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# 1. *Plane Pulse Propagation through a Heterogeneous Medium with a Periodic Structure.*

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## Summary

A plane pulse passing through a heterogeneous medium consists of two parts: an undisturbed and a disturbed pulse. The undisturbed pulse transmits without changing its form. On the other hand, the envelope of the disturbed pulse is of a function $\sin x/x$, its maximum appears to be delayed after the undisturbed pulse, and the period of its carrier wave is $\pi\cdot$*(average velocity)/(wave length of the structure)*. This disturbance is interpreted as scattering by the heterogeneity of the medium.

## 1. Introduction

In the course of the study on the wave propagation in a periodic structure[1),2)], the propagation of plane harmonic waves was discussed, connected with the stability of progressive waves. The wave passing through a heterogeneous medium with a periodic structure is made apparently to attenuate, where the wave length of progressive waves is associated with twice the wave length of the structure. The modulus of transmitted waves is made to attenuate at frequencies for each component in which this condition is fulfilled, if the structure is expressed by Fourier series.

Recently, the wave propagation in such a medium has been discussed by several authors[3)], and almost all of them are based on the characteristic of an infinite determinant, or a continuous fraction.

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1) R. YOSHIYAMA, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction," *Bull. Earthq. Res. Inst.*, **38** (1960), 467-478.

2) I. ONDA, "Propagation and Apparent Attenuation of Elastic Waves in a Heterogeneous Medium with Certain Periodic Structures," *Bull. Earthq. Res. Inst.*, **42** (1964), 427-447.

3) M. K. MILLER, "Acoustic Wave Motion along a Periodic Surface," *J. Acoust. Soc. Amer.*, **36** (1964), 2143-2148.

In this paper, it is shown how some pulses distort during their propagation through a heterogeneous medium fluctuating regularly. In section 2 the stability chart and the applicable solutions of the wave equation in this medium are given. In section 3, in order to discuss disturbance of a progressive wave, the transmission coefficient is calculated. In section 4 the disturbed waves are integrated. In the last section, 5, the mechanism of the wave propagation through a heterogeneous medium is discussed, and an interpretation of the apparent consistency of $Q$ on frequency is stated.

## 2. Stability of solution of the wave equation

For the sake of simplicity, it is assumed that the velocity of a heterogeneous medium varies regularly. The stability of such a wave equation was discussed by Prof. R. Yoshiyama[4]. In that wave equation, an independent variable $x$ is changed into a variable equivalent to a travel time: $\tau=\int dx/c(x)$, and a new function $\varphi$ is introduced in a form of the product of a displacement $u$ and the square root of an impedance: $\varphi=\sqrt{\rho c}u$. The resulting wave equation is written as

$$\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial^2\varphi}{\partial\tau^2}-\alpha^2\varphi\,, \tag{1}$$

where
$$\alpha^2=-\sqrt{\frac{c}{\rho}}\frac{d}{dx}\left(\rho c^2\frac{d}{dx}\frac{1}{\sqrt{\rho c}}\right).$$

If the velocity varies periodically and the density does not vary, we write, respectively,

$$c(x)=c_0(1+\varepsilon\cos\gamma x)\,,\quad\text{and}\quad \rho=\rho_0\,, \tag{2}$$

where $\varepsilon$ denotes a magnitude of a velocity fluctuation. If $a$ is denoted by the ratio of the velocity maximum to one minimum in the medium,

$$a=(1+|\varepsilon|)/(1-|\varepsilon|)\,.$$

As this ratio $a$ as well as its inverse $a^{-1}$ is always positive, $\varepsilon$ is confined between $-1$ and $+1$. Particularly in this investigation, $\varepsilon$ is assumed to be small. The wave equation for plane waves is written, within the order $\varepsilon^4$,

4) R. YOSHIYAMA, *loc. cit.*, 1).

