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44. On Two-Dimensional Elastic Dislocations in an Infinite and Semi-infinite Medium.

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Contents

Introduction

In relation to mathematical treatment of statical deformations of the earth's crust associated with an earthquake, J. A. Steketee (1958 a, b) developed and reviewed the elasticity theory of dislocations. His method of representation of a fault as a dislocation surface in a three-dimensional elastic half-space has been applied to actual faults accompanied by large earthquakes by M. A. Chinnery (1961, 1964, and 1965) and recently by F. Press (1965).

On the other hand, K. Kasahara (1957, 1958a, b, 1959, and 1964) and L. Knopoff (1958) have developed two-dimensional models for very long strike-slip faults on the consideration that the shear stress on the fault surface should vanish in the case of fault formation, with assumed initial stress field in a semi-infinite elastic body.

Recently J. Weertman (1964, 1965) has made use of the concept of continuously distributed infinitesimal dislocations on a plane to study problems of slippage on very long faults with finite friction.

Kasahara's and Knopoff's models, as well as Weertman's can be considered from the view-point of two-dimensional analogue of Steketee's representations.

It may be difficult, at least in general, to obtain the solution to given change in stress on the faults L with various dip angles in a two-dimensional half-space, but if the problem is to compute the displacement and stress field due to given two-dimensional Somigliana dislocation, that is, the displacement discontinuity arbitrarily specified on L, the solution can be easily constructed. Some advantages of the latter formulation can be pointed out from the geophysical view-point (e.g. Press 1965). Besides, once the latter problem is solved, the former can be obtained by numerical calculation with the aid of computers.

It is our main purpose in this paper to obtain and list the fundamental expressions for two-dimensional dislocations to the latter problem for geophysical applications, e.g. for the models of very long strike slip and dip slip faults with various dip angles associated with an earthquake.

Sections 2 and 3 treat of the displacement and stress fields corresponding to some nuclei of strain in a half-plane, two-dimensional analogue of the previously obtained solutions in the three-dimensional case (Maruyama 1964), while Section 4 complex variable representations which may have much merit in problems such as to obtain the stress field along an arbitrary line element, particularly on the faults L, in the form of its normal and tangential components.

Section 5.1 contains some examples of making use of complex representations in Section 4, for fundamental screw and edge dislocations in an unbounded region. In Section 5.2 we shall show some results of calculations of the strain energy changes due to shear cracks under the condition of partial stress relaxation on the crack, with discontinuity in displacement of more realistic type. Some comments on Knopoff's calculation (1958) are included in this Section.

On the reciprocity relations, which will be referred to in Section 3, concerning a force or a nucleus of strain acting at a point and the corresponding displacement or stress field at another ponit, Section 1 treats of the problem in the three-dimensional aeolotropic half-space. Such genelalization will be convenient for understanding of the 'force equivalent' problem in the static case in an aeolotropic inhomogeneous medium (cf. Burridge and Knopoff 1964) as well as for general use.

1. Reciprocity relations in a semi-infinite aeolotropic medium (three-dimensional problem)

In this section while introducing notations such as $G_m^k(P, Q)$ or $G_{kl}^m(P, Q)$, for later treatments, we will show reciprocity relations connecting some of them to one another. Let the medium be a three-dimensional elastic solid which may be aeolotropic.

Rectangular cartesian coordinates are denoted by x_i or ξ_i $(i=1, 2, 3)$. The elastic displacement vector has components u_i. The strain tensor e_{ij} and the stress tensor τ_{ij} are defined by

$$e_{ij}=e_{ji}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{1.1}$$

$$\tau_{ij}=\tau_{ji}=c_{ijkl}\,e_{kl}=c_{ijkl}\frac{\partial u_k}{\partial x_l}. \tag{1.2}$$

In this section the summation convention applies over the whole range (1, 2, 3). The elastic modulus c_{ijkl} (which may be a function of position) is unchanged by interchanging i with j or k with l or the pair (ij) with the pair (kl). In an isotropic medium with Lamé constants λ and μ (which may be also a function of position) we have

$$\tau_{ij}=\lambda\,\delta_{ij}\,e_{kk}+2\mu\,e_{ij}. \tag{1.2$'$}$$

The equations of equilibrium are

$$\frac{\partial\tau_{ij}}{\partial x_j}+\rho f_i=0, \tag{1.3}$$

where $\rho(x_1, x_2, x_3)$ is the mass density of the material and $f_i(x_1, x_2, x_3)$ the body force per unit mass. Elastic potential or strain energy per unit volume is given by

$$w=\frac{1}{2}\tau_{ij}\,e_{ij}. \tag{1.4}$$

For either an internal surface element or boundary surface element of the body with surface normal ν_l, if $\overset{\nu}{T}_k$ denotes the force per unit area, exerted by the positive side of the normal upon the negative side of the normal, it is related with the stress tensor τ_{kl} by the formula

$$\overset{\nu}{T}_k=\tau_{kl}\,\nu_l\,. \tag{1.5}$$

We shall make use of particular solutions of the equations of equilibrium of elastic body which tend to become infinite in the neighborhood of chosen points. The solution having the point $Q(x_1, x_2, x_3)$ as a simple singular point is the one due to the force acting at Q on the body. It may be obtained by considering the case where the body forces per unit mass f_k are different from zero within a finite volume V and vanish outside V and then by passing to a limit by diminishing all the linear dimensions of V indefinitely, but supposing that $\iiint \rho f_k\, dx_1, dx_2\, dx_3$ has a finite limit (Love § 130). In this way a force acting at Q, of magnitude F, in the direction of the axis of x_m is found to be such a limiting case on condition that

$$\iiint \rho f_k\, dx_1\, dx_2\, dx_3=\begin{cases} F & \text{for} \quad k=m \\ 0 & \text{for} \quad k\neq m\,. \end{cases} \tag{1.6}$$

We introduce the notation $G_m^k(P, Q)$ so that the displacement component in the direction of the axis x_k at $P(\xi_1, \xi_2, \xi_3)$ caused by a force, of magnitude F, acting at the point $Q(x_1, x_2, x_3)$ in the direction of the axis of x_m will be expressed by $F\times G_m^k(P, Q)$. In the notation like $G_m^k(P, Q)$, the first letter in the parentheses, P in this case, is used to denote the field point and the second letter, Q in this case, the point at which a force or a force system acts on the body; the superscript, k in this case, is used to denote the component of the quantity and the subscript, m in this case, the component of the force or force system.

If we apply the reciprocal theorem to two sets of body forces and the corresponding displacement fields in a semi-infinite elastic body: a force acting at Q in the direction of x_m-axis and the corresponding displacement field and the force of the same magnitude acting at P in the direction of x_k-axis and the corresponding displacement field, then it is easily verified that

$$G_m^k(P, Q)=G_k^m(Q, P)\,. \tag{1.7}$$

The (kl)-component of the stress field at P, which is denoted by $G_m^{kl}(P, Q)$, is derived from the displacement field $G_m^k(P, Q)$ as

$$G_m^{kl}(P, Q) = G_m^{lk}(P, Q)$$
$$= c_{klrs}(P) \frac{\partial}{\partial \xi_s} G_m^r(P, Q), \tag{1.8}$$

in an isotropic medium

$$= \lambda(P)\delta_{kl} \frac{\partial}{\partial \xi_h} G_m^h(P, Q) + \mu(P)\left\{\frac{\partial}{\partial \xi_k} G_m^l(P, Q) + \frac{\partial}{\partial \xi_l} G_m^k(P, Q)\right\}. \tag{1.8$'$}$$

Consider next the field due to combinations of double forces or strain nuclei. We define the notation $G_{kl}^m(Q, P)$ as follows:

$$G_{kl}^m(Q, P) = G_{lk}^m(Q, P)$$
$$= c_{klrs}(P) \frac{\partial}{\partial \xi_s} G_r^m(Q, P), \tag{1.9}$$

in an isotropic medium

$$= \lambda(P)\delta_{kl} \frac{\partial}{\partial \xi_h} G_h^m(Q, P) + \mu(P)\left\{\frac{\partial}{\partial \xi_k} G_l^m(Q, P) + \frac{\partial}{\partial \xi_l} G_k^m(Q, P)\right\}. \tag{1.9$'$}$$

Now, for example, $F \times \frac{\partial}{\partial \xi_3} G_2^1(Q, P)$ may be considered as the limit value of the superposition of x_1-component of displacement at Q caused by a force $F/\Delta\xi_3$ acting at P' $(\xi_1, \xi_2, \xi_3 + \Delta\xi_3)$ in the direction of x_2 and that caused by the force of the same magnitude at P (ξ_1, ξ_2, ξ_3) in the opposite direction, that is, as the displacement in the direction of x_1-axis at Q caused by a double force at P. Therefore $G_{kl}^m(Q, P)$ defined in equation (1.9) may be considered up to a dimensional constant as the displacement at Q in the x_m-direction caused by a combination, specified by combination (kl), of double forces or strain nuclei which will be referred to simply as a force system (kl). Thus, since the displacement field $G_{kl}^m(Q, P)$ at Q is a limit value of linear combination of displacements satisfying the boundary conditions on the free surface and at infinity, it also satisfies the boundary conditions. The force system (kl) in an isotropic medium can be shown schematically as represented in a figure in Maruyama (1964).

Comparison of equation (1.8) with equation (1.9) and use of equation (1.7) give

$$G_m^{kl}(P, Q) = G_{kl}^m(Q, P). \tag{1.10}$$

The (mn)-component of the stress at Q caused by a force system (kl) located at P is expressed by definition of stress as

$$G_{kl}^{mn}(Q, P)=G_{lk}^{mn}(Q, P)=G_{kl}^{nm}(Q, P)=G_{lk}^{nm}(Q, P)$$
$$=c_{mnrs}(Q)\frac{\partial}{\partial x_s}G_{kl}^{r}(Q, P)\,, \tag{1.11}$$

in an isotropic medium

$$=\lambda(Q)\,\delta_{mn}\frac{\partial}{\partial x_h}G_{kl}^{h}(Q, P)+\mu(Q)\left\{\frac{\partial}{\partial x_m}G_{kl}^{n}(Q, P)+\frac{\partial}{\partial x_n}G_{kl}^{m}(Q, P)\right\}. \tag{1.11$'$}$$

Substitution of equation (1.9) into equation (1.11), use of relation (1.7) and interchange of order of differentiation yield

$$G_{mn}^{kl}(P, Q)=G_{kl}^{mn}(Q, P)\,, \tag{1.12}$$

with the help of expressions to be obtained from (1.9) and (1.11) by interchanging the role of P with that of Q.

2. Two-dimensional relations in an isotropic medium

From this section forward it will be assumed that all quantities depend only on the two coordinates x_2 and x_3, and are independent of x_1, that is, $(\partial/\partial x_1)=0$. Then, instead of considering the entire region occupied by the body, we may restrict the investigation to one of its sections in a plane parallel to (x_2x_3)-plane. Further, it will be assumed that the body is isotropic with Lamé constants λ and μ, which may be functions of coordinates x_2 and x_3 when the body is inhomogeneous.

By these assumptions we find that the system of equations of equilibrium and stress-strain relations (1.1), (1.2)$'$, and (1.3) can be divided into two mutually independent Systems I and II, since u_1 is related only with τ_{12}, τ_{13} and f_1, conversely τ_{12}, τ_{13} and f_1 only with u_1. System I can be expressed by

$$\frac{\partial\tau_{12}}{\partial x_2}+\frac{\partial\tau_{13}}{\partial x_3}+\rho f_1=0 \tag{2.1}$$

and

$$\begin{cases}\tau_{12}=\mu\dfrac{\partial u_1}{\partial x_2}\\[2mm] \tau_{13}=\mu\dfrac{\partial u_1}{\partial x_3}\,,\end{cases} \tag{2.2}$$

while System II by

$$\begin{cases} \dfrac{\partial \tau_{22}}{\partial x_2}+\dfrac{\partial \tau_{23}}{\partial x_3}+\rho f_2=0 \\ \dfrac{\partial \tau_{23}}{\partial x_2}+\dfrac{\partial \tau_{33}}{\partial x_3}+\rho f_3=0 \end{cases} \tag{2.3}$$

and

$$\begin{cases} \tau_{22}=\lambda\Delta+2\mu\dfrac{\partial u_2}{\partial x_2} \\ \tau_{23}=\mu\left(\dfrac{\partial u_3}{\partial x_2}+\dfrac{\partial u_2}{\partial x_3}\right) \\ \tau_{23}=\lambda\Delta+2\mu\dfrac{\partial u_3}{\partial x_3} \end{cases} \tag{2.4}$$

where

$$\Delta=\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\,.$$

The latter is the system of equations in the case of plane strain. In what follows indices such as k and l represent the values only from (2.3). Since considerations are limited to points of the $(x_2\,x_3)$-plane which is assumed to be the plane of one of the normal sections of the body or the cylinder, when talking of a region occupied by the body, we will have in mind a two-dimensional region in the $(x_2\,x_3)$-plane. Surface tractions $\overset{\nu}{T}_1\,d\sigma$ and $\overset{\nu}{T}_k\,d\sigma$, which are the components in x_1- and x_k-directions respectively, of the force acting on a rectangular area perpendicular to the $(x_2\,x_3)$-plane with base $d\sigma$ and unit height, will be considered as the tractions acting on the line element in the $(x_2\,x_3)$-plane, where $\overset{\nu}{T}_1$ and $\overset{\nu}{T}_k$ may be expressed by

$$\overset{\nu}{T}_1=\tau_{1l}\,\nu_l \tag{2.5}$$

$$\overset{\nu}{T}_k=\tau_{kl}\,\nu_l \tag{2.6}$$

instead of by equation (1.5).

In a similar manner to what was taken in the three-dimensional problem (cf. Steketee 1958b, Maruyama 1964), we begin with Betti's reciprocal theorem. By means of divergence theorem in two-dimensional cases we can verify the following relations:

$$\iint_S u_1^{(1)} f_1^{(2)} \rho dS + \int_{L_0} u_1^{(1)} \overset{\nu}{T}{}_1^{(2)} d\sigma = \iint_S u_1^{(2)} f_1^{(1)} \rho dS + \int_{L_0} u_1^{(2)} \overset{\nu}{T}{}_1^{(1)} d\sigma \tag{2.7}$$

and

$$\iint_S u_k^{(1)} f_k^{(2)} \rho dS + \int_{L_0} u_k^{(1)} \overset{\nu}{T}{}_k^{(2)} d\sigma = \iint_S u_k^{(2)} f_k^{(1)} \rho dS + \int_{L_0} u_k^{(2)} \overset{\nu}{T}{}_k^{(1)} d\sigma \tag{2.8}$$

which hold for every pair of scalar fields $u_1^{(1)}$ and $u_1^{(2)}$, vector fields $u_k^{(1)}$ and $u_k^{(2)}$ which satisfy System I and System II respectively and which are single-valued and continuously differentiable for twice in the closed region $S+L_0$ with L_0 as its boundary. In the integrals over L_0 in equations (2.7) and (2.8), $\overset{\nu}{T}_1$ and $\overset{\nu}{T}_k$ are the tractions at the point of the boundary L_0, the superscript ν refers to the normal to L_0 which points outward from S. When μ=constant, equation (2.7) is identical with ordinary Green's theorem concerning $u_1^{(1)}$ and $u_1^{(2)}$.

In either system we can obtain the field due to a force acting at a point or fundamental singularity. It is necessary to be defined in a two-dimensional manner instead of by equation (1.6). We obtain the field caused by a force, of magnitude F per unit length, acting at $Q(x_2, x_3)$ in the direction of axis x_1, by passing to a limit, supposing that

$$\iint \rho f_1 \, dx_2 \, dx_3 = F, \tag{2.9}$$

for System I and that caused by a force F per unit length acting in the direction of axis x_m, by passing to a limit, supposing that

$$\iint \rho f_k \, dx_2 \, dx_3 = \begin{cases} F & \text{for} \quad k=m \\ 0 & \text{for} \quad k \neq m, \end{cases} \tag{2.10}$$

for System II.

In the definition of body force F per unit length acting at a point Q, instead of considering integrals taken over a small area including the point and passing to a limit, as in equations (2.9), (2.10), the resultant of the tractions at the boundary curve σ_ε of the area, which we can choose to be a circle of arbitrary small radius, may be taken so that

$$\int_{\sigma_\varepsilon} \tau_{1l} \nu_l \, d\sigma = F, \tag{2.9$'$}$$

$$\int_{\sigma_\varepsilon} \tau_{kl} \nu_l \, d\sigma = \delta_{mk} F, \tag{2.10$'$}$$

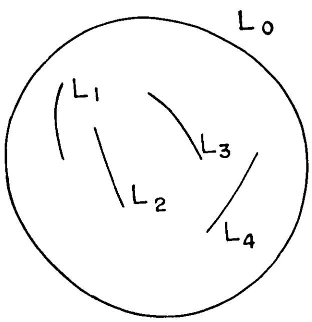


Fig. 1. The outer boundary L_0 and interior boundaries L_1, L_2, ..., L_n.

where the direction ν_l must be to the exterior of the domain excluding the interior of the circle, that is, toward the point Q.

Now we treat of the problem on the displacement field containing discontinuities in displacements or tractions across some arcs or interior boundaries in a region in the $(x_2 x_3)$-plane. In order to apply the reciprocal theorem to the case, we may well proceed by the following technique as suggested by Boley and Weiner (§ 2.7).

We consider a body, with no body forces, occupying a region $S+L_0$ with interior boundaries L across which the displacements and tractions are required to have specified finite discontinuities, $\Delta u_1(P)$ and $\Delta \overset{\nu}{T}_1(P)$ for System I, $\Delta u_k(P)$ and $\Delta \overset{\nu}{T}_k(P)$ for System II, where P is any point on the interior boundaries. That is, it is required that

$$\begin{cases} \lim\limits_{\substack{Q\to P\\ +}} u_1(Q) - \lim\limits_{\substack{Q\to P\\ -}} u_1(Q) = \Delta u_1(P) \\ \lim\limits_{\substack{Q\to P\\ +}} \overset{\nu}{T}_1(Q) - \lim\limits_{\substack{Q\to P\\ -}} \overset{\nu}{T}_1(Q) = \Delta \overset{\nu}{T}_1(P) \end{cases} \tag{2.11}$$

and

$$\begin{cases} \lim\limits_{\substack{Q\to P\\ +}} u_k(Q) - \lim\limits_{\substack{Q\to P\\ -}} u_k(Q) = \Delta u_k(P) \\ \lim\limits_{\substack{Q\to P\\ +}} \overset{\nu}{T}_k(Q) - \lim\limits_{\substack{Q\to P\\ -}} \overset{\nu}{T}_k(Q) = \Delta \overset{\nu}{T}_k(P) \end{cases} \tag{2.12}$$

where the $+$ and $-$ signs denote the limits obtained by approaching P from opposite sides of L. Throughout the remainder of $S+L_0$, the displacements and tractions are required to satisfy the hypotheses of the reciprocal theorem in the case when no interior boundary exists.

