.

Consequently, it is found that the type of solution (21) or (22) is incomplete and inadequate for the problem in hand. It is well-known that if one convergent solution is taken as the form

$$\sum a_n \cos n\gamma x\,,$$

the second solution should be taken as

$$\int \frac{\exp(i\gamma x)\, dx}{[\sum a_n \cos n\gamma x]^2 (1 + 2\varDelta \cos \gamma x)}\,,$$

from calculation of Wronskian.

4. Concluding remarks

As a continuation of the study of a wave passing through a medium with periodic structure, the solution of the wave equation in a medium, where elasticity varies regularly, is obtained in connection with the long wave propagation in a canal with undulatory bed in which the form of the differential equation is similar to that in the elastic wave propagation.

The procedure in this study is similar to that in the previous ones, but the travel time introduced as a variable instead of the spatial coordinate in this study is expressed by means of a Jacobi's elliptic function. When its modulus is restricted between zero and unity and its argument is always real, it can be given accurately in terms of trigonometric functions (see Appendix). The original differential equation involves a polynomial of some elliptic functions, but it is reformed to Hill's equation by means of this approximation. The solution is simultaneously obtained by means of the expression derived in the previous papers. The unstable wave motion (in mathematical sense) appears at the same frequency as that for velocity fluctuation but its magnitude is negligibly small except for the specified frequency.

A solution in the long wave propagation passing through a canal with undulatory bed has been obtained in terms of Fourier series:

$$\sum_{n=1}^{\infty} a_n \cos(n\gamma x+\delta) .$$

Such a solution is considerably simple in solving process, treatment, etc. In addition, it has some characteristics quite different from the solution obtained in section 2. So, it is examined from the stand-point of the theory of differential equations. It was assumed that the solution could be expressed in terms of Fourier series preliminarily and the relation between those coefficients was derived, so as to make its relation converge. If a variable ζ is put for $\exp(i\gamma x)$, Fourier series can also be interpreted by Laurent series around $\zeta=0$. Although ζ^{-1} is taken instead of ζ, the original equation does not alter in form, and then the relation between the coefficients of the series solution around $\zeta=0$ is consistent with that around $\zeta=\infty$. The singular points at zero and infinity of ζ are both regular, and the respective exponents are together zero and unity. The first solution of the fundamental system, therefore, is obtained in terms of Taylor series at $\zeta=0$, and the second one involves a logarithm. In addition, there is the nearest singular point to $\zeta=0$ at $\zeta=\zeta_1$, where

$\zeta_1 = -(1/2\Delta)\sqrt{(1/2\Delta)^2 - 1}$. As ζ_1 is of the order of Δ, such a solution is incomplete and inadequate in the problem on wave motions.

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Appendix: Representation of Jacobi's Eiliptic Functions in terms of Trigonometric Functions

It is assumed that an argument of a Jacobi's function is real and that a modulus k is restricted within zero and unity. Under this condition, the following expressions are valid and convenient to some numerical calculations.

Let one take

$$q = \exp(-\pi K'/K), \tag{A-1}$$

where K and K' are the complete elliptic integrals of the first kind with a modulus k and a complementary modulus $k' = \sqrt{1-k^2}$ respectively, and it can be expanded (Whittaker and Watson, 1935, p. 486) as

$$q = \varepsilon + 2\varepsilon^5 + O(\varepsilon^9), \tag{A-2}$$

where

$$\varepsilon = \frac{1}{2}\frac{1-\sqrt[4]{1-k^2}}{1+\sqrt[4]{1-k^2}} = \frac{k^2}{16} + \frac{k^4}{32} + \frac{9}{512}k^6 + \frac{43}{4096}k^8 + O(k^{10}).$$

Fourier series for the Theta-functions (*loc. cit.*, p. 464) yield power series of q for the Jacobi's functions which are associated with quotients of the Theta-functions (*loc. cit.*, p. 492):

$$\begin{aligned} \operatorname{sn} u &= \frac{\vartheta_3\,\vartheta_1(v)}{\vartheta_2\,\vartheta_4(v)} = \frac{(1+2q)(\sin v + q^2 \sin 3v)}{(1+q)(1-2q\cos 2v)}\{1+O(q^4)\} \\ &= \sin v\{1 + 4q\cos^2 v + 2q^2(\cos 4v + \cos 2v) + O(q^3)\}, \end{aligned} \tag{A-3}$$

$$\mathrm{cn}\, u = \frac{\vartheta_4\, \vartheta_2(v)}{\vartheta_2\, \vartheta_4(v)} = \frac{(1-2q)(\cos v + q^2 \cos 3v)}{(1+q^2)(1-2q\cos 2v)}\{1+O(q^4)\}$$

$$= \cos v\{1-4q\sin^2 v + 2q^2(\cos 4v - \cos 2v) + O(q^3)\}\,, \qquad \text{(A-4)}$$

$$\mathrm{dn}\, u = \frac{\vartheta_4\, \vartheta_3(v)}{\vartheta_3\, \vartheta_4(v)} = \frac{(1-2q)(1+2q\cos 2v)}{(1+2q)(1-2q\cos 2v)}\{1+O(q^4)\}$$

$$= 1 - 8q\sin^2 v + 32q^2 \sin^4 v + O(q^3)\,, \qquad \text{(A-5)}$$

where

$$v = u/\vartheta_3^2,$$

$$\vartheta_3^{-4} = \{1+2q+O(q^4)\}^{-4} = 1 - 8q + 40q^2 + O(q^3)\,. \qquad \text{(A-6)}$$

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41. 弾性が規則的に変化している媒質における波動方程式の解

地震研究所 音田 功

今まで速度が規則的に変わっている様な不均質媒質を伝わる波動の安定性について考察を進めて来た．その研究の続きとして今回は弾性が規則的に変わっている場合を取扱った．

以前に行なったと同様の変数変換を行なうと，独立変数として空間座標から導かれる走時はヤコビ楕円関数で与えられ，方程式にはそれらの多項式が含まれる．今回取扱っている条件の下では母数が (0, 1) であり，そして独立変数は実数に限られているので，Appendix で与えた三角関数による近似公式が用いられる．そして方程式をヒルの方程式に書き換え，前論文で得た表現によって解を与えた (§2).

形式的に全く同じ方程式は，底が規則的に起伏している水道を伝わる長波の問題で現われ，これに関して既にフーリェ級数による解が得られている．それは解の導き方においても又その取扱いにおいても簡単であるが，第2節で得た解とは全く異なった解の性質を有している．それでこれを微分方程式論の立場から吟味を加えた．この解は始めにフーリェ級数 $\sum a_n \cos(n\tau x+\delta)$ で仮定して，その係数 a_n には収束するものがあることがわかっている．上の級数は $\zeta=\exp(i\tau x)$ の変数変換に対して $\zeta=0$ のまわりのローラン級数とも解釈できる．原方程式は ζ を ζ^{-1} に変えても全く同形式をとるから，$\zeta=0$ のまわりの解と $\zeta=\infty$ のまわりの解とは同じ係数間の関係を有している．更に方程式で $\zeta=0$（従って $\zeta=\infty$ においても同様）は確定特異点であり，その指数は 0 と 1 である．従って一般解の基本系を作るべき第一の解は $\zeta=0$ のまわりのテイラー級数で与えられ，第2の解は対数項を含む．その上，$\zeta=0$ に最も近い特異点は $\zeta=\zeta_1=-(1/2\Delta)+\sqrt{(1/2\Delta)^2-1}$ であるから，この様にして求められた級数解は一つの特解であって一般解ではない．従って波動の問題に対する解としては不十分であることが示された (§3).