$$\frac{d^2\varphi}{dz^2}+\left\{\left(\frac{2\omega}{\gamma c_0}\right)^2\frac{1}{1-\varepsilon^2}-\frac{\varepsilon^2}{2}-\frac{1}{4}\varepsilon^4+2\varepsilon\cos 2z+2\varepsilon^2\left(\frac{5}{4}+\frac{\varepsilon^2}{2}\right)\cos 4z\right.$$
$$\left.+2\varepsilon^3\cos 6z+\frac{5}{4}\varepsilon^4\cos 8z+O(\varepsilon^5)\right\}\varphi=0\,, \qquad (3)$$

where
$$z=\frac{c_0\gamma\sqrt{1-\varepsilon^2}}{2}\int\frac{dx}{c(x)}=\tan^{-1}\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\gamma x}{2}\right)$$
$$=\frac{\gamma x}{2}-\frac{\varepsilon}{2}\sin\gamma x+\frac{\varepsilon^2}{8}\sin 2\gamma x-\frac{\varepsilon^3}{8}\left(\frac{\sin 3\gamma x}{3}+\sin\gamma x\right)$$
$$+\frac{\varepsilon^4}{16}\left(\frac{\sin 4\gamma x}{4}+\sin 2\gamma x\right)+O(\varepsilon^5)\,. \qquad (4)$$

The solution $\varphi$ can be written by the product of $\exp(\mu z)$ and some periodic functions, and is unstable and stable, according to $\mu$ being real and imaginary respectively. Condition of the stability can be found by the method stated in the previous paper[5]. The relation of the stability between the frequency $\omega$ and the velocity fluctuation $\varepsilon$ are given as follows:

$$\left.\begin{aligned}\mu&=\frac{\varepsilon}{2}\sin 2\sigma+\frac{17}{128}\varepsilon^3\sin 2\sigma+\frac{7}{1024}\varepsilon^4\sin 4\sigma+O(\varepsilon^5)\,,\\ \frac{2\omega}{\gamma c_0}&=1+\frac{\varepsilon}{2}\cos 2\sigma-\frac{7}{16}\varepsilon^2-\frac{17}{128}\varepsilon^3\cos 2\sigma-\varepsilon^4\left(\frac{239}{1536}-\frac{41}{1024}\cos 4\sigma\right)+O(\varepsilon^5)\,.\end{aligned}\right\}\quad(5.\mathrm{a})$$

$$\left.\begin{aligned}\mu&=\frac{\varepsilon^2}{4}\sin 2\sigma+\varepsilon^4\left(\frac{\sin 2\sigma}{9}+\frac{\sin 4\sigma}{128}\right)+O(\varepsilon^6)\,,\\ \frac{2\omega}{\gamma c_0}&=2\left\{1-\varepsilon^2\left(\frac{5}{12}-\frac{\cos 2\sigma}{8}\right)-\varepsilon^4\left(\frac{223}{1728}+\frac{\cos 2\sigma}{288}-\frac{\cos 4\sigma}{256}\right)+O(\varepsilon^6)\right\}\,.\end{aligned}\right\}\quad(5.\mathrm{b})$$

$$\left.\begin{aligned}\mu&=\frac{35}{128}\varepsilon^3\sin 2\sigma+O(\varepsilon^5)\,,\\ \frac{2\omega}{\gamma c_0}&=3\left\{1-\frac{15}{32}\varepsilon^2+\frac{35}{384}\varepsilon^3\cos 2\sigma-\frac{39047}{368640}\varepsilon^4+O(\varepsilon^5)\right\}\,.\end{aligned}\right\}\quad(5.\mathrm{c})$$

$$\left.\begin{aligned}\mu&=\frac{1283}{9216}\varepsilon^4\sin 2\sigma+O(\varepsilon^6)\,,\\ \frac{2\omega}{\gamma c_0}&=4\left\{1-\frac{479}{960}\varepsilon^2-\varepsilon^4\left(\frac{682267}{6129600}-\frac{727}{24576}\cos 2\sigma\right)+O(\varepsilon^6)\right\}\,.\end{aligned}\right\}\quad(5.\mathrm{d})$$

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5) I. ONDA, *loc. cit.*, 2).