First we take a region $S+L_0$ with only an interior boundary L across which the displacement and the traction have specified discontinuities of $\Delta u_1(P)$ and $\Delta \overset{\nu}{T}_1(P)$ as in equation (2.11) for System I.

If the specified boundary L does not cut the region $S+L_0$ into separate parts, it is convenient to extend it so that it does; $\Delta u_1(P)$ and $\Delta \overset{\nu}{T}_1(P)$ are then also extended to be zero on the portion added to L, so that displacements and tractions are continuous across the added portion. Let S_a and S_b be the two portions of S, L_a and L_b their respective boundaries exclusive of the common boundary L (Fig. 2).

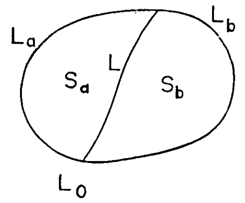


Fig. 2. A region $S+L_0$ with only an interior boundary L.

Assuming the possibility of existence of at most one solution of this problem and choosing another solution $u_1^*(P)$ and $\overset{\nu}{T}{}_1^*(P)$ such that it has a singularity at Q corresponding to a force acting at Q, in the direction of x_1-axis, of magnitude F per unit length, and has no discontinuity in displacements and tractions across L, we apply the reciprocal theorem to these two sets of solutions in the closed region S_a+L_a+L and in the closed region S_b+L_b+L. If the values of the quantities on L in each case are taken as their limits as a point on L is approached from the interior of each respective region, all the requirements of the reciprocity theorem in the case when no interior boundary exists are satisfied. If Q is in S_a we have for S_a+L_a+L

$$\iint_{S_a} u_1 f_1^* \rho\, dS + \int_{L+L_a} u_1 \overset{\nu}{T}{}_1^* d\sigma = \int_{L+L_a} u_1^* \overset{\nu}{T}_1\, d\sigma\,, \tag{2.13}$$

and for S_b+L_b+L

$$\int_{L+L_b} u_1 \overset{\nu}{T}{}_1^* d\sigma = \int_{L+L_b} u_1^* \overset{\nu}{T}_1\, d\sigma\,. \tag{2.14}$$

In the integral over L in equation (2.13), ν_l is the normal to L which points outward from S_a and in the integral over L in equation (2.14), ν_l is the normal to L which points outward from S_b. If the orientation outward from S_b is used in the integral of equation (2.13) as well, and the $+$ and $-$ signs in equation (2.11) are taken as corresponding to the limits from the interior of S_a and S_b respectively, equations (2.13) and (2.14) may be combined as follows:

$$\begin{aligned} F\times u_1(Q) = & \int_L \varDelta u_1 \overset{\nu}{T}{}_1^* d\sigma - \int_L u_1^* \varDelta \overset{\nu}{T}_1 d\sigma \\ & - \int_{L_0} u_1 \overset{\nu}{T}{}_1^* d\sigma + \int_{L_0} u_1^* \overset{\nu}{T}_1 d\sigma\,. \end{aligned} \tag{2.15}$$

The left-hand side can be obtained from the first term on the left-hand side of equation (2.13) by taking first the force acting at Q as body forces distributed in a finite region around the point Q and then by passing to a limit under the condition (2.9). As in all applications of Green's theorem, excluding the point Q from the region and replacing the first term on the left side of (2.13) by a line integral over a small circle surrounding Q, we may arrive at the same result by using the definition (2.9)′.

If Q is in S_b the same formula in equation (2.15) can be obtained. Since the integral over the extended portion of L in equation (2.15) does not affect the result, the tractions as well as displacement fields in S_a and in S_b should be continuous across the extended portion of the boundary.

If the interior boundaries consist of more than one disconnected part $L_1, L_2, \cdots, L_n$, we arrive at the same result as in equation (2.15) by extending the boundaries appropriately so as to cut the region $S+L_0$ into separate parts each of which has no interior boundary, and by applying the reciprocal theorem to each closed region of them. Therefore, we may take the interior boundary L over which integrals are taken in equation (2.15) as the union of all the interior boundaries $L_1, L_2, \cdots, L_n$.

In a completely analogous manner to the above we obtain the formula for System II:

$$F\times u_m(Q)=\int_L \Delta u_k \overset{\nu}{T}{}^*_k d\sigma-\int_L u^*_k \Delta\overset{\nu}{T}_k d\sigma$$

$$-\int_{L_0} u_k \overset{\nu}{T}{}^*_k d\sigma+\int_{L_0} u^*_k \overset{\nu}{T}_k d\sigma\,, \tag{2.16}$$

where u^*_k and $\overset{\nu}{T}{}^*_k$ is a solution which is continuous throughout the region except at Q where it has a singularity corresponding to a body force defined as in equation (2.10) or (2.10)′, while $u_k(P)$ and $\overset{\nu}{T}_k(P)$ is any solution which has discontinuity in displacement $\Delta u_k(P)$ and that in traction $\Delta\overset{\nu}{T}_k(P)$ specified by (2.12) across the interior boundaries $L_1, L_2, \cdots, L_n$, or across L, when the body is free of body forces throughout the region $S+L_0$.

These general relations (2.15) and (2.16) will be used later for explicit expressions of displacement or stress field.

As for the discontinuity in the traction across L, if we consider tensile cracks opened by uniform pressure inside the crack, it should be taken as zero. If it is the case of shear cracks of which two sides are deformed in contact with each other, $\Delta\overset{\nu}{T}_1$ or $\Delta\overset{\nu}{T}_k$ must be zero, owing to the law of equal and opposite action and reaction, since we make no distinction between the coordinates before and after deformation, so far as we are remaining in the classical or infinitesimal theory of elasticity as in this paper.

For ordinary dislocations, or displacement dislocations, it is assumed that the tractions are continuous across L, that is, $\Delta\overset{\nu}{T}_1=\Delta\overset{\nu}{T}_k=0$.

Next we take up the strain energy of a body occupying a region $S+L_0$ with outer boundary L_0 and with interior boundary L across which only displacement discontinuity is given.

From the restriction of our problem $\partial/\partial x_1=0$, the elastic potential or strain energy per unit volume w in equation (1.4) can be divided into two parts $w(I)$ and $w(II)$ which correspond to System I and System II respectively:

$$w=w(I)+w(II)\,. \tag{2.17}$$

From equations (1.1) and (1.4) we have

$$w(I)=\frac{1}{2}\tau_{1l}\frac{\partial u_1}{\partial x_l}\,, \tag{2.18}$$

$$w(II)=\frac{1}{2}\tau_{kl}\frac{\partial u_k}{\partial x_l}\,. \tag{2.19}$$

First consider the case for System I. Assuming that the body is free of body forces and that u_1 and τ_{1k} are continuous with those first derivatives in a closed region $S+L_0$, the total strain energy per unit length of the body $W(I)$, which may be considered as the total strain energy in a plane parallel to the (x_2x_3)-plane, may be written

$$W(I)=\iint_S w\,dx_2dx_3=\frac{1}{2}\iint_S\frac{\partial}{\partial x_l}(u_1\tau_{1l})dx_2\,dx_3=\frac{1}{2}\int_{L_0}u_1\tau_{1l}\nu_l d\sigma\,, \tag{2.20}$$

where equilibrium equation (2.1) and the divergence theorem are used. Applying equation (2.20) to two closed regions S_a+L_a+L and S_b+L_b+L with common boundary L across which discontinuity in displacement Δu_1 is given and proceeding in a similar way to that in the case of obtaining equation (2.15) we find the strain energy per unit length of the body as

$$W(I)=\frac{1}{2}\int_{L_0}u_1\overset{\nu}{T}_1 d\sigma-\frac{1}{2}\int_L \Delta u_1\overset{\nu}{T}_1 d\sigma\,. \tag{2.21}$$

The minus sign is correct since the normal ν is chosen to point outward from S_b, in other words, from $(-)$ side to $(+)$ side, as in equation (2.15).

Similarly we have for System II

$$W(II)=\frac{1}{2}\int_{L_0} u_k \overset{\nu}{T}_k d\sigma-\frac{1}{2}\int_{L} \Delta u_k \overset{\nu}{T}_k d\sigma \,. \tag{2.22}$$

3. Two-dimensional relations in a semi-infinite isotropic medium

In this section let the elastic body be semi-infinite and isotropic. For the time being it may be inhomogeneous, that is, density ρ and Lamé constants λ, μ may be functions of position. Rectangular cartesian coordinate axes are taken so that the x_3-axis penetrates the body from the free surface $x_3=0$ (Fig. 3). It is assumed that all quantities are independent of the coordinate x_1, that is $\partial/\partial x_1=0$, as in Section 2.

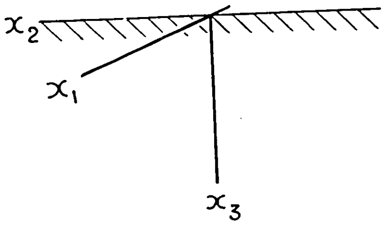


Fig. 3. Rectangular cartesian coordinate axes taken so that x_3-axis penetrates the half-plane with x_2-axis as its free boundary.

For convience' sake we use the same notations $G_m^k(P, Q)$, $G_m^{kl}(P, Q)$ etc. in Section 1 also in two-dimensional cases of this Section. Here P and Q denotes with coordinates (ξ_2, ξ_3) and (x_2, x_3) respectively in the (x_2x_3)-plane. Superscripts and subscripts such as k, l, and m are taken from (2, 3).

On the basis of reciprocal theorem of equations (2.7) and (2.8) we have

$$\begin{cases} G_1^1(P, Q)=G_1^1(Q, P)\,, \\ G_m^k(P, Q)=G_k^m(Q, P)\,, \end{cases} \tag{3.1}$$

corresponding to equation (1.7).

Stress fields at P due to point forces at Q are given by

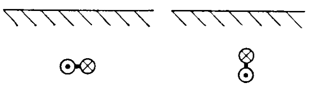
$$\begin{cases} G_1^{1l}(P, Q)=\mu(P)\dfrac{\partial}{\partial \xi_l}G_1^1(P, Q)\,, \\ G_m^{kl}(P, Q)=\lambda(P)\delta_{kl}\dfrac{\partial}{\partial \xi_h}G_m^h(P, Q)+\mu(P)\left\{\dfrac{\partial}{\partial \xi_k}G_m^l(P, Q)+\dfrac{\partial}{\partial \xi_l}G_m^k(P,Q)\right\} \\ \qquad\qquad =G_m^{lk}(P, Q)\,, \end{cases} \tag{3.2}$$

corresponding to equation (1.8).

Displacement fields at Q due to strain nuclei at P which are denoted by $G^{1}_{1l}(Q, P)$ and $G^{m}_{kl}(Q, P)$ may be defined as

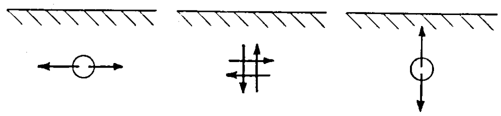
$$\begin{cases} G^{1}_{1l}(Q, P)=\mu(P)\dfrac{\partial}{\partial\xi_l}G^{1}_{1}(Q, P)\,, \\ G^{m}_{kl}(Q, P)=\lambda(P)\delta_{kl}\dfrac{\partial}{\partial\xi_h}G^{m}_{h}(Q, P)+\mu(P)\left\{\dfrac{\partial}{\partial\xi_k}G^{m}_{l}(Q, P)+\dfrac{\partial}{\partial\xi_l}G^{m}_{k}(Q, P)\right\} \\ \qquad\qquad =G^{m}_{lk}(Q, P)\,, \end{cases} \tag{3.3}$$

corresponding to equation (1.9). The strain nuclei at P which produce the displacement fields $G^{1}_{12}(Q, P)$ and $G^{1}_{13}(Q, P)$ at Q may be schematically represented in Fig. 4, and those which produce $G^{m}_{22}(Q, P)$, $G^{m}_{23}(Q, P)$, and $G^{m}_{33}(Q, P)$ in Fig. 5, by the definition (3.3). In Fig. 4, the sign $\odot$ indicates a force acting at the point in the direction of x_1-axis, perpendicular to the paper toward the front side of it, the sign $\otimes$ in the opposite direction, that is, toward the reverse side of paper.



(12) (13)

Fig. 4. Force systems corresponding to G_{1l}.



(22) (23) (33)

Fig. 5. Force systems corresponding to G_{kl}.

The schematical representation of strain nuclei in Fig. 5 is similar to that in the three-dimensional case (e.g. Steketee 1958b, Maruyama 1964), but the representation in Fig. 4 is not.

From equations (3.1), (3.2) and (3.3) we have

$$\begin{cases} G^{1l}_{1}(P, Q)=G^{1}_{1l}(Q, P) \\ G^{kl}_{m}(P, Q)=G^{m}_{kl}(Q, P) \end{cases} \tag{3.4}$$

corresponding to equation (1.10). The $(1n)$-component and the (mn)-component of the stress at Q caused by the force-system $(1l)$ and (kl) respectively located at P may be expressed as

$$\begin{cases} G^{1n}_{1l}(Q, P)=\mu(Q)\dfrac{\partial}{\partial x_n}G^{1}_{1l}(Q, P)\,, \\ G^{mn}_{kl}(Q,P)=\lambda(Q)\delta_{mn}\dfrac{\partial}{\partial x_h}G^{h}_{kl}(Q,P)+\mu(Q)\left\{\dfrac{\partial}{\partial x_m}G^{n}_{kl}(Q, P)+\dfrac{\partial}{\partial x_n}G^{m}_{kl}(Q, P)\right\} \\ \qquad\qquad =G^{mn}_{lk}(Q, P)=G^{nm}_{kl}(Q, P)=G^{nm}_{lk}(Q, P)\,, \end{cases} \tag{3.5}$$

corresponding to equation (1.11). In a similar way to what was taken in the derivation of equation (1.12) we find

$$\begin{cases} G^{1l}_{1n}(P,\ Q)=G^{1n}_{1l}(Q,\ P) \\ G^{kl}_{mn}(P,\ Q)=G^{mn}_{kl}(Q,\ P)\,. \end{cases} \tag{3.6}$$

Now we use equations (2.15) and (2.16) to obtain the field due to dislocations in a semi-infinite isotropic medium. We take in the $(x_2\,x_3)$-plane a region of which outer boundary L_0 consists of the line of free boundary and a semi-circle of radius R_0 which is distant from all the interior boundaries L across which discontinuity in displacement or in stress is specified. As for u^*_1 and $\overset{\nu}{T}{}^*_1$ in equation (2.15); or u^*_k and $\overset{\nu}{T}{}^*_k$ in equation (2.16) we take the field due to a force F per unit length acting at a point Q in the semi-infinite medium.

Assuming that

$$\begin{cases} u_1=O\left(\dfrac{1}{R}\right) \\ u_m=O\left(\dfrac{1}{R}\right) \end{cases}$$

for large R, contributions from the third and fourth terms on the left-hand side of equations (2.15) and (2.16) taken over L_0 tend to zero as R_0 tends to infinity, integrals over the boundary of free surface being zero, we obtain the displacements at Q expressed as

$$\begin{cases} u_1(Q)=\displaystyle\int_L \varDelta u_1(P)\,G^{1l}_1(P,\ Q)\nu_l\,d\sigma-\int_L \varDelta\tau_{1l}(P)G^1_1(P,\ Q)\nu_l d\sigma \\ u_m(Q)=\displaystyle\int_L \varDelta u_k(P)G^{kl}_m(P,\ Q)\nu_l\,d\sigma-\int_L \varDelta\tau_{kl}(P)G^k_m(P,\ Q)\nu_l\,d\sigma\,. \end{cases}$$

These equations may be written, by use of relations (3.1) and (3.4), as follows:

$$\begin{cases} u_1(Q)=\displaystyle\int_L \varDelta u_1(P)G^1_{1l}(Q,\ P)\nu_l d\sigma-\int_L \varDelta\tau_{1l}(P)G^1_1(Q,\ P)\,\nu_l\,d\sigma \\ u_m(Q)=\displaystyle\int_L \varDelta u_k(P)G^m_{kl}(Q,\ P)\nu_l\,d\sigma-\int_L \varDelta\tau_{kl}(P)G^m_k(Q,\ P)\nu_l\,d\sigma\,. \end{cases} \tag{3.7}$$

We observe in equation (3.7) that the discontinuity in displacement

Δu_1 or Δu_k across the line L may be interpreted as distributions along L of a certain type of force system with magnitude corresponding to Δu_1 or Δu_k, while the discontinuity in tractions may be done as distributions along L of body force with magnitude corresponding to $\Delta \overset{\nu}{T}_1$ or $\Delta \overset{\nu}{T}_k$. Further we observe in equation (3.7) that force systems equivalent to discontinuities in displacement Δu_1 perpendicular to the (x_2x_3)-plane are represented as shown in Fig. 4, while those to discontinuities in displacement u_2 and u_3 parallel to the plane are represented as shown in Fig. 5.

It is seen from our derivation that the essential role in the 'force equivalent' problem is played by the relation (3.1) and by the relation (3.4) which is derived from the former. The corresponding relations (1.7) and (1.10) in the three-dimensional aeolotropic medium will clearly play the same role in the three-dimensional corresponding problems. This knowledge will also serve to symplify the computations given in the previous work on the three-dimensional problem in a semi-infinite isotropic body (Maruyama 1964), as in the method of F. Press (1965) who took advantage of the solution by Mindlin and Cheng (1950).

The stress field at the point Q due to dislocations in the semi-infinite medium is given by differentiation with respect to the coordinates of Q and multiplication by elastic constants. If the point Q is not on L, differentiation can be performed in general under the integration sign in equation (3.7), then the stress field may be obtained by means of introduced notations $G_{1l}^{1n}(Q, P)$, $G_{kl}^{mn}(Q, P)$ etc. as

$$\begin{cases} \tau_{1n}(Q)=\int_L \Delta u_1(P)\, G_{1l}^{1n}(Q, P)\nu_l\, d\sigma - \int_L \Delta\tau_{1l}(P) G_1^{1n}(Q, P)\nu_l\, d\sigma \\ \tau_{mn}(Q)=\int_L \Delta u_k(P) G_{kl}^{mn}(Q, P)\nu_l\, d\sigma - \int_L \Delta\tau_{kl}(P) G_k^{mn}(Q, P)\nu_l\, d\sigma\,. \end{cases} \tag{3.8}$$

Let us now compute the explicit expressions of $G_1^1(Q, P)$, $G_k^m(Q, P)$ etc. for isotropic and homogeneous medium, that is, in the case when $\rho, \lambda, \mu=$ const.

Substitution from equation (2.2) into equation (2.1) leads to

$$\nabla^2 u_1 + \frac{1}{\mu}\rho f_1 = 0\,, \tag{3.9}$$

where $\nabla^2 = \partial^2/\partial x_2 + \partial^2/\partial x_3^2$.

If it is the case when the body force of magnitude F per unit length defined as in equation (2.9) is acting at $P(\xi_2, \xi_3)$ in an infinite medium, the solution to equation (3.9) may be given by the well-known logarithmic potential

$$u_1(Q) = -\frac{F}{2\pi\mu}\log R\,, \tag{3.10}$$

where $R = \overline{PQ} = \sqrt{(x_2-\xi_2)^2+(x_3-\xi_3)^2}$. In fact the solution (3.10) satisfies the Laplace equation $\nabla^2 u_1 = 0$, except at $P(\xi_2, \xi_3)$, and the equation corresponding to (2.9)′, concerning the line integral surrounding P.

