BULLETIN OF THE EARTHQUAKE
RESEARCH INSTITUTE
Vol. 43 (1965), pp. 745-754

47. *Tsunami in an L-shaped Canal* [III].

By Takao MOMOI,

Earthquake Research Institute.

(Read July 20, 1965.—Received July 31, 1965.)

Abstract

In a series of works already completed, the theories on a long wave in an L-shaped canal have been developed under the first, second and third approximations. In the present work, these theories are utilized to calculate the wave heights and phases for a canal of uniform width and for discussing the behaviors of the waves in the canal in question. Then the following facts are ascertained: —

When kd (k: a wave number of the incident waves, d: a width of the canal) increases, the wave height of the reflected waves is linearly augmented from zero to amount to about 20 percent of that of the incident waves at $kd=1.0$. Then the advancing waves diminish monotonically in height to reach about 95 percent of the incident waves (if these values are evaluated in energy, the reflected waves for the range $kd<1.0$ are at most 5 percent of the incident waves and the advancing waves are propagated nearly without loss of energy).

As far as the phases are concerned, when kd is small, the reflected waves have a phase difference of $\pi/2$ from the incident waves. As kd increases, such a difference decreases linearly until the value reaches zero at $kd\approx1.5$. In other words, the incident and reflected waves make up quasi-standing waves, instead of complete standing waves, under the condition $\lambda\approx4d$ (λ: a wavelength of the incident waves).

1. Introduction

In a series of works[1),2),3)] on a problem of a long wave in an L-shaped canal, we have developed theories under the first, second and third ap-

1) T. MOMOI, *Bull. Earthq. Res. Inst.*, **40** (1962), 719-732.
2) T. MOMOI, *Bull. Earthq. Res. Inst.*, **41** (1963), 581-587.
3) T. MOMOI, *Bull. Earthq. Res. Inst.*, **42** (1964), 449-463.
The above papers are referred to as papers A, B, and C respectively in the following.

proximations by use of a newly devised method, i.e. the method of the buffer domain. Actual numerical calculations have, however, been reserved for a future paper.

In this paper, the numerical calculations for the hitherto developed theories are carried out in order to discuss the behaviors of the waves in an L-shaped canal, and also mention is made of the conservation of energy.

2. Theory

In this study, the numerical calculations are made only for the case of an L-shaped canal with an *uniform width d* (see Fig. 1).

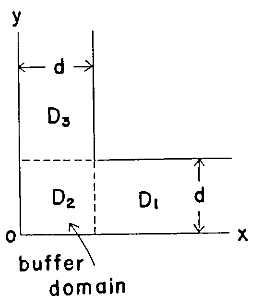

Fig. 1. A geometry of an L-shaped canal.

For convenience of later discussions, the theories obtained under the first, second and third approximations are summarized.

2,1. The first approximation.

According to paper A, the amplitude factors of the zeroth mode of the waves advancing through and reflected from the crooked part of the canal (the buffer domain) are expressed as follows: —

$$A_1^{(2)0}=\frac{i\cdot a_0d}{2-i\cdot a_0d}\cdot A_1^{(1)}e^{-i\cdot 2a_0d} \tag{1}$$

(from (54) of paper A),

$$A_3^0=\frac{2}{2-i\cdot a_0d}\cdot A_1^{(1)}e^{-i\cdot 2a_0d} \tag{2}$$

(from (55) of paper A).

Then the velocity potentials in the domains $D_j(j=1, 3)$ (refer to Fig. 1) are the forms: —

$$\begin{aligned}\Phi_1=&A_1^{(1)}e^{-ia_0x}\cosh a_0(H+z)\\&+A_1^{(2)0}e^{+ia_0x}\cosh a_0(H+z)+(\text{higher modes})\,,\end{aligned} \tag{3}$$

$$\Phi_3=A_3^0e^{+ia_0y}\cosh a_0(H+z)+(\text{higher modes})\,, \tag{4}$$

where Φ_j $(j=1, 3)$ are velocity potentials in the domains D_j $(j=1, 3)$, (x, y, z) the Cartesian coordinates with (x, y) being horizontal components and z a vertical one, a_0 a wave number of the incident waves, H a depth of water, and the time factor $\exp(-i\omega t)$ is omitted as usual (this convention is followed in the subsequent discussions).