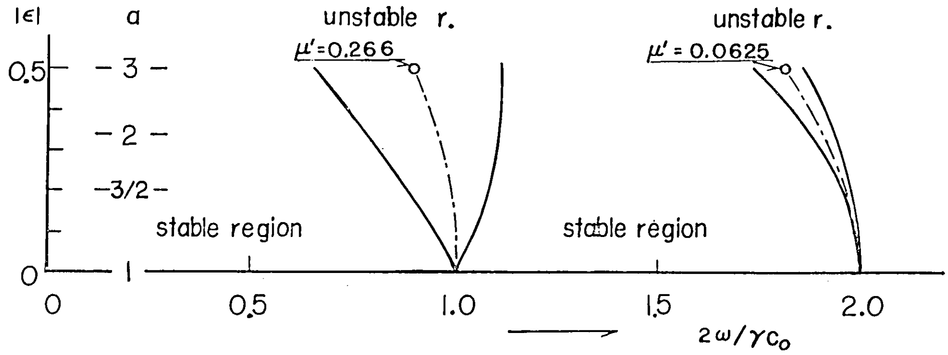


Fig. 1. Stability chart for the solutions of the wave equation: $c=c_0(1+\varepsilon\cos\gamma x)$, $a=(1+|\varepsilon|)/(1-|\varepsilon|)$. $\mu'$ are maximum $\mu$ for $\varepsilon=0.5$.

and $\mu$ is of the order of $\varepsilon^n$ for $2\omega/\gamma c_0$ near $n$, where $\sigma$'s are determined by the lower equations of each. Fig. 1 illustrates the stability chart by means of these relations. From it the solution $\varphi$ is unstable at frequency $\omega$ near $\gamma c_0/2$, and may be regarded as stable at the other frequencies, if $10^{-3}$ is neglected.

In the unstable region, the solutions are expressed by

$$\varphi=Ae^{\mu z}y(z,\sigma)+Be^{-\mu z}y(z,-\sigma)\,, \tag{6}$$

where
$$y(z,\pm\sigma)=\sin(z\mp\sigma)+\frac{\varepsilon}{8}\sin(3z\mp\sigma)$$
$$+\varepsilon^2\left\{\frac{11}{192}\sin(5z\mp\sigma)\mp\frac{7}{64}\sin 2\sigma\cos(3z\mp\sigma)\right.$$
$$\left.-\frac{9}{64}\cos 2\sigma\sin(3z\mp\sigma)\right\}+O(\varepsilon^3)\,,$$

and $\mu$ and $\sigma$ are determined by equations (5). In the stable regions, by means of the solution developed in the appendix of this paper,

$$\varphi=A\varphi_1(z,\nu)+B\varphi_2(z,\nu)\,, \tag{7}$$

where

$$\varphi_{1,2}(z,\nu)=\frac{\cos}{\sin}\nu z+\frac{\varepsilon}{4}\left\{\frac{1}{\nu+1}\frac{\cos}{\sin}(\nu+2)z-\frac{1}{\nu-1}\frac{\cos}{\sin}(\nu-2)z\right\}$$
$$+\frac{\varepsilon^2}{32}\left\{\frac{(5\nu+6)}{(\nu+1)(\nu+2)}\frac{\cos}{\sin}(\nu+4)z-\frac{(5\nu-6)}{(\nu-1)(\nu-2)}\frac{\cos}{\sin}(\nu-4)z\right\}+O(\varepsilon^3)\,,$$

and

$$\left(\frac{2\omega}{\gamma c_0}\right)^2=(1-\varepsilon^2)\left[\nu^2+\frac{\varepsilon^2}{2}\frac{\nu^2}{\nu^2-1}+\frac{\varepsilon^4}{8}\left\{3+\frac{25}{4(\nu^2-4)}+\frac{5\nu^2+7}{4(\nu^2-4)(\nu^2-1)^3}\right\}+O(\varepsilon^6)\right],$$

$\nu$ may be approximately equal to $2\omega/\gamma c_0$, for the factor of $\varepsilon^2$ is smaller than unity except for the frequency $1/\sqrt{2}<2\omega/\gamma c_0<\sqrt{7/6}$. However, in $|\varepsilon|<0.3$, the solution between these frequencies is stable in some part. If the solution in this stable domain be given by equation (7), its convergence is very slow. Therefore, the solution should still be stated by equation (6).