Superposing the field due to an image source at $P'(\xi_2, -\xi_3)$ of the same magnitude with the same sign to the field (3.10), we find that the resultant

$$u_1(Q) = -\frac{F}{2\pi\mu}(\log R + \log S)\,, \tag{3.11}$$

where $S = \overline{P'Q} = \sqrt{(x_2-\xi_2)^2+(x_3+\xi_3)^2}$, satisfies the condition of free surface on the line $x_3 = 0$, hence we can obtain the explicit expression of $G_1^1(Q, P)$.

For the field for System II due to a force acting at a point in an infinite medium we have the solution in the textbook of Love (§ 148). The displacement at $Q(x_2, x_3)$ due to a force F per unit length acting at the origin in the direction of x_2-axis is given by

$$\begin{cases} u_2 = \dfrac{F}{4\pi\mu}\left\{-(2-\alpha)\log R - \alpha\dfrac{x_3^2}{R^2}\right\} \\ u_3 = \dfrac{F}{4\pi\mu}\left\{\alpha\dfrac{x_2\,x_3}{R^2}\right\} \end{cases} \tag{3.12}$$

where $R = \sqrt{x_2^2+x_3^2}$ and $\alpha = (\lambda+\mu)/(\lambda+2\mu)$. The stress components τ_{22}, τ_{23}, τ_{33} are given by the equations

$$\begin{cases} \tau_{22} = \dfrac{F}{2\pi}\left\{-(1+\alpha)\,\dfrac{x_2}{R^2} + 2\alpha\dfrac{x_2 x_3^2}{R^4}\right\} \\ \tau_{23} = \dfrac{F}{2\pi}\left\{-(1-\alpha)\,\dfrac{x_3}{R^2} - 2\alpha\dfrac{x_2^2 x_3}{R^4}\right\} \\ \tau_{33} = \dfrac{F}{2\pi}\left\{(1-\alpha)\,\dfrac{x_2}{R^2} - 2\alpha\dfrac{x_2 x_3^2}{R^4}\right\}. \end{cases} \tag{3.13}$$

We can show that equation (2.10)′ in the case when $m = 2$ is satisfied by these equations.

In order to obtain the corresponding solution in a semi-infinite medium, we may proceed, for instance, as follows. If we replace x_2 by $(x_2-\xi_2)$ and x_3 by $(x_3-\xi_3)$ in equations (3.12) and (3.13), we have the field due to the force acting at (ξ_2, ξ_3) in the infinite medium. Since τ_{23} is odd in x_3 and τ_{33} is even in x_3 in equation (3.13), it is seen that the superposition of two fields, the one due to the force acting at (ξ_2, ξ_3) and the one due to the force at $(\xi_2, -\xi_3)$ of the same sign, makes a field with zero shear stress and doubled normal stress on the line $x_3=0$, with no singularity added in the half-plane $x_3 \geqq 0$. Therefore we can construct the solution in the semi-infinite medium $x_3 \geqq 0$ by adding three fields: the one in the infinite medium, the one due to the force of the same sign at the image point with respect to the line $x_3=0$ and the solution of the boundary problem where the boundary $x_3=0$ is subjected to normal loads, that is, two-dimensional analogue of the solution of the problem of Boussinesq, in which the load is taken as twice the negative of the normal stress on the line $x_3=0$ of the original field.

In the same way we can obtain the solution in the case when the force is acting in the direction of x_3-axis in the semi-infinite medium by starting from the solution in the infinite case which may be obtained by interchanging 2 with 3 in equations (3.12) and (3.13). However, the image source to be added must then be taken as of opposite sign in order to make shear stress zero on the boundary $x_3=0$.

Substitution from equation (2.4) into equation (2.3) leads to

$$(\lambda+\mu)\frac{\partial}{\partial x_k}\left(\frac{\partial u_h}{\partial x_h}\right)+\mu\nabla^2 u_k+\rho f_k=0 . \tag{3.14}$$

In a similar way to what was taken in the three-dimensional problem (Steketee 1958, Maruyama 1964), we may use two-dimensional Galerkin vectors. When no body force is present in equation (3.14), the displacement field may be given from a Galerkin vector $\Gamma=(\Gamma_2, \Gamma_3)$ as

$$u_k=\nabla^2\Gamma_k-\alpha\frac{\partial}{\partial x_k}\left(\frac{\partial \Gamma_h}{\partial x_h}\right). \tag{3.15}$$

Substitution from equation (3.15) into equation (3.14) with $f_k=0$ yields

$$\nabla^2\nabla^2\Gamma_k=0 , \tag{3.16}$$

which shows that each component of the Galerkin vector is biharmonic.

Assuming that the solution of two-dimensional analogue of Boussinesq problem may be obtained from a Galerkin vector $\Gamma=(0, \Gamma)$ which is perpendicular to the free boundary, as in the case of the three-dimensional one, we have from equation (3.15)

$$\left\{\begin{aligned} u_2 &= -\alpha\frac{\partial^2}{\partial x_2 \partial x_3}\Gamma \\ u_3 &= \left(\nabla^2-\alpha\frac{\partial^2}{\partial x_3^2}\right)\Gamma \\ \tau_{23} &= \mu\frac{\partial}{\partial x_2}\left(\nabla^2-2\alpha\frac{\partial^2}{\partial x_3^2}\right)\Gamma \\ \tau_{33} &= \mu\frac{\partial}{\partial x_3}\left(\nabla^2+2\alpha\frac{\partial^2}{\partial x_2^2}\right)\Gamma\,. \end{aligned}\right. \tag{3.17}$$

A biharmonic function $\Gamma(x_2, x_3)$ may be written in the form

$$\Gamma=A(x_3+ix_2)+B(x_3-ix_2)+x_3C(x_3+ix_2)+x_3D(x_3-ix_2)\,, \tag{3.18}$$

where $A(z)$, $B(z)$, $C(z)$, $D(z)$ are arbitrary functions of z. From equation (3.18) we have

$$\nabla^2\Gamma=2\{C'(x_3+ix_2)+D'(x_3-ix_2)\}\,.$$

Functions here shown must be found so that shear stress vanishes and normal stress is equal to the given normal loads on the boundary $x_3=0$.

Stress components on the boundary may be expressed from equations (3.17) and (3.18) as

$$\left\{\begin{aligned} \tau_{23} &= 2\mu i\{-\alpha(A'''-B''')+(1-2\alpha)(C''-D'')\} \\ \tau_{33} &= 2\mu\{-\alpha(A'''+B''')+(1-\alpha)(C''+D'')\}\,. \end{aligned}\right. \tag{3.19}$$

But the normal loads on the boundary of our Boussinesq problem may be given from equation (3.13) in the form

$$P_2(x_2)=\frac{iF}{2\pi}\left[-(1-\alpha)\left\{\frac{1}{\xi_3+ix_2}-\frac{1}{\xi_3-ix_2}\right\}+\alpha\left\{\frac{\xi_3}{(\xi_3+ix_2)^2}-\frac{\xi_3}{(\xi_3-ix_2)^2}\right\}\right]$$

for the force in the direction of x_2-axis at $(0, \xi_3)$ and

$$P_3(x_2)=\frac{F}{2\pi}\left[-\left\{\frac{1}{\xi_3+ix_2}+\frac{1}{\xi_3-ix_2}\right\}-\alpha\left\{\frac{\xi_3}{(\xi_3+ix_2)^2}+\frac{\xi_3}{(\xi_3-ix_2)^2}\right\}\right]$$

for the force in the direction of x_3-axis acting at $(0, \xi_3)$.

The form of each unknown function in the second formula of equations

(3.19) can be obtained as a linear combination of functions that appeared in the expression of $P_2(x_2)$ or $P_3(x_2)$ so that it has no singularity in the half-plane $x_3 \geqq 0$.

Deciding constant coefficients we have in the case of $P_2(x_2)$

$$A'''(x_3+ix_2)=\frac{iF}{4\pi\mu}\left\{-\frac{(1-\alpha)(1-2\alpha)}{\alpha^2}\frac{1}{p+ix_2}+\frac{1-2\alpha}{\alpha}\frac{\xi_3}{(p+ix_2)^2}\right\},$$

$$B'''(x_3-ix_2)=\frac{iF}{4\pi\mu}\left\{\frac{(1-\alpha)(1-2\alpha)}{\alpha^2}\frac{1}{p-ix_2}-\frac{1-2\alpha}{\alpha}\frac{\xi_3}{(p-ix_2)^2}\right\},$$

$$C''(x_3+ix_2)=\frac{iF}{4\pi\mu}\left\{-\frac{1-\alpha}{\alpha}\frac{1}{p+ix_2}+\frac{\xi_3}{(p+ix_2)^2}\right\},$$

$$D''(x_3-ix_2)=\frac{iF}{4\pi\mu}\left\{\frac{1-\alpha}{\alpha}\frac{1}{p-ix_2}-\frac{\xi_3}{(p-ix_2)^2}\right\};$$

and in the case of $P_3(x_2)$

$$A'''(x_3+ix_2)=-\frac{F}{4\pi\mu}\left\{\frac{1-2\alpha}{\alpha^2}\frac{1}{p+ix_2}+\frac{1-2\alpha}{\alpha}\frac{\xi_3}{(p+ix_2)^2}\right\},$$

$$B'''(x_3-ix_2)=-\frac{F}{4\pi\mu}\left\{\frac{1-2\alpha}{\alpha^2}\frac{1}{p-ix_2}+\frac{1-2\alpha}{\alpha}\frac{\xi_3}{(p-ix_2)^2}\right\},$$

$$C''(x_3+ix_2)=-\frac{F}{4\pi\mu}\left\{\frac{1}{\alpha}\frac{1}{p+ix_2}+\frac{\xi_3}{(p+ix_2)^2}\right\},$$

$$D''(x_3-ix_2)=-\frac{F}{4\pi\mu}\left\{\frac{1}{\alpha}\frac{1}{p-ix_2}+\frac{\xi_3}{(p-ix_2)^2}\right\},$$

where $p=x_3+\xi_3$.

Hence, by means of equations (3.17) and (3.18), the displacement field to our Boussinesq problem may be obtained, omitting arbitrary constants, for $P_2(x_2)$

$$\begin{cases} u_2=\dfrac{F}{4\pi\mu}\left\{-2\dfrac{(1-\alpha)^2}{\alpha}\log S-2(1-\alpha)\dfrac{p^2}{S^2}-2\alpha\dfrac{x_3\xi_3}{S^2}+4\alpha\dfrac{x_3\xi_3p^2}{S^4}\right. \\ u_3=\dfrac{F}{4\pi\mu}\left\{2\left(\dfrac{1-\alpha}{\alpha}\right)\arctan\dfrac{x_2}{p}-2\dfrac{x_2\xi_3}{S^2}+2(1-\alpha)\dfrac{x_2x_3}{S^2}-4\alpha\dfrac{x_2x_3\xi_3p}{S^4}\right. \end{cases},$$

and we have for $P_3(x_2)$

$$\begin{cases} u_2 = \dfrac{F}{4\pi\mu}\left\{-2\left(\dfrac{1-\alpha}{\alpha}\right)\arctan\dfrac{x_2}{p} - 2(1-\alpha)\dfrac{x_2\xi_3}{S^2} + 2\dfrac{x_2x_3}{S^2} + 4\alpha\dfrac{x_2x_3\xi_3 p}{S^4}\right\} \\ u_3 = \dfrac{F}{4\pi\mu}\left\{-\dfrac{2}{\alpha}\log S + 2\dfrac{p^2}{S^2} - 2\alpha\dfrac{x_3\xi_3}{S^2} + 4\alpha\dfrac{x_3\xi_3 p^2}{S^4}\right\}, \end{cases}$$

where $S=\sqrt{x_2^2+p^2}$.

Thus the final displacement components at (x_2, x_3) to the semi-infinite medium for a force F per unit length acting at $(0, \xi_3)$ in the direction of x_2-axis become

$$\begin{cases} u_2 = \dfrac{F}{4\pi\mu}\Big[-(2-\alpha)\log R - \alpha\dfrac{(x_3-\xi_3)^2}{R^2} - \dfrac{(2-2\alpha+\alpha^2)}{\alpha}\log S \\ \qquad - \{(2-\alpha)p^2 + 2\alpha p\xi_3 - 2\alpha\xi_3^2\}\dfrac{1}{S^2} + 4\alpha p^2(p-\xi_3)\xi_3\dfrac{1}{S^4}\Big], \\ u_3 = \dfrac{F}{4\pi\mu}\Big[\alpha\dfrac{x_2(x_3-\xi_3)}{R^2} \\ \qquad + 2\left(\dfrac{1-\alpha}{\alpha}\right)\arctan\dfrac{x_2}{p} + (2-\alpha)(p-2\xi_3)\dfrac{x_2}{S^2} - 4\alpha p(p-\xi_3)\xi_3\dfrac{x_2}{S^4}\Big], \end{cases} \tag{3.20}$$

while for a force F per unit length in the direction of x_3-axis we have

$$\begin{cases} u_2 = \dfrac{F}{4\pi\mu}\Big[\alpha\dfrac{x_2(x_3-\xi_3)}{R^2} \\ \qquad - 2\left(\dfrac{1-\alpha}{\alpha}\right)\arctan\dfrac{x_2}{p} + (2-\alpha)(p-2\xi_3)\dfrac{x_2}{S^2} + 4\alpha p(p-\xi_3)\xi_3\dfrac{x_2}{S^4}\Big], \\ u_3 = \dfrac{F}{4\pi\mu}\Big[-(2-\alpha)\log R + \alpha\dfrac{(x_3-\xi_3)^2}{R^2} - \dfrac{(2-2\alpha+\alpha^2)}{\alpha}\log S \\ \qquad + \{(2-\alpha)p^2 - 2\alpha p\xi_3 + 2\alpha\xi_3^2\}\dfrac{1}{S^2} + 4\alpha p^2(p-\xi_3)\xi_3\dfrac{1}{S^4}\Big], \end{cases} \tag{3.21}$$

where $p=\xi_3+x_3$, $R=\sqrt{x_2^2+(x_3-\xi_3)^2}$ and $S=\sqrt{x_2^2+p^2}$.

Omitting the factor F we can obtain explicit expressions for $G_1^1(Q, P)$ from equation (3.11) and for $G_k^m(Q, P)$ from equations (3.20) and (3.21). As for System II we make the assumption $\lambda=\mu$, that is $\alpha=2/3$, for the sake of simplicity in the following. Then we have for System I:

$$G_1^1(Q, P) = \frac{1}{2\pi\mu}\left[\log\frac{1}{R} + \log\frac{1}{S}\right] \tag{3.22}$$

and for System II:

$$
\left\{
\begin{aligned}
G_2^2(Q,\,P)&=\frac{1}{2\pi\mu}\left[\frac{2}{3}\log\frac{1}{R}-\frac{1}{3}\frac{(x_3-\xi_3)^2}{R^2}\right.\\
&\qquad\left.+\frac{5}{6}\log\frac{1}{S}-\frac{2}{3}\frac{(p^2+p\xi_3-\xi_3^2)}{S^2}+\frac{4}{3}\frac{(p^3\xi_3-p^2\xi_3^2)}{S^4}\right]\\
G_2^3(Q,\,P)&=\frac{1}{2\pi\mu}\left[\frac{1}{3}\frac{(x_2-\xi_2)(x_3-\xi_3)}{R^2}+\frac{1}{2}\arctan\left(\frac{x_2-\xi_2}{p}\right)\right.\\
&\qquad\left.+\frac{2}{3}\frac{(x_2-\xi_2)(p-2\xi_3)}{S^2}-\frac{4}{3}\frac{(x_2-\xi_2)(p^2\xi_3-p\xi_3^2)}{S^4}\right]\\
G_3^2(Q,\,P)&=\frac{1}{2\pi\mu}\left[\frac{1}{3}\frac{(x_2-\xi_2)(x_3-\xi_3)}{R^2}-\frac{1}{2}\arctan\left(\frac{x_2-\xi_2}{p}\right)\right.\\
&\qquad\left.+\frac{2}{3}\frac{(x_2-\xi_2)(p-2\xi_3)}{S^2}+\frac{4}{3}\frac{(x_2-\xi_2)(p^2\xi_3-p\xi_3^2)}{S^4}\right]\\
G_3^3(Q,\,P)&=\frac{1}{2\pi\mu}\left[\frac{2}{3}\log\frac{1}{R}+\frac{1}{3}\frac{(x_3-\xi_3)^2}{R^2}\right.\\
&\qquad\left.+\frac{5}{6}\log\frac{1}{S}+\frac{2}{3}\frac{(p^2-p\xi+\xi_3^2)}{S^2}+\frac{4}{3}\frac{(p^3\xi_3-p^2\xi_3^2)}{S^4}\right].
\end{aligned}
\right.
\tag{3.23}
$$

In what follows as well as in equations (3.22) and (3.23), $Q=(x_2,x_3)$, $P=(\xi_2,\xi_3)$, $p=x_3+\xi_3$, $R=\overline{PQ}=\sqrt{(x_2-\xi_2)^2+(x_3-\xi_3)^2}$ and $S=\overline{P'Q}=\sqrt{(x_2-\xi_2)^2+p^2}$.