2,2. The second approximation.

According to paper B, the zeroth modes of the waves in the domains D_j ($j=1, 3$) are described as follows:—

$$A_1^{(2)0}=\frac{\left(+i\cdot\frac{1}{3}a_0d\right)\left(1+\frac{2}{3}a_0^2d^2\right)}{(2-a_0^2d^2)-i\cdot\frac{1}{3}a_0d\left(5-\frac{2}{3}a_0^2d^2\right)}\cdot A_1^{(1)}e^{-i\cdot 2a_0d} \tag{5}$$

and

$$A_3^0=\frac{2-\frac{1}{3}a_0^2d^2}{(2-a_0^2d^2)-i\cdot\frac{1}{3}a_0d\left(5-\frac{2}{3}a_0^2d^2\right)}\cdot A_1^{(1)}e^{-i\cdot 2a_0d}, \tag{6}$$

where the expressions (5) and (6) are derived from (17) and (18) of paper B respectively, after a few algebraic manipulations for the case $d_1=d_3=d$.

Then the complete forms of the velocity potentials in each domain are given by the foregoing expressions (3) and (4).

2,3. The third approximation.

According to paper C, the solutions of the zeroth modes of the waves in the straight parts of the canals are as follows:—

$$\zeta_1^{(0)}=\frac{ikd\cdot(P^2-Q\cdot R)}{(P-ikd\cdot Q)(R-ikd\cdot P)}\cdot\zeta_0e^{-i\cdot 2kd} \tag{7}$$

(from (47) of paper C),

$$\zeta_3^{(0)}=\frac{P\cdot(R+k^2d^2\cdot Q)}{(P-ikd\cdot Q)(R-ikd\cdot P)}\cdot\zeta_0e^{-i\cdot 2kd} \tag{8}$$

(from (46) of paper C),

where

$$P=1-\frac{1}{3!}k^2d^2,$$

$$Q=\frac{2}{3!}-\frac{4}{5!}k^2d^2,$$

$$
\left.
\begin{aligned}
&R=S-T\,,\\
&S=2-\frac{4}{3!}k^2d^2+\frac{6}{5!}k^4d^4\,,\\
&T=\frac{4}{5!}k^2d^2\cdot ik^{(1)}d\cdot\frac{W}{V}\\
&\qquad\qquad\left(ik^{(1)}d=-\sqrt{\pi^2-(kd)^2}\right),\\
&V=1-\left(\frac{1}{\pi^2}+\frac{2}{3!}\right)\cdot ik^{(1)}d\,,\\
&W=1+\left(\frac{1}{\pi^2}-\frac{1}{3!}\right)\cdot k^2d^2\,,
\end{aligned}
\right\}
\tag{9}
$$

and where k is a wave number of the incident waves.

Then the wave heights ζ_j $(j=1, 3)$ in the domains D_j $(j=1, 3)$ are expressed by

$$
\zeta_1=\zeta_0e^{-ikx}+\zeta_1^{(0)}e^{+ikx}+(\text{higher modes}) \tag{10}
$$

and

$$
\zeta_3=\zeta_3^{(0)}e^{+iky}+(\text{higher modes})\,. \tag{11}
$$

2,4. Identification of the notations.

Since we have used different notations in deriving the theories under the first, second and third approximations, the identification of these notations is made in the following for convenience of comparison of three cases.

Comparing (3) and (4) with (10) and (11), the coefficients of the zeroth modes of the velocity potentials and those of the wave heights are related with, through the surface condition

$$
\zeta=\frac{-1}{g}\left(\frac{\partial\Phi}{\partial t}\right)_{z=0},
$$

as follows: —

$$
\left.
\begin{aligned}
&\zeta_0=\frac{i\omega}{g}A_1^{(1)}\cosh a_0H\,,\\
&\zeta_1^{(0)}=\frac{i\omega}{g}A_1^{(2)}\cosh a_0H\,,\\
&\zeta_3^{(0)}=\frac{i\omega}{g}A_3^{0}\cosh a_0H\,.
\end{aligned}
\right\}
\tag{12}
$$