## 3. Transmission of some waves

Fourier transform of a displacement $u(x, t)$ is written as follows;

$$\bar{u}(x, t)=\int_{-\infty}^{\infty} e^{-i\omega t}u(x, t)dt\,, \tag{8}$$

which gives a complex spectrum of the displacement specified by $u$. If the complex transmission coefficient of waves passing through a medium is specified by $T(x_0, \omega)$, where $x_0$ is the distance travelled, multiplying the spectrum of an incident wave $\bar{u}(0, \omega)$ by a transmission coefficient yields the complex one of the transmitted wave $\bar{u}(x_0, \omega)$:

$$\bar{u}(x_0, \omega)=T(x_0, \omega)\cdot\bar{u}(0, \omega)\,. \tag{9}$$

Therefore, wave forms of the transmitted wave can be calculated by means of the inverse transform

$$u(x_0, t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}T(x_0, \omega)\bar{u}(0, \omega)d\omega\,, \tag{10}$$

and by using the convolution formula

$$u(x_0 t)=\int_{-\infty}^{\infty} u(0, t-\tau)\cdot g(x_0, \tau)d\tau\,, \tag{11}$$

where

$$g(x_0, \tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}T(x_0, \omega)d\omega\,,$$

which corresponds to the response to a pulse passing through the medium.

Velocity and its derivative are assumed as always continuous, to discuss the effect of the periodic structure alone on the wave propagation. The transmission coefficient $T$ is derived from these solutions (6) and (7), by means of the formulae (17) of the previous paper[6]: In the unstable region,

$$T=\frac{\exp(-iz_0-i\zeta)\cos\zeta}{\cosh(\mu z_0\sin 2\sigma)} \tag{12.a}$$

where $$z_0=\gamma x_0/2\,, \qquad \mu=\varepsilon/2\,.$$

$$\tan\zeta=\tanh(\mu z_0\sin 2\sigma)\cdot(\cos 2\sigma-2\delta-2\delta^2\cos 2\sigma)/\sin 2\sigma\,,$$

$$\delta=\frac{\varepsilon}{8}+\varepsilon^2\left(\frac{1}{6}-\frac{\cos 2\sigma}{32}\right),$$

the relation between the frequency $\omega$ and $\sigma$ being given by equations (5). In the stable regions,

$$T=\cos\zeta'\exp(-i\nu z_0-i\zeta') \tag{12.b}$$

where $$\tan\zeta'=\frac{\varepsilon}{2}\frac{1}{\nu^2-1}\sin 2\nu z_0\,.$$

The expressions (12) give the transmission coefficient in a whole frequency domain, and its modulus is shown in Fig. 2, as an example, for $z_0=10\pi$

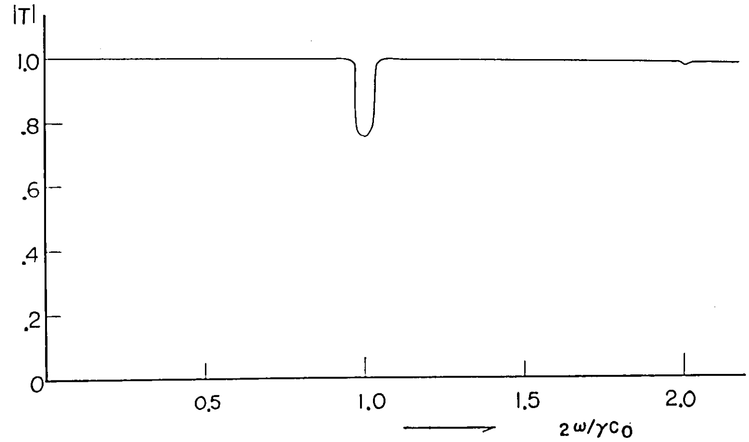


Fig. 2. Modulus of the transmission coefficient for $z_0=10\pi$ and $\varepsilon=0.05$ or $a=1.1053$.

and $\varepsilon=0.05$ for which the velocity ratio $a$ is 1.1053. Its modulus except for frequency near $\gamma c_0/2$ can be regarded as unity.

Substitution of these into (10) and a comparison with the initial wave form yield to find the distortion of the wave form.