By use of equation (3.3) we obtain explicit expressions for $G_{1l}^1(Q,\,P)$ and $G_{kl}^m(Q,\,P)$ as follows. For System I:

$$
\left\{
\begin{aligned}
G_{12}^1(Q,\,P)&=\frac{1}{2\pi}\left[\frac{1}{R^2}+\frac{1}{S^2}\right](x_2-\xi_2)\\
G_{13}^1(Q,\,P)&=\frac{1}{2\pi}\left[\frac{(x_3-\xi_3)}{R^2}-\frac{p}{S^2}\right]
\end{aligned}
\right.
\tag{3.24}
$$

and for System II:

$$
\left\{
\begin{aligned}
G_{22}^2(Q,\,P)&=\frac{1}{2\pi}\left[\frac{5}{3}\frac{1}{R^2}-\frac{4}{3}\frac{(x_3-\xi_3)^2}{R^4}+\frac{7}{3}\frac{1}{S^2}\right.\\
&\qquad\left.-\frac{4}{3}\frac{(3p^2+p\xi_3-2\xi_3^2)}{S^4}+\frac{32}{3}\frac{(p^3\xi_3-p^2\xi_3^2)}{S^6}\right](x_2-\xi_2)\\
G_{22}^3(Q,\,P)&=\frac{1}{2\pi}\left[\frac{(x_3-\xi_3)}{R^2}-\frac{4}{3}\frac{(x_3-\xi_3)^3}{R^4}+\frac{1}{3}\frac{(p-10\xi_3)}{S^2}\right.\\
&\qquad\left.-\frac{4}{3}\frac{(3p^3+p^2\xi_3-6p\xi_3^2)}{S^4}+\frac{32}{3}\frac{(p^4\xi_3-p^3\xi_3^2)}{S^6}\right]
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
G_{23}^{2}(Q,P)&=\frac{1}{2\pi}\left[\frac{5}{3}\frac{(x_3-\xi_3)}{R^2}-\frac{4}{3}\frac{(x_3-\xi_3)^3}{R^4}-\frac{1}{3}\frac{(5p+2\xi_3)}{S^2}\right.\\
&\qquad\left.+\frac{4}{3}\frac{(p^3+7p^2\xi_3-6p\xi_3^2)}{S^4}-\frac{32}{3}\frac{(p^4\xi_3-p^3\xi_3^2)}{S^6}\right]\\
G_{23}^{3}(Q,P)&=\frac{1}{2\pi}\left[\frac{1}{3}\frac{1}{R^2}+\frac{4}{3}\frac{(x_3-\xi_3)^2}{R^4}-\frac{1}{3}\frac{1}{S^2}\right.\\
&\qquad\left.-\frac{4}{3}\frac{(p^2-p\xi_3-2\xi_3^2)}{S^4}+\frac{32}{3}\frac{(p^3\xi_3-p^2\xi_3^2)}{S^6}\right](x_2-\xi_2)\\
G_{33}^{2}(Q,P)&=\frac{1}{2\pi}\left[-\frac{1}{3}\frac{1}{R^2}+\frac{4}{3}\frac{(x_3-\xi_3)^2}{R^4}+\frac{1}{3}\frac{1}{S^2}\right.\\
&\qquad\left.-\frac{4}{3}\frac{(p^2-5p\xi_3+2\xi_3^2)}{S^4}-\frac{32}{3}\frac{(p^3\xi_3-p^2\xi_3^2)}{S^6}\right](x_2-\xi_2)\\
G_{33}^{3}(Q,P)&=\frac{1}{2\pi}\left[\frac{1}{3}\frac{(x_3-\xi_3)}{R^2}+\frac{4}{3}\frac{(x_3-\xi_3)^3}{R^4}\right.\\
&\qquad\left.-\frac{1}{3}\frac{(p-2\xi_3)}{S^2}-\frac{4}{3}\frac{(p^3-5p^2\xi_3+6p\xi_3^2)}{S^4}-\frac{32}{3}\frac{(p^4\xi_3-p^3\xi_3^2)}{S^6}\right]
\end{aligned}
\right.
\tag{3.25}
$$

By the help of equation (3.4) we may easily obtain explicit expressions for $G_1^{1l}(Q,P)$ and $G_m^{kl}(Q,P)$.

From equation (3.5) we have the following expressions for System I:

$$
\left\{
\begin{aligned}
G_{12}^{12}(Q,P)&=\frac{\mu}{2\pi}\left[-\frac{1}{R^2}+2\frac{(x_3-\xi_3)^2}{R^4}-\frac{1}{S^2}+2\frac{p^2}{S^4}\right]\\
G_{12}^{13}(Q,P)&=\frac{\mu}{2\pi}\left[-2\frac{(x_3-\xi_3)}{R^4}-2\frac{p}{S^4}\right](x_2-\xi_2)\\
G_{13}^{12}(Q,P)&=\frac{\mu}{2\pi}\left[-2\frac{(x_3-\xi_3)}{R^4}+2\frac{p}{S^4}\right](x_2-\xi_2)\\
G_{13}^{13}(Q,P)&=\frac{\mu}{2\pi}\left[\frac{1}{R^2}-2\frac{(x_3-\xi_3)^2}{R^4}-\frac{1}{S^2}+2\frac{p^2}{S^4}\right]
\end{aligned}
\right.
\tag{3.26}
$$

and for System II:

$$
\left\{
\begin{aligned}
G_{22}^{22}(Q,P)&=\frac{\mu}{2\pi}\left[-4\frac{1}{R^2}+16\frac{(x_3-\xi_3)^2}{R^4}-\frac{32}{3}\frac{(x_3-\xi_3)^4}{R^6}-\frac{20}{3}\frac{1}{S^2}\right.\\
&\qquad+\frac{16}{3}\frac{(7p^2+3p\xi_3-3\xi_3^2)}{S^4}-32\frac{(p^4+4p^3\xi_3-4p^2\xi_3^2)}{S^6}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&\qquad +128\frac{(p^5\xi_3-p^4\xi_3^2)}{S^8}\Bigg]\\
&G^{23}_{22}(Q,\,P)=\frac{\mu}{2\pi}\Bigg[-8\frac{(x_3-\xi_3)}{R^4}+\frac{32}{3}\frac{(x_3-\xi_3)^3}{R^6}-\frac{8}{3}\frac{(5p-2\xi_3)}{S^4}\\
&\qquad +\frac{32}{3}\frac{(3p^3+4p^2\xi_3-6p\xi_3^2)}{S^6}-128\frac{(p^4\xi_3-p^3\xi_3^2)}{S^8}\Bigg](x_2-\xi_2)\\
&G^{33}_{22}(Q,\,P)=\frac{\mu}{2\pi}\Bigg[\frac{4}{3}\frac{1}{R^2}-\frac{32}{3}\frac{(x_3-\xi_3)^2}{R^4}+\frac{32}{3}\frac{(x_3-\xi_3)^4}{R^6}-\frac{4}{3}\frac{1}{S^2}\\
&\qquad -\frac{16}{3}\frac{(4p^2-3p\xi_3-3\xi_3^2)}{S^4}+\frac{32}{3}\frac{(3p^4+8p^3\xi_3-12p^2\xi_3^2)}{S^6}\\
&\qquad -128\frac{(p^5\xi_3-p^4\xi_3^2)}{S^8}\Bigg]\\
&G^{22}_{23}(Q,\,P)=\frac{\mu}{2\pi}\Bigg[-8\frac{(x_3-\xi_3)}{R^4}+\frac{32}{3}\frac{(x_3-\xi_3)^3}{R^6}+\frac{8}{3}\frac{(3p+2\xi_3)}{S^4}\\
&\qquad -\frac{32}{3}\frac{(p^3+8p^2\xi_3-6p\xi_3^2)}{S^6}+128\frac{(p^4\xi_3-p^3\xi_3^2)}{S^8}\Bigg](x_2-\xi_2)\\
&G^{23}_{23}(Q,\,P)=\frac{\mu}{2\pi}\Bigg[\frac{4}{3}\frac{1}{R^2}-\frac{32}{3}\frac{(x_3-\xi_3)^2}{R^4}+\frac{32}{3}\frac{(x_3-\xi_3)^4}{R^6}-\frac{4}{3}\frac{1}{S^2}\\
&\qquad +\frac{16}{3}\frac{(2p^2+3p\xi_3-3\xi_3^2)}{S^4}-\frac{32}{3}\frac{(p^4+12p^3\xi_3-12p^2\xi_3^2)}{S^6}\\
&\qquad +128\frac{(p^5\xi_3-p^4\xi_3^2)}{S^8}\Bigg]\\
&G^{33}_{23}(Q,\,P)=\frac{\mu}{2\pi}\Bigg[\frac{8}{3}\frac{(x_3-\xi_3)}{R^4}-\frac{32}{3}\frac{(x_3-\xi_3)^3}{R^6}-\frac{8}{3}\frac{(p-2\xi_3)}{S^4}\\
&\qquad +\frac{32}{3}\frac{(p^3+4p^2\xi_3-6p\xi_3^2)}{S^6}-128\frac{(p^4\xi_3-p^3\xi_3^2)}{S^8}\Bigg](x_2-\xi_2)\\
&G^{22}_{33}(Q,\,P)=\frac{\mu}{2\pi}\Bigg[\frac{4}{3}\frac{1}{R^2}-\frac{32}{3}\frac{(x_3-\xi_3)^2}{R^4}+\frac{32}{3}\frac{(x_3-\xi_3)^4}{R^6}-\frac{4}{3}\frac{1}{S^2}\\
&\qquad +\frac{16}{3}\frac{(2p^2-9p\xi_3+3\xi_3^2)}{S^4}-\frac{32}{3}\frac{(p^4-16p^3\xi_3+12p^2\xi_3^2)}{S^6}\\
&\qquad -128\frac{(p^5\xi_3-p^4\xi_3^2)}{S^8}\Bigg]
\end{aligned}
\right.
\tag{3.27}
$$

$$\left\{\begin{aligned}
G_{33}^{23}(Q,P)=&\frac{\mu}{2\pi}\Big[\frac{8}{3}\frac{(x_3-\xi_3)}{R^4}-\frac{32}{3}\frac{(x_3-\xi_3)^3}{R^6}-\frac{8}{3}\frac{(p-2\xi_3)}{S^4}\\
&+\frac{32}{3}\frac{(p^3-8p^2\xi_3+6p\xi_3^2)}{S^6}+128\frac{(p^4\xi_3-p^3\xi_3^2)}{S^8}\Big](x_2-\xi_2)\\
G_{33}^{33}(Q,P)=&\frac{\mu}{2\pi}\Big[\frac{4}{3}\frac{1}{R^2}+\frac{16}{3}\frac{(x_3-\xi_3)^2}{R^4}-\frac{32}{3}\frac{(x_3-\xi_3)^4}{R^6}-\frac{4}{3}\frac{1}{S^2}\\
&-\frac{16}{3}\frac{(p^2-3p\xi_3+3\xi_3^2)}{S^4}+\frac{32}{3}\frac{(p^4-12p^3\xi_3+12p^2\xi_3^2)}{S^6}\\
&+128\frac{(p^5\xi_3-p^4\xi_3^2)}{S^8}\Big]
\end{aligned}\right.$$

These expressions are the same as those to be obtained by integration with respect to ξ_1 from W_{1l}^1, W_{kl}^m, W_{1l}^{1n} and W_{kl}^{mn} which are listed in the previous paper (Maruyama 1964), though calculations are lengthy, as

$$G_{1l}^1(Q,P)=\int_{-\infty}^{\infty}W_{1l}^1 d\xi_1\,,$$

$$G_{kl}^m(Q,P)=\int_{-\infty}^{\infty}W_{kl}^m d\xi_1\,,\quad \text{etc.}$$

So far we have chosen the orientation of the line element $d\sigma$ of L arbitrarily. If we take the positive direction of the line element ds of L such that the tangent t and the normal ν to the line are oriented with each other as x_2- and x_3-axis, then we have

$$\left\{\begin{aligned}
\nu_2&=-\cos(t,\xi_3)=-\frac{d\xi_3}{ds}\,,\\
\nu_3&=\cos(t,\xi_2)=\frac{d\xi_2}{ds}\,.
\end{aligned}\right.\tag{3.28}$$

If we use ds in place of $d\sigma$ in equations (3.7) and (3.8), in the cases of continuous tractions across L, they may be written in the forms

$$\left\{\begin{aligned}
u_1(Q)&=\int\Delta u_1\{G_{13}^1(Q,P)\,d\xi_2-G_{12}^1(Q,P)\,d\xi_3\}\\
u_m(Q)&=\int\Delta u_k\{G_{k3}^m(Q,P)\,d\xi_2-G_{k2}^m(Q,P)\,d\xi_3\}\,,
\end{aligned}\right.\tag{3.29}$$

$$\left\{\begin{aligned}
\tau_{1n}(Q)&=\int\Delta u_1\{G_{13}^{1n}(Q,P)\,d\xi_2-G_{12}^{1n}(Q,P)\,d\xi_3\}\\
\tau_{mn}(Q)&=\int\Delta u_k\{G_{k3}^{mn}(Q,P)\,d\xi_2-G_{k2}^{mn}(Q,P)\,d\xi_3\}\,,
\end{aligned}\right.\tag{3.30}$$

where the repeated suffix k is to be summed over the values 2, 3 as before.

4. Complex representations for an isotropic and homogeneous medium

In this section we consider two-dimensional problems as in Section 3, by means of complex representations. The body will be assumed to be isotropic and homogeneous from the beginning. Cartesian coordinate system of reference in the plane will be denoted by (x, y) for convenience of the familiar expression of complex variables, with the y-axis taken downward so that the semi-infinite body occupies the half-space $y \geqq 0$. It will be only necessary for reference to Section 3 to replace

$$\begin{array}{c} (x_2, x_3),\ (\xi_2, \xi_3),\ \text{and}\ (u_2, u_3) \\ \text{by}\ (x,\ \ y),\ (\xi,\ \ \eta),\ \text{and}\ (u,\ \ v) \end{array}$$

respectively.

The first part of this section treats of the field with displacements perpendicular to the (x, y)-plane, that is, for System I, while the second part the field with displacements parallel to the plane, that is, for System II.

4. 1. Expressions for System I

Euilibrium equations when no body forces are present for System I may be written in terms of new variables

$$\frac{\partial \tau_{1x}}{\partial x}+\frac{\partial \tau_{1y}}{\partial y}=0 \tag{4.1.1}$$

$$\begin{cases} \tau_{1x}=\mu \dfrac{\partial u_1}{\partial x} \\ \tau_{1y}=\mu \dfrac{\partial u_1}{\partial y}\ , \end{cases} \tag{4.1.2}$$

where retaining u_1 and suffix 1 in τ_{12} and τ_{13} in equations (2.1) and (2.2), we reserved the use of z corresponding to the third axis that is perpendicular to the plane, for the familiar expression of a complex variable $x+iy$.

Substitution from equation (4.1.2) into (4.1.1) yields the harmonic equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u_1=0\,.$$

We assume that u_1 is an analytic function of variables x, y inside some region. If we introduce new variables

$$\begin{cases} z=x+iy \\ \bar{z}=x-iy\,, \end{cases} \tag{4.1.3}$$

then differentiations with respect to x and y being combined as

$$\begin{cases} \dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}=2\dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial x}+i\dfrac{\partial}{\partial y}=2\dfrac{\partial}{\partial \bar{z}}\,, \end{cases} \tag{4.1.4}$$

we have

$$\frac{\partial^2}{\partial z\,\partial \bar{z}}u_1=0\,,$$

from which it follows that

$$u_1=\chi_1(z)+\chi_2(\bar{z})\,,$$

where χ_1 and χ_2 are arbitrary functions. Since u_1 is a real function it is seen that we must put

$$\overline{\chi_1(z)}=\chi_2(\bar{z})\,,$$

where $\overline{\chi_1(z)}$ is the complex conjugate of $\chi_1(z)$, therefore u_1 is taken as the real part of a holomorphic function of the complex variable z.

It may be more convenient to adopt the expression

$$\begin{aligned} \mu u_1&=\mathrm{Re}\{\chi(z)\} \\ \text{or}\quad 2\mu u_1&=\chi(z)+\overline{\chi(z)} \end{aligned} \tag{4.1.5}$$

where Re denotes the real part.

From equations (4.1.2) and (4.1.5), with the help of equation (4.1.4), we obtain

$$\tau_{1x}-i\tau_{1y}=\chi'(z) \tag{4.1.6}$$

where $\chi'(z)=d\chi(z)/dz$.

Consider some curve AB in the (xy)-plane and suppose that positive direction along the curve is from A to B. We draw the normal ν at any point of AB to the left when looking along the curve in the positive direction. Then the tangent and the positive direction of the normal are oriented with respect to each other as the x- and y-axis (cf. Fig. 6). It should be noted that the orientation of normal ν is opposite with respect to the normal n in Muskhelishvili (1953a, § 32).

As in the preceding sections, $\overset{\nu}{T}_1 ds$ is understood to be the force acting on element ds of the arc AB, from the side of the positive normal, and we have as in equation (2.5)

$$\overset{\nu}{T}_1=\tau_{1x}\cos(\nu, x)+\tau_{1y}\cos(\nu, y)$$

which may be written

$$\overset{\nu}{T}_1=-\mathrm{Im}\{(\tau_{1x}-i\tau_{1y})(\cos(\nu, y)-i\cos(\nu, x))\},$$

where Im denotes the imaginary part. From this, with (4.1.6) and the equation

$$\cos(\nu, y)-i\cos(\nu, x)=\frac{dz}{ds}$$

which is easily verified by the relations

$$\cos(\nu, x)=-\cos(t, y)=-\frac{dy}{ds}, \qquad \cos(\nu, y)=\cos(t, x)=\frac{dx}{ds},$$

where t is the positive direction of the tangent, ds the line element with the orientation of t, we obtain

$$\begin{aligned}\overset{\nu}{T}_1&=-\mathrm{Im}\left\{\chi'(z)\frac{dz}{ds}\right\}\\&=-\frac{d}{ds}\mathrm{Im}\{\chi(z)\}.\end{aligned} \tag{4.1.7}$$

Equation (4.1.7) may also be written in the forms

$$\begin{aligned}\overset{\nu}{T}_1&=-\mathrm{Im}\{e^{i(t,x)}\chi'(z)\}\\&=\mathrm{Re}\{e^{i(\nu,x)}\chi'(z)\},\end{aligned} \tag{4.1.8}$$

where (t, x), (ν, x) denotes the angle between the tangent and the x-axis, or the normal and the x-axis, as usual, measured counterclockwise from the x-axis in the plane.

Let X_1 be the total force in the direction of x_1-axis, measured per unit length of the x_1-axis, exerted across AB by the region on the positive side of normal upon the region on the negative side. Then we have from equation (4.1.7)

$$X_1 = \int_A^B \overset{\nu}{T}_1 \, ds = \Big[-\mathrm{Im}\,\{\chi(z)\} \Big]_A^B$$

where $[\quad]_A^B$ denotes the change in value of the function in the brackets as the point z passes along the arc from A to B. It follows that this resultant force does not depend on the shape of the arc joining A and B except that it leaves the region. Therefore if we write the force simply

$$X_1 = -\mathrm{Im}\{\chi(z)\}\,, \tag{4.1.9}$$

it can be understood to be the resultant force exerted by the positive side of normal to an arbitrary arc connecting the variable point B, the coordinate of which is z, with some fixed point A.

We now note some results in complex variable theory which will be used in what follows (cf. Muskhelishvili 1953 a, b; Green and Zerna 1954).

Let L be a simple smooth contour, a simple smooth arc or the union of a finite number of such disconnected arcs and contours in the (xy)-plane. If L contains arcs, their ends will be called ends of the line L. When L consists of disconnected parts a positive direction must be chosen on each of these parts.

The arc with the ends A, B will be denoted by AB, where the positive direction along the arc is from A to B (Fig. 6). If we draw around any point on L not coinciding with an end, a circle of sufficient small radius, this circle will be divided by L into two parts, one of which will lie on the left and the other on the right when looking in the positive direction of L. They will be regarded as the left and right neighborhoods and denoted by $(+)$ and $(-)$ respectively. We draw the normal ν at any point on L to the left when looking along L in the positive direction. The orientation of the normal ν is such that it points to the positive side $(+)$ of two neighborhoods and such that the sense is consistent with the orientation of the normal in previous works (e.g. Steketee 1958; Maruyama 1963, 1964) as well as in the previous sections of this paper.

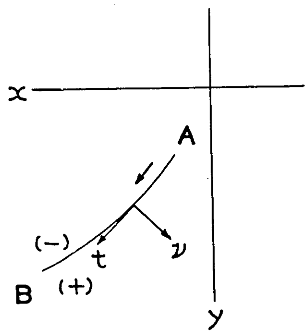


Fig. 6. $(+)$ and $(-)$ neighborhoods and the orientation of normal ν to the curve AB.