As shown in the above, the amplitude factors of the incident, advancing and reflected waves derived by use of an equation of a long wave differ from those obtained by the expression of a velocity potential by the additive factor

$$\frac{i\omega}{g}\cosh a_0H.$$

Therefore, if the ratios of the amplitude factors divided by that of the incident waves are discussed, of which the forms are given as

$$|A_1^{(2)0}/A_1^{(1)}|,\quad |A_3^0/A_1^{(1)}|,\quad |\zeta_1^{(0)}/\zeta_0| \text{ and } |\zeta_3^{(0)}/\zeta_0|, \tag{13}$$

the troubles due to the difference of notations are avoided, and the whole of the present work is directed in this way.

2,5. Conservation of energy.

Before proceeding with numerical calculations, a conservation of energy between the waves incoming, advancing and reflected in two branches of the canals is examined.

In (3), (4), (10) and (11), each term of the higher modes has the exponential factors

$$e^{+ik^{(m)}x}(x>0) \text{ and } e^{+ik^{(m)}y}(y>0),$$

where

$$k^{(m)}=\sqrt{a_0^2-\left(\frac{m\pi}{d}\right)^2} \text{ or } \sqrt{k^2-\left(\frac{m\pi}{d}\right)^2}$$

for positive integers m (refer to the complete forms of the waves in papers A, B, C).

Since the developments of our approximated theories, instead of a general theory, have been carried out under the condition a_0d (or kd)$<\pi$, the higher modes in (3), (4), (10) and (11) are found to be damping waves and hence, at the points distant from the crooked part of the canal, the only zeroth modes of the waves remain.

Accordingly, the conservation of energy, if it holds, must be expressed by the zeroth modes alone, i.e.,

$$|A_1^{(2)0}|^2+|A_3^0|^2=|A_1^{(1)}|^2 \tag{14}$$

or

$$|\zeta_1^{(0)}|^2+|\zeta_3^{(0)}|^2=|\zeta_0|^2. \tag{15}$$

Substituting (1) and (2) (for the first approximation), (5) and (6) (for the second approximation), and (7) and (8) (for the third approximation) into (14) (for the first and second approximations) and (15) (for the third approximation), these equations are, after some algebraic reductions, found to be held for the theories under the first, second and third approximations respectively.

3. Numerical calculations and discussions

In this section, the numerical calculations of three kinds of the theories for the canal of uniform width developed under the first, second and third approximations are carried out to discuss the behaviors of the waves in the canal.

Using the expressions (1), (2), (5), (6), (7) and (8), the variations of the relative amplitudes

$$|A_1^{(2)0}/A_1^{(1)}|, \quad |A_3^0/A_1^{(1)}|, \quad |\zeta_1^{(0)}/\zeta_0| \text{ and } |\zeta_3^{(0)}/\zeta_0|$$

of the zeroth modes of the waves are visualized in Figs. 2 and 3.

According to the discussion in *2,4* of the previous section, the different notations for the amplitude factors used in the theories of the first, second, and third approximations become out of the question if the discussion is made through the relative quantities of the modes, as shown in the above.

As shown in Fig. 2, when the approximation proceeds, the curves of the advancing and the reflected waves become slighter in gradient and seem to tend to certain asymptotic curves.

If the theory is developed by use of Lamb's idea, i.e., a continuity of flux, no reflected waves be expected in a canal of uniform width. And the relative amplitude of the advancing waves is expressed by a line

$$|\zeta_3^{(0)}/\zeta_0| = 1.0$$

in Fig. 2.

Referring to Fig. 2, the application ranges of the theories are at most $a_0d<$ about 0.1 for the first approximation, $a_0d<$ about 0.5 for the second approximation and, for the third approximation, such a range cannot be set definitely. For the determination of the application range of the last approximation, a further development of a more generalized theory is required, but this range might be set supposedly at 0 to about 1.0.