6) I. ONDA, *loc. cit.*, 2), p. 437.

$$u(x_0\,,\,t)=\frac{1}{2\pi}\int_{-\infty}^{\frac{\gamma c_0}{2}\left(1-\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}\bar{u}(0,\,\omega)e^{i\omega t'}d\omega$$

$$+\frac{1}{2\pi}\int_{\frac{\gamma c_0}{2}\left(1-\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}^{\frac{\gamma c_0}{2}\left(1+\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}\frac{\bar{u}(0\,,\,\omega)\cos\zeta}{\cosh(\mu z_0\sin 2\sigma)}e^{i(\omega t-z_0-\zeta)}d\omega$$

$$+\frac{1}{2\pi}\int_{\frac{\gamma c_0}{2}\left(1+\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}^{\infty}\bar{u}(0,\,\omega)e^{i\omega t'}d\omega$$

$$\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{u}(0,\,\omega)e^{i\omega t'}d\omega+u_a(t',\,\varepsilon)$$

$$=u(0,\,t')+u_a(t',\,\varepsilon)\,, \tag{13}$$

where

$$u_a(t',\,\varepsilon)=-\frac{1}{2\pi}\int_{\frac{\gamma c_0}{2}\left(1-\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}^{\frac{\gamma c_0}{2}\left(1+\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}\bar{u}(0,\,\omega)\left\{e^{i\omega t'}-\frac{\cos\zeta\exp\{i(\omega t-z_0-\zeta)\}}{\cosh(\mu z_0\sin 2\sigma)}\right\}d\omega \tag{14}$$

and $t'=t-x_0/c_0$. The term $u(0,t')$ means that waves are propagated without any distortion of the incident wave forms and are delayed in time by $x_0/c_0$, which is regarded as the travel time through the heterogeneous medium, while the integral $u_a$ is related with the disturbance of the incident wave resulted from heterogeneity of the medium.

## 4. Forms of some disturbed pulse

If we know the response to an initial wave of the delta function type, the response to incidence of an arbitrary wave form can be synthesized by the theorem of convolution mentioned above. Since Fourier transform of the delta function is constant, the integral

$$I_a(t)=-\frac{1}{2\pi}\int_{\omega_0\left(1-\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}^{\omega_0\left(1+\frac{\varepsilon}{2}-\frac{7}{16}\varepsilon^2\right)}\left\{e^{i\omega(t-x_0/c_0)}-\frac{\cos\zeta}{\cosh(\mu z_0\sin 2\sigma)}e^{i(\omega t-z_0-\zeta)}\right\}d\omega \tag{15}$$

is discussed in this section, where $\omega_0=\gamma c_0/2$.

$\zeta$ can be expressed by means of Taylor expansion

$$\zeta=-\frac{\varepsilon}{4}\tanh\mu z_0+\left(\frac{\omega-\omega_0}{\varepsilon\omega_0/2}\right)\frac{(1-\varepsilon^2/3)\tanh\mu z_0}{1+(\varepsilon^2/16)\tanh^2\mu z_0}$$

$$+\left(\frac{\omega-\omega_0}{\varepsilon\omega_0/2}\right)^2\frac{\varepsilon}{8}\cdot\frac{\tanh\mu z_0-\mu z_0\,\mathrm{sech}^2\mu z_0}{\{(1+\varepsilon^2/16)\tanh^2\mu z_0\}^2}+\cdots. \tag{16}$$

$(\omega-\omega_0)\tanh(\varepsilon z_0/2)/(\varepsilon\omega_0/2)$ is much smaller than unity within the considering frequency, and then the third term appears to be negligibly small. The modulus of the second term of integrand (15) varies as U-shape, so that we now assumed it as being a constant in this frequency interval. Substitution of them into (15) yields the disturbance of the pulse. The result is

$$I_d(t)=-\frac{\varepsilon\omega_0}{2\pi}e^{i\omega_0(t-x_0/c_0)}$$
$$\times\left[\frac{\sin\tau_1}{\tau_1}-\frac{\exp\{i(\varepsilon/4)(\tanh\mu z_0-7\varepsilon\omega_0 t/4)\}}{\cosh\mu z_0}\frac{\sin\tau_2}{\tau_2}+O(\varepsilon^2)\right], \tag{17}$$

where
$$\tau_1=\frac{\varepsilon\omega_0}{2}(t-x_0/c_0)\,,$$

and
$$\tau_2=\frac{\varepsilon\omega_0 t}{2}-\frac{(1-\varepsilon^2/3)\tanh\mu z_0}{1+(\varepsilon^2/16)\tanh^2\mu z_0}\,.$$