Denote by S' the part of the plane which contains all the points not belonging to L; in other words, S' is the (xy)-plane cut along L. If L consists only of arcs, then S' is a connected region, while if L includes contours, S' consists of several connected regions bounded by these contours.

Let $F(z)$ be some function given in S' (but not on L) and satisfying the following conditions:

1) The function $F(z)$ is holomorphic everywhere in S'.

2) It is continuous from the left and from the right at all points of L, other than the ends.

3) Near the ends c

$$|F(z)| < \frac{C}{|z-c|^{\alpha}} \tag{4.1.10}$$

where C and α are positive constants and $0 \leqq \alpha < 1$.

In 2) $F(z)$ is said to be continuous at the point ζ on L from the left (or from the right) if $F(z)$ tends to a definite limit $F^{+}(\zeta)$ (or $F^{-}(\zeta)$) when z approaches ζ along any path which remains on the left (or on the right) of L.

Such a function $F(z)$ will be called sectionally holomorphic in the entire plane. The line L will be called the line of discontinuity of $F(z)$.

The problem in which we are concerned is to find the sectionally holomorphic function $F(z)$ with the line of discontinuity L, the boundary values of which from the left and from the right satisfy, except at the ends, the condition

$$F^{+}(\zeta) - F^{-}(\zeta) = f(\zeta)\,, \tag{4.1.11}$$

where $f(\zeta)$ is a function given on L. If we assume that the function $f(\zeta)$ satisfies the H condition on L, the general solution of the problem is given by the integral of the Cauchy type

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta + C\,, \tag{4.1.12}$$

where C is an arbitrary constant. Here, a function of position on an arc or contour $f(\zeta)$ is said to satisfy the H condition or Hölder condition, if for any two points ζ_1, ζ_2 on the arc or contour the inequality

$$|f(\zeta_2) - f(\zeta_1)| \leqq A\,|\zeta_2 - \zeta_1|^{\mu} \tag{4.1.13}$$

holds, where A and μ are positive constants and $0 < \mu \leqq 1$.

As regards the boundary values $F^{+}(\zeta_0)$ and $F^{-}(\zeta_0)$ at any point ζ_0 of L, other than its ends, when $C=0$ in equation (4.1.12), they are given by the Plemelj formulae

$$\begin{cases} F^{+}(\zeta_0)=\dfrac{1}{2}f(\zeta_0)+\dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{f(\zeta)}{\zeta-\zeta_0}d\zeta \\ F^{-}(\zeta_0)=-\dfrac{1}{2}f(\zeta_0)+\dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{f(\zeta)}{\zeta-\zeta_0}d\zeta\,, \end{cases} \tag{4.1.14}$$

where the principal values of the integrals must be taken on the right-hand sides.

As an application of this solution we can treat of the problem of dislocations stated in previous sections.

Now we denote by L the union of non-intersecting smooth arcs in the plane on which the discontinuity in displacement is given, since the field must be obtained by the help of sectionally holomorphic functions.

If the discontinuity in displacement is expressed by

$$u_1^{+}(\zeta)-u_1^{-}(\zeta)=\varDelta u_1(\zeta) \quad \text{on } L\,, \tag{4.1.15}$$

we have from equation (4.1.5)

$$\chi^{+}(\zeta)+\overline{\chi^{+}(\zeta)}=\chi^{-}(\zeta)+\overline{\chi^{-}(\zeta)}+2\mu\varDelta u_1 \quad \text{on } L\,. \tag{4.1.16}$$

Since we are concerned in the case of continuous tranctions across L in this section, from equation (4.1.9)

$$\chi^{+}(\zeta)-\overline{\chi^{+}(\zeta)}=\chi^{-}(\zeta)-\overline{\chi^{-}(\zeta)} \quad \text{on } L\,. \tag{4.1.17}$$

Adding (4.1.16) and (4.1.17) we find

$$\chi^{+}(\zeta)-\chi^{-}(\zeta)=\mu\varDelta u_1(\zeta) \quad \text{on } L\,. \tag{4.1.18}$$

First we consider the case when the body occupies the infinite region. Assuming that $\chi(\zeta)$ is sectionally holomorphic in the entire plane and vanishes at infinity, the solution to the condition (4.1.18) may be obtained from equation (4.1.12) as

$$\chi(z)=\frac{\mu}{2\pi i}\int_L \frac{\varDelta u_1(\zeta)}{\zeta-z}d\zeta\,. \tag{4.1.19}$$

Next, the solution to the problem in the semi-infinite medium which includes L on which the discontinuity in displacement is given may be easily constructed by means of this solution.

In fact, defining $\chi_0(z)$ as in equation (4.1.19) by

$$\chi_0(z)=\frac{\mu}{2\pi i}\int_L \frac{\Delta u_1(\zeta)}{\zeta-z}d\zeta \tag{4.1.20}$$

and putting

$$\chi(z)=\chi_0(z)+\overline{\chi_0(\bar{z})}, \tag{4.1.21}$$

we find that $\chi(z)$ is sectionally holomorphic in the semi-infinite region $y \geqq 0$ with the line of discontinuity L and vanishes at infinity. The displacement field satisfying equation (4.1.15) may be written by equation (4.1.5) as

$$\mu u_1 = \mathrm{Re}\,\{\chi_0(z)+\overline{\chi_0(\bar{z})}\}$$

or

$$u_1=\frac{1}{2}\left[\frac{1}{2\pi i}\int_L \frac{\Delta u_1}{\zeta-z}d\zeta+\overline{\left\{\frac{1}{2\pi i}\int_L \frac{\Delta u_1}{\zeta-z}d\zeta\right\}}\right. \tag{4.1.22}$$

$$\left.+\frac{1}{2\pi i}\int_L \frac{\Delta u_1}{\zeta-\bar{z}}d\zeta+\overline{\left\{\frac{1}{2\pi i}\int_L \frac{\Delta u_1}{\zeta-\bar{z}}d\zeta\right\}}\right].$$

By use of the rule

$$\frac{d}{dz}\left\{\overline{\chi_0(\bar{z})}\right\}=\overline{\left\{\frac{d}{d\bar{z}}\chi_0(\bar{z})\right\}},$$

the expression including the stress components may be obtained from (4.1.6) as

$$\tau_{1x}-i\tau_{1y}=\chi_0'(z)+\overline{\chi_0'(\bar{z})} \tag{4.1.23}$$

from which it is seen that the requirement $\tau_{1y}=0$ on the boundary $z=\bar{z}$ is satisfied. Thus we have the required solution in the semi-infinite medium.

Starting from the first equation in (3.29), we can arrive at the expression (4.1.22) by the help of (3.24) and the relations

$$\begin{cases} z=x_2+ix_3\,, & \bar{z}=x_2-ix_3 \\ \zeta=\xi_2+i\xi_3\,, & \bar{\zeta}=\xi_2-i\xi_3\,. \end{cases} \tag{4.1.24}$$

4.2. Expressions for System II

Equilibrium equations when no body forces are present for System II are written in this section, in place of equations (2.3) and (2.4),

$$\begin{cases} \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y} = 0 \end{cases} \tag{4.2.1}$$

and

$$\begin{cases} \tau_{xx} = \lambda \Delta + 2\mu \dfrac{\partial u}{\partial x} \\ \tau_{xy} = \mu \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \right) \\ \tau_{yy} = \lambda \Delta + 2\mu \dfrac{\partial v}{\partial y}, \end{cases} \tag{4.2.2}$$

where

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

From these equations it can be derived that there always exists some biharmonic function $A(x, y)$ of x, y called a stress function or Airy function by the help of which stresses may be expressed as

$$\tau_{xx} = \frac{\partial^2 A}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 A}{\partial x \partial y}, \quad \tau_{yy} = \frac{\partial^2 A}{\partial x^2}. \tag{4.2.3}$$

As in Section 4.1, assuming that A is an analytic function of variables x, y inside some region and introducing new variables z, $\bar{z}$ by equation (4.1.3), we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 A = 16 \frac{\partial^4}{\partial z^2 \partial \bar{z}^2} A = 0.$$

Hence A may be written

$$A = \bar{z}\varphi_1(z) + z\varphi_2(\bar{z}) + \omega_1(z) + \omega_2(\bar{z}),$$

where φ_1, φ_2, ω_1, and ω_2 are arbitrary functions. A being real we must put

$$\varphi_2(\bar{z}) = \overline{\varphi_1(z)}, \qquad \omega_2(\bar{z}) = \overline{\omega_1(z)}.$$

Thus we have A in the form

$$\begin{aligned} A &= \mathrm{Re}\{\bar{z}\varphi(z)+\omega(z)\} \\ &= \frac{1}{2}\{\bar{z}\varphi(z)+z\overline{\varphi(z)}+\omega(z)+\overline{\omega(z)}\}\,. \end{aligned} \tag{4.2.4}$$

Using (4.1.4), from equations in (4.2.3)

$$\begin{cases} \tau_{xx}+\tau_{yy}=2(\lambda+\mu)\varDelta=4\dfrac{\partial^2}{\partial z\partial\bar{z}}A \\ \tau_{yy}-\tau_{xx}+2i\tau_{xy}=4\dfrac{\partial^2}{\partial z^2}A\,. \end{cases}$$

These are given in terms of $\varphi'(z)$, $\psi'(z)$ and their complex conjugates as

$$\begin{cases} \tau_{xx}+\tau_{yy}=2(\lambda+\mu)\varDelta=2\{\varphi'(z)+\overline{\varphi'(z)}\} \\ \tau_{yy}-\tau_{xx}+2i\tau_{xy}=2\{\bar{z}\varphi''(z)+\psi'(z)\} \end{cases} \tag{4.2.5}$$

where

$$\psi(z)=\omega'(z)\,. \tag{4.2.6}$$

In order to obtain an expression for displacement we may proceed in a similar way to what is taken in Green and Zerna (§ 6.4). Introducing new variables $D, \bar{D}$ defined by

$$D=u+iv\,,\quad \bar{D}=u-iv$$

and noting that

$$\varDelta=\frac{\partial D}{\partial z}+\frac{\partial\bar{D}}{\partial\bar{z}}$$

$$\tau_{yy}-\tau_{xx}+2i\tau_{xy}=-4\mu\frac{\partial\bar{D}}{\partial z}\,,$$

we have from equation (4.2.5)

$$\begin{cases} 2\mu\left(\dfrac{\partial D}{\partial z}+\dfrac{\partial\bar{D}}{\partial\bar{z}}\right)=\dfrac{2\mu}{\lambda+\mu}\{\varphi'(z)+\overline{\varphi'(z)}\} \\ 2\mu\dfrac{\partial\bar{D}}{\partial z}=-\{\bar{z}\varphi''(z)+\psi'(z)\}\,. \end{cases}$$

These equations may be integrated to give

$$2\mu(u+iv)=\kappa\varphi(z)-z\,\overline{\varphi'(z)}-\overline{\psi(z)}\,, \tag{4.2.7}$$

where

$$\kappa=\frac{\lambda+3\mu}{\lambda+\mu}=\frac{2}{\alpha}-1\,. \tag{4.2.8}$$

In the same way as in Section 4.1, taking some arc AB in the region, we consider the force acting on element ds of the arc, from the positive side of normal, which is expressed by $(\overset{\nu}{T}_x ds,\ \overset{\nu}{T}_y ds)$. By the help of relations in Section 4.1 we have

$$\begin{aligned}
\overset{\nu}{T}_x &= \tau_{xx}\cos(\nu,\,x)+\tau_{xy}\cos(\nu,\,y)\\
&= -\frac{\partial^2 A}{\partial y^2}\frac{dy}{ds}-\frac{\partial^2 A}{\partial x\partial y}\frac{dx}{ds}\\
&= -\frac{d}{ds}\left(\frac{\partial A}{\partial y}\right)\\
\overset{\nu}{T}_y &= \tau_{xy}\cos(\nu,\,x)+\tau_{yy}\cos(\nu,\,y)\\
&= \frac{\partial^2 A}{\partial x\partial y}\frac{dy}{ds}+\frac{\partial^2 A}{\partial x^2}\frac{dx}{ds}\\
&= \frac{d}{ds}\left(\frac{\partial A}{\partial x}\right),
\end{aligned}$$

in the complex form

$$\overset{\nu}{T}_x+i\overset{\nu}{T}_y=2i\frac{d}{ds}\left(\frac{\partial A}{\partial \bar{z}}\right).$$

Substitution from equations (4.2.4) and (4.2.6) into this equation leads to

$$\overset{\nu}{T}_x+i\overset{\nu}{T}_y=i\frac{d}{ds}\{\varphi(z)+z\,\overline{\varphi'(z)}+\overline{\psi(z)}\}\,. \tag{4.2.9}$$

It is sometimes required to be concerned with the force acting on an element ds of the arc as divided into the normal component Nds and the tangential component Tds with respect to the arc. If these components are again understood to be of the force exerted by the region on the positive side of the normal, the complex form $(T+iN)ds$ is to be obtained from $(\overset{\nu}{T}_x+i\overset{\nu}{T}_y)ds$ by rotation of coordinate axes, we then have easily

$$T+iN=ie^{-i(t,x)}\frac{d}{ds}\{\varphi(z)+z\,\overline{\varphi'(z)}+\overline{\psi(z)}\}\,. \tag{4.2.10}$$

From equation (4.2.9) the total force $(X,\,Y)$ per unit length, exerted across AB by the region on the positive side of the normal upon the region on the negative side of it can be obtained by integration with respect to s from A to B along the arc. It is convenient to write the force in the form

$$X+iY=i\{\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}\}\,, \tag{4.2.11}$$

for the variable point B with coordinate z and some fixed point A. This is a similar manner adopted in equation (4.1.9), and we have

$$\overset{\nu}{T}_x+i\overset{\nu}{T}_y=\frac{d}{ds}(X+iY)\,, \tag{4.2.9$'$}$$

$$T+iN=e^{-i(t,x)}\frac{d}{ds}(X+iY)\,. \tag{4.2.10$'$}$$

Applications of theory of sectionally holomorphic functions to the dislocation problems as plane strain deformation will be made in the same way as in Muskhelishvili (1953 a, § 109) in which an application to the problem of insertion of the same material into a hole is treated of as due to D. I. Sherman.

As before we denote by L the union of non-intersecting smooth arcs in the plane on which the discontinuity in displacement is given as

$$g(\zeta)=\Delta(u+iv)=\{u^+(\zeta)+iv^+(\zeta)\}-\{u^-(\zeta)+iv^-(\zeta)\}\,, \tag{4.2.12}$$

then from equation (4.2.7)

$$\kappa\varphi^+(\zeta)-\zeta\,\overline{\varphi'^+(\zeta)}-\overline{\psi^+(\zeta)}=\kappa\varphi^-(\zeta)-\zeta\,\overline{\varphi'^-(\zeta)}-\overline{\psi^-(\zeta)}+2\mu g(\zeta) \quad \text{on } L. \tag{4.2.13}$$

Since we are concerned in the case of continuous tractions across L in this section, we have the condition

$$\varphi^+(\zeta)+\zeta\,\overline{\varphi'^+(\zeta)}+\overline{\psi^+(\zeta)}=\varphi^-(\zeta)+\zeta\,\overline{\varphi'^-(\zeta)}+\overline{\psi^-(\zeta)} \quad \text{on } L, \tag{4.2.14}$$

from equation (4.2.11).

Adding (4.2.13) and (4.2.14), we find

$$\varphi^+(\zeta)-\varphi^-(\zeta)=\frac{2\mu}{\kappa+1}g(\zeta) \quad \text{on } L. \tag{4.2.15}$$

Taking the complex conjugate of equation (4.2.14), we have

$$\psi^+(\zeta)-\psi^-(\zeta)=-\overline{\{\varphi^+(\zeta)-\varphi^-(\zeta)\}}-\bar{\zeta}\{\varphi'^+(\zeta)-\varphi'^-(\zeta)\}\,. \tag{4.2.16}$$

In the first place we treat of the case when the body occupies the entire plane.

We assume that the field will be expressed by means of sectionally holomorphic functions in the entire plane with the line of discontinuity L, and that $u+iv$ in (4.2.7) and $X+iY$ in (4.2.11) have no poles and vanish at infinity. On this assumption $\varphi(z)$ must be a sectionally holomorphic function in the entire plane with line of discontinuity L and from (4.2.15)

$$\varphi(z)=\frac{2\mu}{\kappa+1}\frac{1}{2\pi i}\int_L \frac{g(\zeta)}{\zeta-z}d\zeta \tag{4.2.17}$$

and hence

$$\begin{aligned}\varphi'(z)&=\frac{2\mu}{\kappa+1}\frac{1}{2\pi i}\int_L \frac{g(\zeta)}{(\zeta-z)^2}d\zeta\\ &=\frac{2\mu}{\kappa+1}\left\{-\frac{1}{2\pi i}\left[\frac{g(\zeta)}{\zeta-z}\right]_{\zeta=a}^{\zeta=b}+\frac{1}{2\pi i}\int_L \frac{g'(\zeta)}{\zeta-z}d\zeta\right\},\end{aligned} \tag{4.2.18}$$

where $\quad g'(\zeta)=\frac{d}{d\zeta}g(\zeta)$.