As shown in Fig. 2, when $a_0d(=kd)$ increases, the wave height of the reflected waves is linearly augmented (from the curve for the third approximation) to amount to about 20 percent of that of the incident

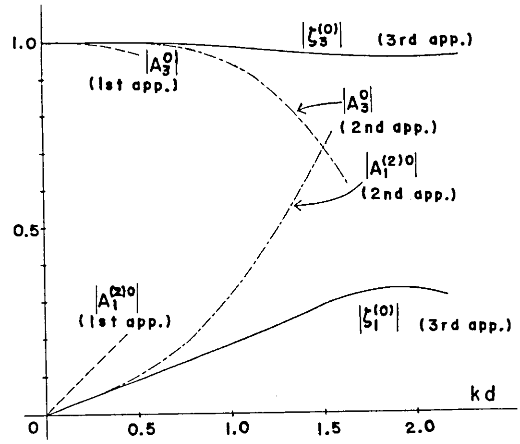

Fig. 2. Variations of amplitudes. (unit: $A_1^{(1)}$ or ζ_0)

waves and the advancing waves diminish monotonically in height to reach about 95 percent of the incident waves at $kd=1.0$. However, if these values are evaluated in energy, the reflected waves for the range $a_0d(=kd)<1.0$ are at most 5 percent of the incident waves and the advancing waves are propagated nearly without loss of energy through the crooked part of the canal. Inasmuch as the relation $a_0d=1.0$ corresponds to that of $\lambda\approx 6d$ (λ: a wavelength of the incident waves, d: a width of the canal), the transmission of energy through the corner is found to be remarkably good for the range $a_0d<1.0$.

As far as the higher modes are concerned these are, judging from the variation of the zeroth modes shown in Fig. 2, small in amount for the range $a_0d<1.0$ and a further development of the theory under the more generalized approximation is considered rather more required than the numerical calculation for the higher modes up to the third approximation. Therefore, the computation of the latter is postponed until the former development is considered, and with regard to the higher modes, the author contents himself with saying that:—

(1) the higher modes in the first approximation remain to estimate the order of them because the development of the theory is carried out under the linear approximation of the sine and cosine functions and the higher modes are found to be the second order of this approximation;

(2) on the contrary, since the theories of the second and third approximations are developed under the approximations of the second and third order of kd, the higher modes for these approximations are no longer the expressions to evaluate the order and the non-negligible values which might be involved in an examination of the behaviors of the waves in the neighbouring regions of the crooked part of the canal.

Next, consider the phase lags of the advancing and reflected waves from the incident waves.

The variations of the phases are depicted in Fig. 3, where the curves

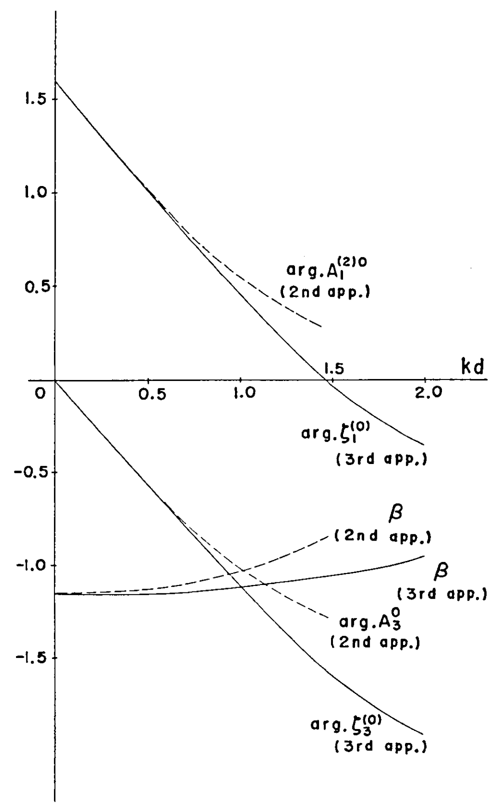

Fig. 3. Variation of phases.

for the theory of the first approximation are omitted and those derived from the second and third approximations are only visualized by use of the expressions (5), (6), (7) and (8).