The disturbance of the pulse $I_d(t)$ represents a carrier wave with the frequency $\omega_0$ and a modulation which behaves like $\sin x/x$, the maximum of which appears later than the arrival time; we approximate as

$$I_d(t)=-Ae^{i\omega_0(t-x_0/c_0)}\frac{\sin(\varepsilon\omega_0/2)(t-t_1)}{(t-t_1)}\,, \tag{18}$$

where $t_1$ is a retarded time of the modulation with maximum amplitude of $(\varepsilon\omega_0/2\pi)\ \{\sin\tau_1/\tau_1-\operatorname{sech}(\varepsilon z_0/2)\sin\tau_2/\tau_2\}$. The disturbance resulted from an arbitrary incident wave is expressed by means of relation (11):

$$u_d(x_0\,,\ t)=-A\int_{-\infty}^{\infty}e^{i\omega_0(\tau-x_0/c_0)}\frac{\sin(\varepsilon\omega_0/2)(\tau-t_1)}{(\tau-t_1)}u(0,\ t-\tau)d\tau\,. \tag{19}$$

Particularly, if the initial wave is given in duration $0<t<T$, it is written as

$$u_d(x_0\,,t)=-Ae^{i\omega_0(t-x_0/c_0)}\int_0^T e^{-i\omega_0\tau}\frac{\sin(\varepsilon\omega_0/2)(t-t_1-\tau)}{t-t_1-\tau}u(0,\ \tau)d\tau\,,$$
$$=-Ae^{i\omega_0(t-x_0/c_0)}\left[\frac{\sin(\varepsilon\omega_0/2)(t-t_1)}{t-t_1}\int_0^T e^{-i\omega_0\tau}u(0,\ \tau)d\tau\right.$$
$$+\left\{\frac{\varepsilon\omega_0}{2}\frac{\cos(\varepsilon\omega_0/2)(t-t_1)}{(t-t_1)}-\frac{\sin(\varepsilon\omega_0/2)(t-t_1)}{(t-t_1)^2}\right\}\int_0^T e^{i\omega_0\tau}u(0,\ \tau)\tau d\tau$$
$$\left.+\cdots\right]. \tag{20}$$

*Example* 1. If an incident pulse is rectangular, we get

$$u(0, t)=u_0 \quad (\text{for } 0<t<T),$$

$$\quad =0 \quad (\text{for } t<0, \text{ and } t>T).$$

$$u_d(x_0, t)=2Au_0 e^{i\omega_0(t-x_0/c_0-T/2)} \frac{\sin(\varepsilon\omega_0/2)(t-t_1)}{\omega_0(t-t_1)} \sin\frac{\omega_0 T}{2}\{1+O(\varepsilon)\}.$$

This represents a carrier wave with the frequency $\omega_0$ and a modulation $2Au_0 \sin(\omega_0 T/2) \sin(\varepsilon\omega_0/2)(t-t_1)/\omega_0(t-t_1)$.

*Example* 2. If an incident wave is sinusoidal, we get

$$u(0, t)=u_0 e^{i\omega t} \quad \text{for } 0<t<T,$$

$$=0 \quad \text{for } t<0, \text{ and } t>T.$$

$$u_d(x_0, t)=2Au_0 e^{i\omega_0(t-x_0/c_0)+i(\omega_0-\omega)T/2} \frac{\sin(\varepsilon\omega_0/2)(t-t_1)\sin(\omega_0-\omega)T/2}{(\omega_0-\omega)(t-t_1)}.$$

The disturbance seems to resonate at the frequency $\omega$ which is near to $\omega_0$.

## 5. Concluding remarks

Some finite wave trains passing through a heterogeneous medium varying periodically are treated. The method used in this paper is the same as that in the previous paper[7], in which a harmonic wave was treated. In particular, the solution convergent in as few terms as possible must be selected, in the neighbourhood of boundaries between the stable and unstable regions of the wave equation. The modulus of the transmission coefficient in the entire frequency domain is shown in Fig. 2 as an example. It follows from this figure that the dissipation of the spectrum appears near the specified wave length $\lambda$ that is twice the wave length of a structural heterogeneity $L$: $\lambda=2L$.