From (4.2.15) and (4.2.18) the left-hand side of equation (4.2.16) may be found to be in the form

$$\psi^+(\zeta)-\psi^-(\zeta)=\frac{2\mu}{\kappa+1}\{-\overline{g(\zeta)}-\bar{\zeta}g'(\zeta)\}. \tag{4.2.19}$$

Owing to the assumption that the expressions of $u+iv$ and $X+iY$ contain no poles in the entire plane, $\bar{z}\varphi'(z)+\psi(z)$ as well as $\varphi(z)$ should have no poles, as seen from (4.2.7) and (4.2.11). Therefore the sectionally holomorphic function which will be obtained on the formula (4.2.19) should be added by the terms which cancel the poles of $\varphi'(z)$ on the right-hand side of equation (4.2.18). Thus we obtain

$$\psi(z)=\frac{2\mu}{\kappa+1}\frac{1}{2\pi i}\left\{-\int_L \frac{\overline{g(\zeta)}}{\zeta-z}d\zeta-\int_L \frac{\bar{\zeta}g'(\zeta)}{\zeta-z}d\zeta+\left[\frac{\bar{\zeta}g(\zeta)}{\zeta-z}\right]_{\zeta=a}^{\zeta=b}\right\},$$

which may be written in the form

$$\psi(z)=\frac{2\mu}{\kappa+1}\left\{-\frac{1}{2\pi i}\int_L \frac{\overline{g(\zeta)}}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_L g(\zeta)d\left(\frac{\bar{\zeta}}{\zeta-z}\right)\right\}. \tag{4.2.20}$$

With (4.2.18) and (4.2.20) we have

$$\bar{z}\varphi'(z)+\psi(z)=\frac{2\mu}{\kappa+1}\left[-\frac{1}{2\pi i}\int_L\frac{\overline{g(\zeta)}}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_L g(\zeta)d\left(\frac{\bar{\zeta}-\bar{z}}{\zeta-z}\right)\right]. \quad (4.2.21)$$

Finally, for the displacement and the resultant force on arcs we obtain from equations (4.2.7) and (4.2.11)

$$u+iv=\frac{1}{\kappa+1}\left[\frac{\kappa}{2\pi i}\int_L\frac{g(\zeta)}{\zeta-z}d\zeta+\overline{\left\{\frac{1}{2\pi i}\int_L\frac{\overline{g(\zeta)}}{\zeta-z}d\zeta\right\}}-\overline{\left\{\frac{1}{2\pi i}\int_L g(\zeta)d\left(\frac{\bar{\zeta}-\bar{z}}{\zeta-z}\right)\right\}}\right] \quad (4.2.22)$$

$$X+iY=\frac{2\mu i}{\kappa+1}\left[\frac{1}{2\pi i}\int_L\frac{g(z)}{\zeta-z}d\zeta-\overline{\left\{\frac{1}{2\pi i}\int_L\frac{\overline{g(\zeta)}}{\zeta-z}d\zeta\right\}}+\overline{\left\{\frac{1}{2\pi i}\int_L g(\zeta)d\left(\frac{\bar{\zeta}-\bar{z}}{\zeta-z}\right)\right\}}\right]. \quad (4.2.23)$$

The last term in the spuare brackets on the right-hand side of (4.2.22) or (4.2.23) is the complex conjugate of the last one in equation (4.2.21); if we denote the latter by $F(z)$,

$$F(z)=\frac{1}{2\pi i}\int_L g(\zeta)d\left(\frac{\bar{\zeta}-\bar{z}}{\zeta-z}\right). \quad (4.2.24)$$

Putting

$$\zeta-z=Re^{i\alpha}, \quad (4.2.25)$$

$F(z)$ may be written in the form

$$F(z)=\frac{1}{2\pi i}\int_L g(\zeta)de^{-2i\alpha}. \quad (4.2.26)$$

It may be shown that $F(z)$ has no discontinuity when z passes across the line L. Take a point z in a vicinity of L and consider the integral (4.2.26) when z passes across L to the other side along a line l through ζ_0 on L, other than its ends. If we denote by A_1B_1 a sufficiently small part of L including ζ_0 in it, it is clear that the contribution to the integral (4.2.26) from the part of L other than A_1B_1 is continuous with respect to z. On the other hand owing to the assumption that $g(\zeta)$ satisfies H condition on L, it is clear that $g(\zeta)$ is bounded on the part A_1B_1, and we have

$$\left|\int_{\zeta=a_1}^{\zeta=b_1} g(\zeta)de^{-2i\alpha}\right|\leqq C\left|\left[e^{-2i\alpha}\right]_{\alpha=\alpha_a}^{\alpha=\alpha_b}\right|,$$

where C is a constant and α_a, α_b are the values of α at $\zeta=a_1$, b_1

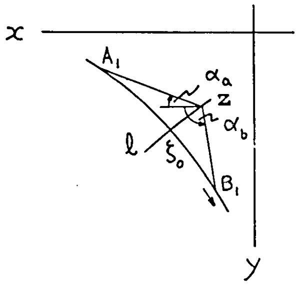


Fig. 7. Az z tends to ζ_0, α_b tends to $\alpha_a \pm \pi$ and $F(z)$ shows no discontinuity across the smooth arc.

respectively. Since α_b tends to $\alpha_a \pm \pi$ in this inequality, as z tends to ζ_0 on the smooth line L (Fig. 7), the contribution to the integral (4.2.26) from the arc A_1B_1 must become zero. Hence $F(z)$ shows no discontinuity across L.

It is also seen from equation (4.2.26) that $F(z)$ vanishes at infinity, since the angle subtended by L tends to zero as z tends to infinity.

Thus $u+iv$ in equation (4.2.22) and $X+iY$ in equation (4.2.23) satisfy the required conditions on L, that is, the given discontinuity in equation (4.2.12) for $u+iv$ and the continuity for $X+iY$ in the limit values from the left and right sides of L, as well as the condition at infinity.

We now consider the case when the body occupies the semi-infinite plane $y \geqq 0$ with a free boundary $y=0$, using the method in Moriguti (1957, p. 71).

On the functions $\varphi(z)$ and $\psi(z)$ obtained in the case of infinite region, which we write as $\varphi_0(z)$ and $\psi_0(z)$ in what follows, we introduce a function $\chi_0(z)$ which must be holomorphic whenever both $\varphi_0(z)$ and $\psi_0(z)$ are holomorphic, by the formula

$$\chi_0(z) = z\varphi_0'(z) + \psi_0(z)\,. \tag{4.2.27}$$

Defining $\varphi_1(z)$ and $\chi_1(z)$ by

$$\begin{cases} \varphi_1(z) = -\overline{\chi_0(\bar{z})} \\ \chi_1(z) = -\overline{\varphi_0(\bar{z})} \end{cases} \tag{4.2.28}$$

and a function $\psi_1(z)$, which must be holomorphic whenever both $\varphi_1(z)$ and $\chi_1(z)$ are holomorphic, in the formula

$$\chi_1(z) = z\varphi_1'(z) + \psi_1(z)\,, \tag{4.2.29}$$

we have the equation

$$\begin{aligned} &\{\varphi_0(z)+\varphi_1(z)\} + z\{\overline{\varphi_0'(z)+\varphi_1'(z)}\} + \{\overline{\psi_0(z)+\psi_1(z)}\} \\ &= \{\varphi_0(z)-\varphi_0(\bar{z})\} + \{\overline{\chi_0(z)}-\overline{\chi_0(\bar{z})}\} + (z-\bar{z})\overline{\{\varphi_0'(z)+\varphi_1'(z)\}} \end{aligned}$$

which vanishes on the boundary where $z=\bar{z}$.

The solution in the semi-infinite plane may then be obtained by putting

$$\begin{cases} \varphi(z) = \varphi_0(z) + \varphi_1(z) \\ \psi(z) = \psi_0(z) + \psi_1(z) , \end{cases} \tag{4.2.30}$$

by which the boundary condition for $y=0$ is satisfied in the form

$$X + iY = 0 . \qquad (y=0)$$

From equations (4.2.27), (4.2.28), and (4.2.29) we have

$$\begin{aligned} \varphi_1(z) &= -\{z\overline{\varphi_0'(\bar{z})} + \overline{\psi_0(\bar{z})}\} , \\ z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} &= \chi_1(z) + (z-\bar{z})\overline{\varphi_1'(z)} \\ &= -\varphi_0(\bar{z}) - (z-\bar{z})\{\varphi_0'(\bar{z}) + \bar{z}\varphi_0''(\bar{z}) + \psi_0'(\bar{z})\} , \end{aligned}$$

and obtain from (4.2.7) and (4.2.11),

$$\begin{aligned} 2\mu(u+iv) = \kappa\{\varphi_0(z) - z\overline{\varphi_0'(\bar{z})} - \overline{\psi_0(\bar{z})}\} \\ -\{-\varphi_0(\bar{z}) + z\overline{\varphi_0'(z)} + \overline{\psi_0(z)}\} \\ +(z-\bar{z})\{\varphi_0'(\bar{z}) + \bar{z}\varphi_0''(\bar{z}) + \psi_0'(\bar{z})\} , \end{aligned} \tag{4.2.31}$$

$$\begin{aligned} X + iY = i[\{\varphi_0(z) - z\overline{\varphi_0'(\bar{z})} - \overline{\psi_0(\bar{z})}\} \\ +\{-\varphi_0(\bar{z}) + z\overline{\varphi_0'(z)} + \overline{\psi_0(z)}\} \\ -(z-\bar{z})\{\varphi_0'(\bar{z}) + \bar{z}\varphi_0''(\bar{z}) + \psi_0'(\bar{z})\}] , \end{aligned} \tag{4.2.32}$$

where

$$\begin{cases} \varphi_0(z) = \dfrac{2\mu}{\kappa+1}\dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{g}{\zeta - z}d\zeta , \\ \psi_0(z) = \dfrac{2\mu}{\kappa+1}\left\{-\dfrac{1}{2\pi i}\displaystyle\int_L \dfrac{\bar{g}}{\zeta - z}d\zeta + \dfrac{1}{2\pi i}\displaystyle\int_L g d\left(\dfrac{\bar{\zeta}}{\zeta - z}\right)\right\} . \end{cases} \tag{4.2.33}$$

The results may also be written in the forms as

$$\begin{aligned} u+iv = \frac{\kappa}{\kappa+1}\left[\frac{1}{2\pi i}\int_L \frac{g}{\zeta-z}d\zeta + \overline{\left\{\frac{1}{2\pi i}\int_L \frac{\bar{g}}{\zeta-\bar{z}}d\zeta\right\}} + \frac{1}{2\pi i}\int_L \bar{g}d\,\frac{\zeta-\bar{\zeta}}{\bar{\zeta}-z}\right] \\ + \frac{1}{\kappa+1}\left[\frac{1}{2\pi i}\int_L \frac{g}{\zeta-\bar{z}}d\zeta + \overline{\left\{\frac{1}{2\pi i}\int_L \frac{\bar{g}}{\zeta-z}d\zeta\right\}} + \frac{1}{2\pi i}\int_L \bar{g}d\,\frac{\zeta-\bar{\zeta}}{\bar{\zeta}-\bar{z}}\right. \\ \left. +(z-\bar{z})\left\{\frac{1}{2\pi i}\int_L \bar{g}d\,\frac{1}{\zeta-\bar{z}} - \frac{1}{2\pi i}\int_L \bar{g}d\,\frac{1}{\bar{\zeta}-\bar{z}} - \frac{1}{2\pi i}\int_L gd\,\frac{\zeta-\bar{\zeta}}{(\zeta-\bar{z})^2}\right\}\right] \end{aligned} \tag{4.2.34}$$

$$X+iY=\frac{2\mu i}{\kappa+1}\left[\frac{1}{2\pi i}\int_L\frac{g}{\zeta-z}d\zeta+\overline{\left\{\frac{1}{2\pi i}\int_L\frac{\bar{g}}{\zeta-\bar{z}}d\zeta\right\}}+\frac{1}{2\pi i}\int_L\bar{g}d\frac{\zeta-\bar{\zeta}}{\bar{\zeta}-z}\right]$$
$$-\frac{2\mu i}{\kappa+1}\left[\frac{1}{2\pi i}\int_L\frac{g}{\zeta-\bar{z}}d\zeta+\overline{\left\{\frac{1}{2\pi i}\int_L\frac{\bar{g}}{\zeta-z}d\zeta\right\}}+\frac{1}{2\pi i}\int_L\bar{g}d\frac{\zeta-\bar{\zeta}}{\bar{\zeta}-\bar{z}}\right.$$
$$\left.+(z-\bar{z})\left\{\frac{1}{2\pi i}\int_L\bar{g}d\frac{1}{\zeta-\bar{z}}-\frac{1}{2\pi i}\int_L\bar{g}d\frac{1}{\bar{\zeta}-\bar{z}}-\frac{1}{2\pi i}\int_L gd\frac{\zeta-\bar{\zeta}}{(\zeta-\bar{z})^2}\right\}\right]. \tag{4.2.35}$$

Starting from the second equation in (3.29), we can arrive at the expression (4.2.34) in the case when $\kappa=2$, i.e. $\lambda=\mu$, by the help of (3.25), (4.1.24) and the relations

$$\begin{cases} g=\Delta u_2+i\Delta u_3 \\ \bar{g}=\Delta u_2-i\Delta u_3\,. \end{cases} \tag{4.2.36}$$

5. Some Typical Problems in an Infinite Medium

In this section we collect and arrange some simple typical two-dimensional problems concerning the fields caused by shearings or ruptures on the basis of previous expressions.

We take coordinate axes of reference (x, y) in an infinite isotropic and homogeneous medium, and assume that the discontinuity in displacement takes place along the real axis.

For System I we have from equations (4.1.5), (4.1.8), (4.1.6) and (4.1.19), with $z=x+iy$,

$$\mu u_1=\mathrm{Re}\{\chi(z)\} \tag{5.1}$$

$$\overset{\nu}{T}_1=-\mathrm{Im}\{e^{i(t,x)}\chi'(z)\}=\mathrm{Re}\{e^{i(\nu,x)}\chi'(z)\} \tag{5.2}$$

$$\tau_{1x}-i\tau_{1y}=\chi'(z) \tag{5.3}$$

and

$$\chi(z)=\frac{\mu}{2\pi i}\int_L\frac{\Delta u_1(\xi)}{\xi-z}d\xi \tag{5.4}$$

where the integral is taken over a segment L of real axis on which the discontinuity in displacement $\Delta u_1(\xi)$ is given as

$$\Delta u_1(\xi)=u_1^+(\xi)-u_1^-(\xi) \tag{5.5}$$

corresponding to equation (4.1.15).

Since $\chi'(z)$ may be transformed as

$$\begin{aligned}\chi'(z)&=\frac{\mu}{2\pi i}\int_L \frac{\Delta u_1(\xi)}{(\xi-z)^2}d\xi\\&=-\frac{\mu}{2\pi i}\int_L \Delta u_1(\xi)d\frac{1}{\xi-z}\\&=-\frac{\mu}{2\pi i}\left[\frac{\Delta u_1}{\xi-z}\right]_{\xi=a}^{\xi=b}+\frac{\mu}{2\pi i}\int_L\frac{\Delta u_1'}{\xi-z}d\xi\,,\end{aligned}$$

with the help of the derivative of $\Delta u(\xi)$ with respect to ξ

$$\Delta u_1'=\frac{d}{d\xi}\Delta u_1(\xi)\,, \tag{5.6}$$

stress components may be expressed in the form

$$\tau_{1x}-i\tau_{1y}=-\frac{\mu}{2\pi i}\left[\frac{\Delta u_1}{\xi-z}\right]_{\xi=a}^{\xi=b}+\frac{\mu}{2\pi i}\int_L\frac{\Delta u_1'}{\xi-z}d\xi\,. \tag{5.7}$$

For System II, we have from equations (4.2.22) and (4.2.23) the expressions

$$u+iv=\frac{1}{\kappa+1}\left[\frac{\kappa}{2\pi i}\int_L\frac{g}{\xi-z}d\xi-\frac{1}{2\pi i}\int_L\frac{g}{\xi-\bar z}d\xi-\frac{(z-\bar z)}{2\pi i}\int_L \bar g d\left(\frac{1}{\xi-\bar z}\right)\right] \tag{5.8}$$

$$X+iY=\frac{2\mu i}{\kappa+1}\left[\frac{1}{2\pi i}\int_L\frac{g}{\xi-z}d\xi+\frac{1}{2\pi i}\int_L\frac{g}{\xi-\bar z}d\xi+\frac{(z-\bar z)}{2\pi i}\int_L \bar g d\left(\frac{1}{\xi-\bar z}\right)\right] \tag{5.9}$$

where integrals are taken over a segment L of real axis on which the discontinuity in displacement is given by

$$g=g(\xi)=u^+(\xi)-u^-(\xi)+i\{v^+(\xi)-v^-(\xi)\}=\Delta u+i\Delta v\,, \tag{5.10}$$

corresponding to equation (4.2.12).

On the left-hand sides of equations (5.8) and (5.9), however, it is rather convenient to write in the form

$$\begin{cases}2\mu(u+iv)=\kappa\varphi(z)-\varphi(\bar z)-(z-\bar z)\overline{\varphi'(z)}\\ X+iY=i\{\varphi(z)+\varphi(\bar z)+(z-\bar z)\overline{\varphi'(z)}\}\end{cases} \tag{5.11}$$

where

$$\varphi(z)=\frac{2\mu}{\kappa+1}\frac{1}{2\pi i}\int_L \frac{g}{\xi-z}d\xi \tag{5.12}$$

as in equation (4.2.17). These expressions are of the type in the case of tensile cracks appearing in the book of Green and Zerna.

As in equation (4.2.5) we have

$$\tau_{xx}+\tau_{yy}=2\{\varphi'(z)+\overline{\varphi'(z)}\}\,. \tag{5.13}$$

Taking a line element ds at the point (x, y) in the direction of y-axis or x-axis and applying equation (4.2.9)$'$ to the case, we obtain

$$\begin{cases} \tau_{xx}+i\tau_{xy}=-\dfrac{\partial}{\partial y}(X+iY) \\ \tau_{yy}-i\tau_{xy}=-i\dfrac{\partial}{\partial x}(X+iY)\,. \end{cases} \tag{5.14}$$

By use of these formulae, from (5.11) we obtain

$$\begin{cases} \tau_{xx}+i\tau_{xy}=\varphi'(z)-\varphi'(\bar{z})+2\overline{\varphi'(z)}-(z-\bar{z})\overline{\varphi''(z)} \\ \tau_{yy}-i\tau_{xy}=\varphi'(z)+\varphi'(\bar{z})+(z-\bar{z})\overline{\varphi''(z)}\,. \end{cases} \tag{5.15}$$

It is sometimes more useful to have stress components expressed in integral forms containing the derivative of $g(\xi)$ with respect to ξ,

$$g'=\frac{d}{d\xi}g(\xi)=\frac{d}{d\xi}(\Delta u+i\Delta v)\,. \tag{5.16}$$

Writing $\varphi'(z)$ in the form

$$\varphi'(z)=\frac{2\mu}{\kappa+1}\left\{-\frac{1}{2\pi i}\left[\frac{g}{\xi-z}\right]_{\xi=a}^{\xi=b}+\frac{1}{2\pi i}\int_L \frac{g'}{\xi-z}d\xi\right\},$$

or differentiating (5.9) with respect to x, with transformation by integration by parts, we obtain from equations (5.13) and (5.15)

$$\begin{aligned} \tau_{xx}+\tau_{yy}=\frac{4\mu}{\kappa+1}\Bigg\{&-\frac{1}{2\pi i}\left[\frac{g}{\xi-z}\right]_{\xi=a}^{\xi=b}+\frac{1}{2\pi i}\left[\frac{\bar{g}}{\xi-\bar{z}}\right]_{\xi=a}^{\xi=b} \\ &+\frac{1}{2\pi i}\int_L \frac{g'}{\xi-z}d\xi-\frac{1}{2\pi i}\int_L \frac{\bar{g}'}{\xi-\bar{z}}d\xi\Bigg\}, \end{aligned} \tag{5.17}$$

$$\begin{aligned} \tau_{yy}-i\tau_{xy}=\frac{2\mu}{\kappa+1}\Bigg\{&-\frac{1}{2\pi i}\left[\frac{g}{\xi-z}\right]_{\xi=a}^{\xi=b}-\frac{1}{2\pi i}\left[\frac{g}{\xi-\bar{z}}\right]_{\xi=a}^{\xi=b}+\frac{(z-\bar{z})}{2\pi i}\left[\frac{\bar{g}}{(\xi-\bar{z})^2}\right]_{\xi=a}^{\xi=b} \\ &+\frac{1}{2\pi i}\int_L \frac{g'}{\xi-z}d\xi+\frac{1}{2\pi i}\int_L \frac{g'}{\xi-\bar{z}}d\xi-\frac{(z-\bar{z})}{2\pi i}\int_L \frac{\bar{g}'}{(\xi-\bar{z})^2}d\xi\Bigg\}, \end{aligned} \tag{5.18}$$

where

$$\bar{g}'=\overline{\left(\frac{dg}{d\xi}\right)}=\frac{d}{d\xi}\,\bar{g}=\frac{d}{d\xi}(\Delta u-i\Delta v)\,.$$