If the waves in two branches of the canals are described in a complete form including the time factor, these are expressed as (the notation of a long wave is used here): —

in the domain D_1 (from (10)),

$$\zeta_1=\zeta_0 \cos(\omega t+kx) + |\zeta_1^{(0)}| \cos(\omega t-kx-\text{arg.}\,\zeta_1^{(0)}) + (\text{higher modes})\,; \tag{16}$$

in the domain D_3 (from (11)),

$$\zeta_3=|\zeta_3^{(0)}| \cos(\omega t-ky-\text{arg.}\,\zeta_3^{(0)}) + (\text{higher modes})\,; \tag{17}$$

where the only real parts are retained.

Introducing a quantity

$$\beta=(\text{arg.}\,\zeta_3^{(0)})/kd\,,$$

the first term (the transmitted waves) of (17) is transformed as

$$|\zeta_3^{(0)}| \cos\{\omega t-kd(y/d+\beta)\}\,. \tag{18}$$

Then the variation of β is also depicted in Fig. 3.

According to Fig. 3, β is about -1.2 for small kd. From the expression (18), $\{-\beta\}$ implies the supposed origin of the $\cos(\omega t-ky)$ type transmitted waves, so that the $\cos(\omega t-ky)$ type waves advancing through the crooked part of the canal are propagated as if they originate at the point y=about 1.2.

As far as the reflected waves are concerned, arg. $\zeta_1^{(0)}$ (or $A_1^{(2)0}$) is about $\pi/2$ for small kd. Accordingly, using this value, the expression of the reflected waves (the second term of (16)) is approximated sufficiently by

$$|\zeta_1^{(0)}| \sin(\omega t-kx) \tag{19}$$

for the range of small kd.

Comparing the expressions of the incident waves (the first term of (16)) and reflected waves (19), it is discovered that, when kd is small, both waves are expressed by the orthogonal functions, i.e., cosine and sine functions.

As kd increases, arg. $\zeta_1^{(0)}$ diminishes in amount to reach zero value at kd=about 1.5. The last equation is substituted for the relation

$$\lambda \approx 4d \tag{20}$$

(λ: a wavelength of the incident waves, d: a width of the canal). Then the waves in the domain D_1 (the domain in which the incident waves invade), excepting the damping modes, become, from (16),

$$\zeta_0 \cos(\omega t + kx) + |\zeta_1^{(0)}| \cos(\omega t - kx). \tag{21}$$

The above expression does not imply a standing wave, since $|\zeta_1^{(0)}| \approx 0.3\,\zeta_0$ from the curve of the reflected waves in Fig. 2.

The relation (20) is equal to that derived from an *eigenvalue* problem in a model of a rectangular bay leading to the open sea, where the longitudinal length of the bay is d and λ a wavelength of the seiche of the first mode. Therefore, similar mechanism is interpreted to take place in the crooked part of the canal. Then the waves in the domain D_1, as shown in (21), become, in a sense, quasi-standing waves, instead of rigorous standing waves.

---

47. L字水路における津波 [III]

地震研究所　桃　井　高　夫

今までおこなわれてきた一連の研究で，筆者はL字水路における長波の理論を第1，第2，第3近似までの範囲で展開を試みた．

本報告では，これらの理論を用い，水路の幅が不変なモデルについて数値解析をおこない，L字水路における長波の状態を調べた．そしてつぎのようなことを知つた．

(1) 波高の変化については，kd (k: 進入波の波数，d: 水路の幅) が0のとき，反射波は全くなく，kdが増加するにつれてその波高はほとんど線型的に増加する，しかしその量は非常に小さく kd=1.0 の近くで，進入波の波高の 20 パーセント程度である (エネルギーに換算すると5パーセントにすぎない)．

そして進入波の波高は，kd が 0 から 1.0 の範囲でほとんど屈曲部の影響をうけることなくそのまま透過波の波高となつている．

(2) 位相については，kd が小さいとき反射波の位相は進入波のそれと $\pi/2$ のずれをもち，kd が増

加するにつれてほとんど線型的に減少し，kd がおよそ 1.5 のところ，換言すれば $\lambda\approx 4d$ (λ: 進入波の波長) のところで反射波の位相のずれは 0 になる．そして進入波と準定常波を構成する．また透過波については $\cos(\omega t-ky)$ (ω: 進入波の角振動数，t: 時間の変数，y: 水路の軸にそう座標軸) 型の進行波の仮想的な原点はほぼ $1.2d$ 位のところである．