Next, the response to the incident wave of a delta function is calculated. If the incident wave or its spectrum is given, the transmitted wave can be simply synthesized by means of convolution of Fourier transform.

As the result, the transmitted wave consists of two parts: an undisturbed wave and a disturbed one. The wave form of the undisturbed wave agrees with the incident one after the time passing through this

7) I. ONDA, *loc. cit.*, 2).

medium in phase. On the other hand, the disturbed wave has a period of that specified wave mentioned above, and appears after the arrival of the undisturbed wave. The envelope of this distrubance is spindle-shaped. It seems that a portion of energy of that specified wave is stored in the medium, and is spread out after the arrival of the undisturbed waves. Such resonant-like phenomena arise for the wave with the corresponding components[8], when the structural heterogeneity consists of many structural wave lengths, so that the spectrum of transmitted waves is dissipated at frequency resonated by each of the structural heterogeneity. If the heterogeneity is finite in some range of the structural wave length, the apparent dissipation is proportional to the frequency because $\operatorname{sech} x=\exp(-x)\ \{1+\sinh x \exp(-x)+\cdots\}$, and then it can be interpreted that the measure of internal friction $Q$ is independent of the frequency.

## Acknowledgements

The author wishes to express his sincere thanks to Professor Ryoichi Yoshiyama for his encouragement given during the progress of this work.

## Appendix; General Solutions of Hill's Equation in Stable Regions

In a previous paper[9], we obtained the solutions of Hill's equation which were convenient to discuss the stability, according to the method of Whittaker. It is to be noted that these solutions are very slowly or not convergent in the stable regions. In this Appendix, the solutions in the stable regions are introduced in a similar series as the previous calculation. The equation is

$$\frac{d^2\varphi}{dz^2}+(\theta_0+2\sum_r \theta_r \cos 2z)\varphi=0\,. \tag{A-1}$$

Let a solution $\varphi$ and a parameter $\theta_0$ be expanded as, respectively,

$$\varphi=\sin \nu z+\sum_k \theta_k A_k(z)+\sum_{k,l} \theta_k \theta_l A_{kl}(z)+\cdots\,, \tag{A-2}$$

8) I. ONDA, *loc. cit.*, 2).

9) I. ONDA, "Propagation and Apparent Attenuation of Elastic Waves in a Heterogeneous Medium with Certain Periodic Structures, Appendix; General Solutions of Hill's Equation," *Bull. Earthq. Res. Inst.*, **42** (1964), 441-447.

and $$\theta_0=\nu^2+\sum_k \theta_k q_k+\sum_{k,l}\theta_k\theta_l q_{kl}+\cdots . \tag{A-3}$$

Substituting these relations into equation (A-1), it is expressed by a form of power series of $\theta$, and the condition under which this solution must identically satisfy all the values of $\theta$ is given by equalizing coefficients of each term to zero. In addition, the solution $\varphi$ and therefore $A_k(z)$, $A_{kl}(z)$, etc. must be periodic functions, respectively. Since the process of calculations is the same as in the previous paper, we indicate merely the final results:

$$\theta_0=\nu^2+\sum_{k=1}^{\infty}\frac{\theta_k^2}{2(\nu^2-k^2)}+\sum_{k=1}^{\infty}\frac{(5\nu^2+7k^2)\theta_k^4}{32(\nu^2-k^2)^3(\nu^2-4k^2)}+O(\theta^6), \tag{A-4}$$