5.1. Screw and edge dislocations

First we consider the cases of constant discontinuity in displacements. If we put

$$\begin{aligned}\Delta u_1&=0 &&(\xi<-R,\ 0<\xi)\\ &=\text{const.}\neq 0 &&(-R\leqq\xi\leqq 0)\end{aligned}\tag{5.1.1}$$

in equation (5.4), we have

$$\chi(z)=\frac{\mu\Delta u_1}{2\pi i}\Big[\log(\xi-z)\Big]_{\xi=-R}^{\xi=0}.$$

As seen from this equation and the equation (5.3), the contribution to the stress field from the term for $\xi=-R$ tends to zero as R tends to infinity. Putting

$$B=\Delta u_1\tag{5.1.2}$$

and neglecting the term for $\xi=-R$, we have

$$\chi(z)=\frac{\mu B}{2\pi i}\log z\,,\tag{5.1.3}$$

omitting a constant term. Hence we obtain from equation (5.1)

$$u_1=\frac{B}{2\pi}\arg z=\frac{B}{2\pi}\arctan\frac{y}{x}\,,\tag{5.1.4}$$

which is the displacement field referred to as due to a screw dislocation with dislocation line along x_1-axis with the Burgers vector B. From equation (5.1.3) we have

$$\tau_{1x}-i\tau_{1y}=\frac{\mu B}{2\pi i}\frac{1}{z}\,,$$

and hence

$$\begin{cases} \tau_{1x} = -\dfrac{B}{2\pi}\dfrac{y}{x^2+y^2} \\ \tau_{1y} = \dfrac{B}{2\pi}\dfrac{x}{x^2+y^2} \end{cases} \tag{5.1.5}$$

Similarly, putting

$$\begin{aligned} g(\xi) = \varDelta u &= 0 \qquad\qquad (\xi < -R,\ 0 < \xi) \\ &= \text{const.} \neq 0 \quad (-R \leqq \xi \leqq 0) \end{aligned} \tag{5.1.6}$$

in equation (5.12), we have

$$\varphi(z) = \frac{2\mu}{\kappa+1}\frac{\varDelta u}{2\pi i}\Big[\log(\xi - z)\Big]_{\xi=-R}^{\xi=0}.$$

Neglecting the term for $\xi = -R$ in this expression for the reason that the contribution to stress field from the term for $\xi = -R$ tends to zero as R tends to infinity, and putting

$$B = \varDelta u \tag{5.1.7}$$

we obtain, omitting a constant term,

$$\varphi(z) = \frac{2\mu}{\kappa+1}\frac{B}{2\pi i}\log z\,, \tag{5.1.8}$$

$$\begin{cases} 2\mu(u+iv) = \dfrac{2\mu}{\kappa+1}\dfrac{B}{2\pi i}\left(\kappa\log z - \log\bar{z} + \dfrac{z}{\bar{z}}\right) \\ X+iY = \dfrac{2\mu}{\kappa+1}\dfrac{B}{2\pi}\left(\log z + \log\bar{z} - \dfrac{z}{\bar{z}}\right). \end{cases} \tag{5.1.9}$$

If we separate the expression for $u+iv$ into the real and imaginary parts, we have

$$\begin{cases} u = \dfrac{B}{2\pi}\left(\dfrac{2}{\kappa+1}\dfrac{xy}{x^2+y^2} + \arctan\dfrac{y}{x}\right) \\ v = \dfrac{B}{2\pi}\left(-\dfrac{1}{\kappa+1}\dfrac{x^2-y^2}{x^2+y^2} - \dfrac{\kappa-1}{\kappa+1}\log\sqrt{x^2+y^2}\right) \end{cases} \tag{5.1.10}$$

which is the displacement field referred to as due to an edge dislocation along x_1-axis with the Burgers vector B.

For the stress components due to this edge dislocation, according to equations (5.15) and (5.1.8), for instance, we obtain

$$\begin{cases} \tau_{xx} = -\dfrac{\mu}{\kappa+1}\dfrac{B}{2\pi}\dfrac{y(3x^2+y^2)}{(x^2+y^2)^2} \\ \tau_{xy} = \dfrac{\mu}{\kappa+1}\dfrac{B}{2\pi}\dfrac{x(x^2-y^2)}{(x^2+y^2)^2} \\ \tau_{yy} = \dfrac{\mu}{\kappa+1}\dfrac{B}{2\pi}\dfrac{y(x^2-y^2)}{(x^2+y^2)^2}\,. \end{cases} \tag{5.1.11}$$

If we put

$$\Delta u_1(a) = \Delta u_1(b) = 0$$

in equation (5.7), the shear stress τ_{1y} in the slip plane due to $\Delta u_1(\xi)$ is obtained as

$$\tau_{1y} = \frac{\mu}{2\pi}\int\frac{\Delta u_1'}{\xi - x}d\xi\,; \tag{5.1.12}$$

similarly, putting

$$g(a) = g(b) = 0$$

in equation (5.18), the shear stress τ_{xy} in the slip plane due to $g = \Delta u(\xi)$ is obtained as

$$\tau_{xy} = \frac{2}{\kappa+1}\frac{\mu}{2\pi}\int\frac{\Delta u'}{\xi - x}d\xi\,. \tag{5.1.13}$$

These equations (5.1.12) and (5.1.13) are referred to as the shear stresses due to the continuous distribution of infinitesimal screw and edge dislocations lying on the slip plane. The total length of the Burgers vectors of the infinitesimal dislocations lying between ξ and $\xi + d\xi$ are represented by $\Delta u_1' d\xi$ and $\Delta u' d\xi$ for screw and edge dislocations respectively. In these cases, the reverse problem how to calculate the dislocation distribution corresponding to the specified shear stress over the slip plane has been solved (e.g. Leibfried 1951, Head and Louat 1955) and used by Weertman (1964) on geophysical problems.

5.2. Strain energy changes due to shear cracks

J. A. Steketee (1958 b) has shown that the crack problems can be put in terms of dislocation theory. He constructed, as an example, the formulae for the displacement and stress fields of the tensile crack of Griffith type, by starting from the computation of the two-dimensional Somigliana tensor.

Here we shall take up the problem of strain energy changes due to two-dimensional shear cracks by utilizing the foregoing expressions. Of two-dimensional shear cracks, L. Knopoff's and A. T. Starr's which belong to the cases of System I and System II respectively are the most typical ones.

In the calculation of the field containing a surface on which shear stress vanishes, Knopoff (1958) took advantage of the analogy with electrostatics. He evaluated the energy difference between two static states: the state of the solid in a uniform, flawless condition and the state of the solid after a crack has been introduced, by use of a formula from the book of Stratton (1941).

If we follow Knopoff's analogy, however, a question arises in the calculation of strain energy difference: Why it contains no mention about the boundary condition at infinity? Since the formula which Knopoff (1958) referred to, equation (20), can be derived for a field caused by a fixed set of electric charges as seen in Stratton (1941), it seems that the source or the boundary condition that caused the initial elastic field should be considered when applying the formula to the case. In reality, Starr (1928) showed in the first part of the paper, the result of calculation that the strain energy difference between the state of the solid in a uniform, flawless condition and the state of the solid with a crack is in the direction to be increased by the existence of the crack. In order to avoid this result, Starr takes the second approach to the problem in the paper, by considering that the boundary at infinity is fixed while the stress on the crack slowly diminished. If we proceed along the Knopoff's method of calculation, the same as the first way in Starr, we are to obtain the result that the energy difference between the two states is zero, in the case of System I.

In connection with the boundary condition, we recall the theorem of Colonnetti which states that the strain energy associated with the dislocation is independent of the initial state of stress which may exist in the body, if no additional forces are applied on the outer surface when the dislocation was produced (cf. Steketee 1958 b, Nabarro 1952).

Concerning this theorem and its application to the problem of facture in geophysics, Steketee (1958 b) gives a very interesting account as follows:

"...we have to recognize that the surface of the earth is essentially free and if a dislocation is made under those circumstances, Colonnetti's Theorem shows that the strain energy can only increase.

"In this dilemma one may perhaps see an additional argument for the generally accepted idea that earthquakes are not world-wide phenomena but that their cause lies in local conditions; if one considers the whole earth it is difficult to see any restriction on the displacement of its surface; on the other hand if a particular region is considered it is easy to imagine that certain parts may obstruct or prevent the displacement of others and give rise in this way to boundary conditions of the type in which parts of the surface are no longer free, creating in that way the possibility of escaping Collonnetti's Theorem."

If we take up a bounded region including the crack and consider the decrease in strain energy plus the work done slowly on this region by the outer region, we can get out of the consideration of the boundary condition at infinity. We shall denote this sum by A in what follows. This quantity A is the proper one that may be interesting when compared with the energy radiated in waves in the case of rapid formation of the crack. The increase of A can be considered to be balanced by the increase of surface energy of the crack in the theory of Griffith crack (e.g. Yokobori § 9.3).

In the case of calculation of A, that is $A(I)$ and $A(II)$, the initial fields are taken as being the same as Konoff's and Starr's for System I and System II respectively.

As for the additional fields to be superposed to these initial fields in order to obtain the fields containing cracks, we shall treat of two types of discontinuity in displacement on the crack: one of which is the type generating the additional field to make the fields of Knopoff's and Starr's for System I and System II respectively, but generating the stress field one component of which tends to infinity at the ends of the crack when the approach is from the outside of the crack; while the other type of discontinuity in displacement is the type generating finite stresses everywhere.

Now we consider the strain energy of a body occupied by a region $S+L_0$ containing interior boundary or the crack L across which discontinuity in displacement is specified.

In the same way as in (1.1) and (1.2), if e^0_{ij}, τ^0_{ij}, and w^0 define the initial equilibrium state and e_{ij}, τ_{ij}, and w the state due to dislocations in the three-dimensional case, we have, for the strain energy per unit volume w' in the final state, the expression

$$
\begin{aligned}
w' &= \frac{1}{2}(\tau^0_{ij}+\tau_{ij})(e^0_{ij}+e_{ij}) \\
&= w^0+w+\frac{1}{2}(\tau^0_{ij}e_{ij}+\tau_{ij}e^0_{ij})\,.
\end{aligned}
\tag{5.2.1}
$$

The expression between parentheses in (5.2.1) may be written, in the cases we shall treat of, in the form

$$\frac{1}{2}(\tau_{ij}^0 e_{ij} + \tau_{ij} e_{ij}^0) = \tau_{ij}^0 e_{ij} = \tau_{ij} e_{ij}^0 . \tag{5.2.2}$$

In our two-dimensional problem w' may be divided into two parts

$$w' = w'(I) + w'(II) , \tag{5.2.3}$$

where $w'(I)$ and $w'(II)$ are the strain energy per unit volume in the final state for System I and System II respectively.

Corresponding to equation (5.2.1) $w'(I)$ may be written

$$\begin{aligned} w'(I) &= \frac{1}{2}(\tau_{1l}^0 + \tau_{1l})(e_{1l}^0 + e_{1l}) \\ &= \frac{1}{2}(\tau_{1l}^0 + \tau_{1l})\left(\frac{\partial u_1^0}{\partial x_l} + \frac{\partial u_1}{\partial x_l}\right) . \end{aligned}$$

Applying this formula to a region $S + L_0$ with the interior boundary or the crack L, in the same manner as in the derivation of equation (2.21), and using the equation of equilibrium and the general relation (5.2.2), we obtain, by the help of divergence theorem, the total strain energy per unit length as

$$\begin{aligned} W'(I) &= W^0(I) + W(I) + \int_{L_0} u_1 \tau_{1l}^0 \nu_l d\sigma - \int_{L} \Delta u_1 \tau_{1l}^0 \nu_l d\sigma \\ &= W^0(I) + W(I) + \int_{L_0} u_1^0 \tau_{1l} \nu_l d\sigma , \end{aligned} \tag{5.2.4}$$

where $W^0(I)$ and $W'(I)$ are the total strain energies per unit length in the initial and final states respectively, and where the continuity of tractions across the boundary L is considered.

Using the expression of W in equation (2.21), from the first of equation (5.2.4) we have

$$\begin{aligned} W^0(I) - W'(I) &+ \frac{1}{2}\int_{L_0} u_1 \{\tau_{1l}^0 + (\tau_{1l}^0 + \tau_{1l})\} \nu_l d\sigma \\ &= \frac{1}{2}\int_{L} \Delta u_1 \{\tau_{1l}^0 + (\tau_{1l}^0 + \tau_{1l})\} \nu_l d\sigma . \end{aligned} \tag{5.2.5}$$

Since $\tau_{1l}^0 \nu_l$ and $(\tau_{1l}^0 + \tau_{1l})\nu_l$ in the integrand on the left-hand side of this equation are the forces in the x_1-direction per unit length of the boundary L_0 exerted upon the region in the initial and the final states, the integral on the left-hand side is the work done by the outer region; accordingly, the left-hand side is nothing but the quantity $A(I)$ previously defined. On the other hand if we write the integral of the right-hand side of equation (5.2.5) in the form

$$-\left[\frac{1}{2}\int_L u_1^+\{-\tau_{1l}^0-(\tau_{1l}^0+\tau_{1l})\}\nu_l d\sigma+\frac{1}{2}\int_L u_1^-\{\tau_{1l}^0+(\tau_{1l}^0+\tau_{1l})\}\nu_l d\sigma\right],$$

since $-\tau_{1l}^0\nu_l$ and $-(\tau_{1l}^0+\tau_{1l})\nu_l$ are the forces per unit length exerted upon the positive side of L, while $\tau_{1l}\nu_l$ and $(\tau_{1l}^0+\tau_{1l})\nu_l$ are those exerted upon the negative side of L, the right-hand side of equation (5.2.5) may be considered as the negative of the work done on the region by the crack, which we may well take simply as the work done on the crack. Hence we have

$$\begin{aligned} A(I)&=\text{Work done on the crack}\\ &=\frac{1}{2}\int_L \Delta u_1\{\tau_{1l}^0+(\tau_{1l}^0+\tau_{1l})\}\nu_l d\sigma\,. \end{aligned}\tag{5.2.6}$$

In a similar manner, we obtain for System II

$$\begin{aligned} A(II)&=\text{Work done on the crack}\\ &=\frac{1}{2}\int_L \Delta u_k\{\tau_{kl}^0+(\tau_{kl}^0+\tau_{kl})\}\nu_l d\sigma\,. \end{aligned}\tag{5.2.7}$$

Now we compute explicit expressions for $A(I)$ and $A(II)$. The initial uniform fields to be taken are

$$u_1^0=\frac{S}{\mu}y\,,\quad \tau_{1x}^0=0\quad\text{and}\quad \tau_{1y}^0=S\tag{5.2.8}$$

for System I and

$$\left\{\begin{aligned} &u^0=\frac{S}{2\mu}y\,,\quad v^0=\frac{S}{2\mu}x\\ &\tau_{xx}=\tau_{yy}=0\,,\quad \tau_{xy}=S \end{aligned}\right.\tag{5.2.9}$$

for System II.

For a segment of x-axis $L=(-a, a)$, as the first type of dislocation

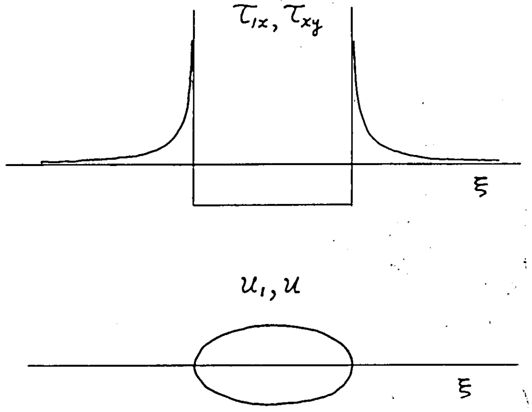


Fig. 8. Shear stress on the slip plane due to the displacement discontinuity in the form $\left\{1-\left(\frac{\xi}{a}\right)^2\right\}^{1/2}$.

we take

$$\begin{cases} \Delta u_1 = U_1\sqrt{1-\left(\dfrac{\xi}{a}\right)^2} & \text{on } L \quad \text{for System I,} \\ \Delta u = U\sqrt{1-\left(\dfrac{\xi}{a}\right)^2} & \text{on } L \quad \text{for System II,} \end{cases} \tag{5.2.10}$$

and obtain as in Appendix

$$\begin{aligned} \chi(z) &= \frac{\mu}{2\pi i}\int_L \frac{\Delta u_1}{\xi - z} d\xi \\ &= \frac{\mu U_1}{2a}\{\sqrt{a^2-z^2}+iz\} \end{aligned} \tag{5.2.11}$$

and

$$\begin{aligned} \varphi(z) &= \frac{2\mu}{\kappa+1}\frac{1}{2\pi i}\int_L \frac{\Delta u}{\xi - z} d\xi \\ &= \frac{2\mu}{\kappa+1}\frac{U}{2a}\{\sqrt{a^2-z^2}+iz\} \end{aligned} \tag{5.2.12}$$

where $\sqrt{a^2-z^2}$ is that branch, holomorphic in the plane cut along $(-a, a)$, taking the positive value on the upper side; for large $|z|$ this branch is $-iz+O(1)$. Displacements and stresses due to these dislocations may be obtained for System I by

$$\begin{cases} \mu u_1 = \mathrm{Re}\,\{\chi(z)\} \\ \tau_{1x} - i\tau_{1y} = \chi'(z) \end{cases} \tag{5.2.13}$$

as in equations (5.1) and (5.3), we have on the x-axis

$$\begin{cases} u_1^+ = \dfrac{U_1}{2a}\sqrt{a^2-x^2} & \\ & (|x| \leqq a) \\ u_1^- = -\dfrac{U_1}{2a}\sqrt{a^2-x^2} & \\ u_1 = 0 & (|x| > a) \end{cases}$$

$$\begin{cases} \tau_{1x}^+ = -\dfrac{\mu U_1}{2a}\dfrac{x}{\sqrt{a^2-x^2}} & \\ & (|x| \leqq a) \\ \tau_{1x}^- = \dfrac{\mu U_1}{2a}\dfrac{x}{\sqrt{a^2-x^2}} & \\ \tau_{1x} = 0 & (|x| > a) \end{cases}$$

$$\begin{cases} \tau_{1y} = -\dfrac{\mu U_1}{2a} & (|x| \leqq a) \\ \tau_{1y} = \dfrac{\mu U_1}{2a}\left\{\dfrac{|x|}{\sqrt{x^2-a^2}} - 1\right\} & (|x| > a) \end{cases}$$

while from equations (5.11) and (5.15) for System II we obtain on the x-axis

$$\begin{cases} 2\mu(u+iv) = \kappa\varphi(z) - \varphi(\bar{z}) \\ \tau_{xx} + i\tau_{xy} = \varphi'(z) - \varphi'(\bar{z}) + 2\overline{\varphi'(z)} \\ \tau_{yy} - i\tau_{xy} = \varphi'(z) + \varphi'(\bar{z})\,, \end{cases} \tag{5.2.14}$$

which give

$$\begin{cases} u^+ = \dfrac{U}{2a}\sqrt{a^2-x^2} & \\ & (|x| \leqq a) \\ u^- = -\dfrac{U}{2a}\sqrt{a^2-x^2} & \\ u\ = 0 & (|x| > a) \end{cases}$$

$$\begin{aligned} v &= \frac{\kappa-1}{\kappa+1}\,\frac{U}{2a}x & (|x| \leqq a) \\ &= \frac{\kappa-1}{\kappa+1}\,\frac{U}{2a}\left\{-\frac{x}{|x|}\sqrt{x^2-a^2} + x\right\} & (|x| > a) \end{aligned}$$

$$\begin{cases} \tau_{xx}^{+}=-\dfrac{4\mu}{\kappa+1}\dfrac{U}{a}\dfrac{x}{\sqrt{a^2-x^2}} & \\ & (|x|\leqq a) \\ \tau_{xx}^{-}=\dfrac{4\mu}{\kappa+1}\dfrac{U}{a}\dfrac{x}{\sqrt{a^2-x^2}} & \\ \tau_{xx}=0 & (|x|>a) \end{cases}$$

$$\begin{cases} \tau_{xy}=-\dfrac{2\mu}{\kappa+1}\dfrac{U}{a} & (|x|\leqq a) \\ \tau_{xy}=\dfrac{2\mu}{\kappa+1}\dfrac{U}{a}\left\{\dfrac{|x|}{\sqrt{x^2-a^2}}-1\right\} & (|x|>a) \\ \tau_{yy}=0\,. & \end{cases}$$