$$\begin{aligned}\varphi= \sin \nu z&+\frac{1}{4}\sum_{k=1}^{\infty}\left\{\frac{\sin(\nu+2k)z}{k(\nu+k)}-\frac{\sin(\nu-2k)z}{k(\nu-k)}\right\}\theta_k\\
&+\frac{1}{32}\sum_{k}\left\{\frac{\sin(\nu+4k)z}{k^2(\nu+k)(\nu+2k)}+\frac{\sin(\nu-4k)z}{k^2(\nu-k)(\nu-2k)}\right\}\theta_k^2\\
&+\frac{1}{16}\sum_{k\neq l}\left\{\frac{\nu(k+l)+k^2+l^2}{kl(k+l)(\nu+k)(\nu+l)(\nu+k+l)}\sin(\nu+2k+2l)z\right.\\
&\qquad-\frac{\nu(k-l)-(k^2+l^2)}{kl(k-l)(\nu-k)(\nu+l)(\nu-k+l)}\sin(\nu-2k+2l)z\\
&\qquad-\frac{\nu(k-l)+(k^2+l^2)}{kl(k-l)(\nu+k)(\nu-l)(\nu+k-l)}\sin(\nu+2k-2l)z\\
&\qquad\left.+\frac{\nu(k+l)-(k^2+l^2)}{kl(k+l)(\nu-k)(\nu-l)(\nu-k-l)}\sin(\nu-2k-2l)z\right\}\theta_k\theta_l+O(\theta^3).\end{aligned} \tag{A-5}$$

The second solution is obtained by writing cosine for sine in (A-5). If all the $\theta_k$'s besides $k=1$ vanish, these solutions are agreement with those defined as the Mathieu functions of the fractional order, $\mathrm{se}_\nu(z,\ \theta_1)$ and $\mathrm{ce}_\nu(z,\ \theta_1)$[10].

**Errata:** Propagation and Apparent Attenuation of Elastic Waves in a Heterogeneous Medium with Certain Periodic Structures. By I. Onda, *Bull. Earthq. Res. Inst.*, **42** (1964), 427-447.

Eq. (12) of p. 432 should be omitted, and The 4-th and 5-th lines of p. 446 should read

$$\sum_{k\neq n}\theta_k\left[\frac{\sin\{(n+2k)z-\sigma\}}{4k(n+k)}-\frac{\sin\{(n-2k)z-\sigma\}}{4k(n-k)}\right] \text{ for } \sum_{k\neq n}\theta_k\left[\frac{\sin\{(n+2k)z-\sigma\}}{2k(n+k)}-\frac{\sin\{(n-2k)z-\sigma\}}{2k(n-k)}\right]$$

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10) N. W. McLachlan, *Theory and Application of Mathieu Functions* (Oxford, 1951), § 2.16, p. 19.

# 1. 周期的に変わる不均質媒質を伝わる平面波パルス

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周期的に変わる不均質媒質が透過する波に与える影響を議論するために，特定の有限長波形をもつ波が入射したときの透過波の波形を計算した．方法は入射調和波について行なつたものと同じであるが，運動方程式の解の不安定領域と安定領域の境の近くでは，解の選択に特に注意を払つた．このようにして収束のよい解を継ぎ合わせて，全周波数領域にわたつて透過係数の大きさを求めた例を第2図に示した．この例では $z_0=\gamma x_0/2=10\pi$ および $\varepsilon=1/20$ 言いかえると速度の最大最小の値の比が約 1.105 にとつてある．

まず，入射波が時間に関するデルタ関数で与えられた場合の応答を計算した．入射波の波形が与えられると，それに対する応答はフーリエ変換のたたみ込みの原理によつて簡単に合成することができる．

計算の結果次のことがわかつた．透過波は2つの部分から成り立つている： 入射波の波形がそのまま乱されずに透過する部分と，丁度与えられた構造と共鳴する周期をもつ波が第一の波に少し遅れて伝わる部分とである．後者のいわゆる擾乱波の包絡線はほぼ紡錘形をしている．いいかえると，構造の不均質性によつてそれに共鳴する波のエネルギーの一部分が媒質の中に取り残されて，この媒質から少しずつ遅れて滲み出して行くように見える．ここに構造に共鳴する波はその波長 $\lambda$ が構造の不均質性の波長 $L$ の2倍に相当している．$\lambda=2L$．

従つて，構造の不均質性の大きさが，種々の波長のものを含んでいる場合には，それぞれに応じた成分波について上記の共鳴現象が起るので，透過波のスペクトルには吸収が現われる．そしてある構造上の波長域でほぼ一定の不均質性を有しているならば，吸収の見かけの量は波の周波数に比例，あるいは内部摩擦の濁度 $Q$ は周波数に依存しない量として説明することができる．