Assuming that the shear stress is only partially relieved on the crack, to a fraction γ of its initial uniform value S, as in Burridge and Knopoff (1966), we have the relations

$$S-\frac{\mu U_1}{2a}=\gamma S \qquad \text{for System I}, \tag{5.2.15}$$

$$S-\frac{2\mu}{\kappa+1}\frac{U}{a}=\gamma S \qquad \text{for System II}. \tag{5.2.16}$$

From equations (5.2.6) and (5.2.7) we obtain the work done on the crack as follows:

$$\begin{aligned} A(I)&=\int_L \Delta u_1\,\tau_{1y}^0\,dx+\frac{1}{2}\int_L \Delta u_1\tau_{1y}dx \\ &=\frac{\pi}{2}(1-\gamma^2)\frac{a^2S^2}{\mu} \\ &=\frac{\pi}{8}\left(\frac{1+\gamma}{1-\gamma}\right)\mu U_1^2\,, \end{aligned} \tag{5.2.17}$$

$$\begin{aligned} A(II)&=\int_L \Delta u\tau_{xy}^0\,dx+\frac{1}{2}\int_L \Delta u\tau_{xy}dx \\ &=\frac{\pi}{8}(1-\gamma^2)(\kappa+1)\frac{a^2S^2}{\mu} \\ &=\frac{\pi}{2}\left(\frac{1+\gamma}{1-\gamma}\right)\frac{1}{\kappa+1}\mu U^2 \end{aligned} \tag{5.2.18}$$

As seen from previous equations or in Fig. 8, the discontinuity in

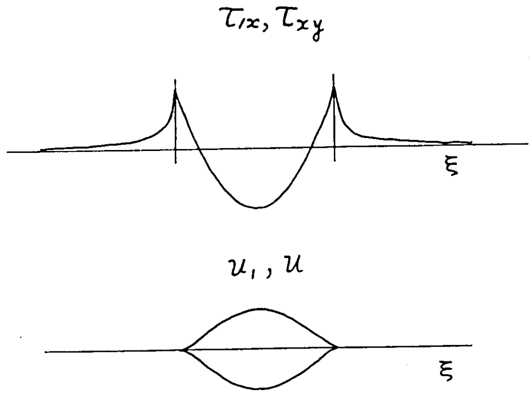


Fig. 9. Shear stress on the slip plane due to the displacement discontinuity in the form $\left\{1-\left(\frac{\xi}{a}\right)^2\right\}^{3/2}$.

displacement Δu_1 or Δu starts too sharply from the ends to be expected in actual circumstances, correspondingly the stress τ_{1y} or τ_{xy} tends to infinity as the approach to the end is from the outside of the crack on the x-axis.

Next we take an example of the crack with discontinuity in displacement starting gradually on the crack from the ends and with finite stresses everywhere; in these cases, however, the shear stress due to this type of dislocation is not constant on the crack and the resultant shear stress cannot at every point vanish.

The discontinuity in displacement is now given by

$$\left\{\begin{array}{lll} \Delta u_1 = U_1\left\{1-\left(\dfrac{\xi}{a}\right)^2\right\}^{\frac{3}{2}} & \text{on } L & \text{for System I} \\ \Delta u \ = U \left\{1-\left(\dfrac{\xi}{a}\right)^2\right\}^{\frac{3}{2}} & \text{on } L & \text{for System II}. \end{array}\right. \tag{5.2.19}$$

In place of equations (5.2.11) and (5.2.12) we then have

$$\left\{\begin{array}{l} \chi(z) = \dfrac{\mu U_1}{2a^3}\left\{(a^2-z^2)\sqrt{a^2-z^2} - i\left(z^3 - \dfrac{3}{2}a^2 z\right)\right\}, \\ \varphi(z) = \dfrac{2\mu}{\kappa+1}\ \dfrac{U}{2a^3}\left\{(a^2-z^2)\sqrt{a^2-z^2} - i\left(z^3 - \dfrac{3}{2}a^2 z\right)\right\}, \end{array}\right. \tag{5.2.20}$$

and from equation (5.2.13) we obtain on the x-axis for System I:

$$\begin{cases} u_1^+ = \dfrac{U_1}{2a^3}(a^2-x^2)^{\frac{3}{2}} & \\ & (|x| \leqq a) \\ u_1^- = -\dfrac{U_1}{2a^3}(a^2-x^2)^{\frac{3}{2}} & \\ u=0 & (|x|>a) \end{cases}$$

$$\begin{cases} \tau_{1x}^+ = -\dfrac{3\mu U_1}{2a^3} x\sqrt{a^2-x^2} & \\ & (|x| \leqq a) \\ \tau_{1x}^- = \dfrac{3\mu U_1}{2a^3} x\sqrt{a^2-x^2} & \\ \tau_{1x}=0 & (|x|>a) \end{cases}$$

$$\begin{aligned} \tau_{1y} &= \frac{3\mu U_1}{2a^3}\left(x^2-\frac{1}{2}a^2\right) && (|x| \leqq a) \\ &= \frac{3\mu U_1}{2a^3}\left\{-|x|\sqrt{x^2-a^2}+\left(x^2-\frac{1}{2}a^2\right)\right\} && (|x|>a) \end{aligned}$$

From equation (5.2.14) for System II, on the x-axis we have

$$\begin{cases} u^+ = \dfrac{U}{2a^3}(a^2-x^2)^{\frac{3}{2}} & \\ & (|x| \leqq a) \\ u^- = -\dfrac{U}{2a^3}(a^2-x^2)^{\frac{3}{2}} & \\ u=0 & (|x|>a) \end{cases}$$

$$\begin{aligned} v &= \frac{\kappa-1}{\kappa+1}\,\frac{U}{2a^3}\left(-x^3+\frac{3}{2}a^2x\right) && (|x| \leqq a) \\ &= \frac{\kappa-1}{\kappa+1}\,\frac{U}{2a^3}\left\{\frac{x}{|x|}(x^2-a^2)^{\frac{3}{2}}+\left(-x^3+\frac{3}{2}a^2x\right)\right\} && (|x|>a) \end{aligned}$$

$$\begin{cases} \tau_{xx}^+ = -\dfrac{2\mu}{\kappa+1}\,\dfrac{6U}{a^3} x\sqrt{a^2-x^2} & \\ & (|x| \leqq a) \\ \tau_{xx}^- = \dfrac{2\mu}{\kappa+1}\,\dfrac{6U}{a^3} x\sqrt{a^2-x^2} & \\ \tau_{xx}=0 & (|x|>a) \end{cases}$$

$$\tau_{xy} = \left(\frac{2\mu}{\kappa+1}\right)\frac{3U}{a^3}\left(x^2-\frac{1}{2}a^2\right) \qquad (|x| \leqq a)$$

$$=\left(\frac{2\mu}{\kappa+1}\right)\frac{3U}{a^3}\left\{-|x|\sqrt{x^2-a^2}+\left(x^2-\frac{1}{2}a^2\right)\right\} \qquad (|x|>a)$$

$$\tau_{yy}=0\,.$$

In an analogous manner to what was taken in equation (5.2.15), assuming that the shear stress on the crack is partially relieved and at the center it becomes to a fraction γ of its initial uniform value S, we have the relations

$$S-\frac{3}{4}\frac{\mu U_1}{a}=\gamma S \qquad \text{for System I} \qquad (5.2.21)$$

$$S-\frac{3}{\kappa+1}\frac{\mu U}{a}=\gamma S \qquad \text{for System II}, \qquad (5.2.22)$$

From equations (5.2.6) and (5.2.7) we then obtain the work done on the crack as follows:

$$\begin{aligned} A(I)&=\int_L \Delta u_1\,\tau_{1y}^0\,dx+\frac{1}{2}\int_L \Delta u_1\tau_{1y}dx \\ &=\frac{\pi}{6}(1-\gamma)(2+\gamma)\frac{a^2S^2}{\mu} \qquad (5.2.23)\\ &=\frac{3\pi}{32}\left(\frac{2+\gamma}{1-\gamma}\right)\mu U_1^2\,, \end{aligned}$$

$$\begin{aligned} A(II)&=\int \Delta u\,\tau_{xy}^0dx+\frac{1}{2}\int \Delta u\,\tau_{xy}dx \\ &=\frac{\pi}{24}(1-\gamma)(2+\gamma)(\kappa+1)\frac{a^2S^2}{\mu} \qquad (5.2.24)\\ &=\frac{3\pi}{8}\left(\frac{2+\gamma}{1-\gamma}\right)\left(\frac{1}{\kappa+1}\right)\mu U^2. \end{aligned}$$

$A(I)$ or $A(II)$ may also be considered as the decrease in the strain energy on condition that the boundary at infinity is fixed while the shear stress over the inner boundary is slowly diminished.

We obtain the same results as in Knopoff (1958) and in Starr (1928), if we put $\gamma=0$ in the second equations in (5.2.17) and in (5.2.18) respectively; the same result as in Burridge and Knopoff (1966) in the third equation in (5.2.17), paying due regard to the fact that the corresponding equation in the former is formulated (for the vertical strike slip fault) in the half-space.

It can be seen from equations (5.2.17), (5.2.18), (5.2.23), and (5.2.24) that if we estimate the energy release from the relative displacement U_1 or U, the estimate depends very sensitively upon γ, as pointed out by Burridge and Knopoff (1966).

Appendix

We can easily evaluate the integral

$$I(z)=\frac{1}{2\pi i}\int_L \frac{P(\xi)}{\sqrt{a^2-\xi^2}\,(\xi-z)}d\xi \qquad \text{(a-1)}$$

where L is the segment of real axis $(-a, a)$ in the direction from $\xi=-a$ to $\xi=a$ and where $P(\xi)$ is a polynomial

$$P(\xi)=A_m\xi^m+A_{m-1}\xi^{m-1}+\cdots+A_0 \qquad (A_m \neq 0) \qquad \text{(a-2)}$$

(e.g. Muskhelishvili 1953 a).

For the integral (a-1) we consider another integral

$$\Omega(z)=\frac{1}{2\pi i}\int_\Lambda \frac{P(\zeta)}{\sqrt{a^2-\zeta^2}\,(\zeta-z)}d\zeta\,, \qquad \text{(a-3)}$$

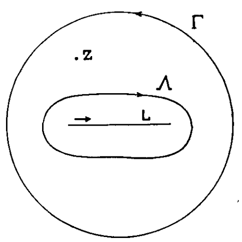


Fig. 10. The line segment L, a contour Λ and a circle Γ with large radius.

where Λ is a contour surrounding the segment L in clockwise direction as shown in Fig. 10. and assume that the point z remains outside the contour Λ. As for the function $\sqrt{a^2-\zeta^2}$ appearing in the integrand of equation (a-3), a definite branch is selected, the branch for which the function takes positive values on the left side of the segment L, i.e., the branch which has for large $|\zeta|$ the form

$$\sqrt{a^2-\zeta^2}=-i\zeta\left(1-\frac{a^2}{\zeta^2}\right)^{\frac{1}{2}}$$
$$=-i\zeta\left(1-\frac{1}{2}\frac{a^2}{\zeta^2}-\frac{1}{8}\frac{a^4}{\zeta^4}-\cdots\right),$$

therefore for large $|\zeta|$

$$\frac{P(\zeta)}{\sqrt{a^2-\zeta^2}}=\alpha_q\zeta^q+\alpha_{q-1}\zeta^{q-1}+\cdots+\alpha_0+\frac{\alpha_{-1}}{\zeta}+\cdots \qquad \text{(a-4)}$$

with $q=m-1$. Then, according to a theorem, to be derived later,

$$\Omega(z)=\frac{P(z)}{\sqrt{a^2-z^2}}-(\alpha_q z^q+\alpha_{q-1}z^{q-1}+\cdots+\alpha_0)\,, \tag{a-5}$$

where, on the right-hand side, the second term in the parentheses is nothing but the principal part of the first function at the point $z=\infty$.

On the other hand, letting the contour Λ shrink into L and noting that $\sqrt{a^2-\zeta^2}$ in equation (3-a) will then tend to $+\sqrt{a^2-\xi^2}$ or $-\sqrt{a^2-\xi^2}$, depending on the position of ζ with respect to L, we have

$$\Omega(z)=2\,I(z)\,.$$

The function $P(\zeta)/\sqrt{a^2-\zeta^2}$ may be unbounded near the ends of L, but this equation holds, since integrals taken over a small circle surrounding the ends tend to zero.

Hence the result is

$$\frac{1}{2\pi i}\int_L \frac{P(\xi)}{\sqrt{a^2-\xi^2}\,(\xi-z)}d\xi=\frac{1}{2}\left\{\frac{P(z)}{\sqrt{a^2-z^2}}-(\alpha_q z^q+\alpha_{q-1}z^{q-1}+\cdots+\alpha_0)\right\}. \tag{a-6}$$

In order to prove the formula (a-5) let Γ be a circle with center at the origin and with radius R so large that Λ and the point z lie inside Γ. Applying Cauchy's formula to the function

$$\omega(z)=\frac{P(z)}{\sqrt{a^2-z^2}}-(\alpha_q z^q+\alpha_{q-1}z^{q-1}+\cdots+\alpha_0)$$

which is holomorphic outside Λ, we have

$$\omega(z)=\frac{1}{2\pi i}\int_\Lambda \frac{\omega(\zeta)}{\zeta-z}d\zeta+\frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta-z}d\zeta\,,$$

where the positive direction on Γ is assumed to be counterclockwise. If we write the second integral on the right-hand side as I_1, the value of I_1 does not change, if R is arbitrarily increased, since the function $\omega(\zeta)$ is holomorphic outside Λ. On the other hand, we have for sufficiently large $|\zeta|$, $|\omega(\zeta)|<C/|\zeta|$, and hence

$$|I_1|\leqq\frac{C}{R-|z|}\,,$$

with a positive constant C; thus when $R\to\infty$, $I_1\to 0$. But since I_1 does not depend on R, $I_1=0$. Hence

$$\omega(z)=\frac{1}{2\pi i}\int_{\Lambda}\frac{P(\zeta)}{\sqrt{a^2-\zeta^2}\,(\zeta-z)}d\zeta-\frac{1}{2\pi i}\int_{\Lambda}\frac{(\alpha_q\zeta^q+\cdots+\alpha_0)}{\zeta-z}d\zeta\,.$$

The second integral on the right-hand side again vanishes, since the integrand is holomorphic inside Λ. Thus the formula (a–5) is proved.

With the aid of the formula (a–6) we obtain the following results:

$$\frac{1}{2\pi i}\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}}{\xi-z}d\xi=\frac{1}{2}\{\sqrt{a^2-z^2}+iz\}\,, \tag{a–7}$$

$$\frac{1}{2\pi i}\int_{-a}^{a}\frac{(\sqrt{a^2-\xi^2})^3}{\xi-z}d\xi=\frac{1}{2}\left\{(a^2-z^2)\sqrt{a^2-z^2}-i\left(z^3-\frac{3}{2}a^2z\right)\right\}. \tag{a–8}$$

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44. 無限および半無限媒質における二次元的くいちがいについて

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Steketee 以来地震にともなう断層に対して，1 つの面の両側でくいちがいの存在することによる半無限弾性体の変形を想定し，地表での変位や傾斜から断層の広がりを推定する試みが行なわれている．

一方笠原，Knopoff は長い断層を，その上で剪断応力が 0 になるような 2 次元の問題として弾性論によって取り扱っており，Weertman は転位論で用いられる式を長い断層に適用しているが，これらはいずれも strike slip fault の問題であるかまたは非常に深い部分での dip slip fault の問題であるかに限られている．

実際半無限弾性体内部の任意の曲面上に応力の減少が与えられたとき，これに応ずる周囲の変形を求めることは初期応力を想定しても一般には困難であろう．しかしその上にくいちがいが与えられたときこれに応ずる周囲の変形を得ることは容易である．

本論は，地震にともなう任意の傾き・深さ・広がりをもつ水平方向に長い断層およびその集まりの弾性論的模型のために，この第 2 の問題を 2 次元問題として取り扱ったものである．

第 2 節・第 3 節では，くいちがいを歪核でおきかえることに対応する表現を求め，第 4 節では複素数による表現を求めた．前者は筆者の前の論文（1964）の主題の 2 次元化であり，後者は主として Muskhelishvili の弾性論 § 109 Sherman の方法を端点をもつ弧の場合へ適用したものである．複素数による方が表現が簡単で一般に便利であるが，ある種の数値計算に際しては前者の方が有用であろう．

第 5 節は第 4 節の応用例である．初めの部分で Weertman などの方法に触れ，次の部分で Knopoff, Starr の shear crack およびそれらより実際に近いと思われる型の shear crack に対して，Knopoff, Starr と同じ初期応力の下で，応力の解消が crack の上で完全でない場合を含めて，crack 形成の際 crack に対して（場によって）なされた仕事を計算した．ここで，crack に対してなされた仕事は，無限媒質の中に crack を含む有限の領域を考えるとき，この領域の歪エネルギーの減少分にこの領域に対して周囲からなされた仕事を加えたものにあたり，これはまた無限遠方を固定しておくと考えた場合，shear crack 形成によって解消された歪エネルギーとみなされる．

第 1 節は，第 3 節で用いる相反定理が実は 3 次元の一般の非等方不均質の媒質について同じ形で成立するので，これを一般の形で述べたもので，層状の媒質などでの同種の問題の取り扱いに一つの基礎を与える．