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19. *The Method of the "Buffer Domain" in Water with a Step Bottom [I].*

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Abstract

In the previous works, the author has developed a new method which is very useful in treating the problem with the abruptly varying boundaries. In the neighbouring parts of the abrupt changing boundaries, new domains named "buffer domains" are set up for easy corrections of the boundary conditions. Hence, this method is called the method of the buffer domain.

In this paper, this method is verified as being applicable to the case where the boundary changes suddenly in the vertical direction, while the problems hitherto treated are the cases of horizontally changing boundaries.

The primary purpose of the present study is to prove that the method of the buffer domain is still applicable to the problem which has an abrupt irregularity of the boundary in the vertical direction. The present method is found to be very practicable to the problem mentioned above.

The secondary purpose of the work is to examine the behaviors of the waves in the surrounding regions of the irregular boundary. As a result of a computation under the first approximation, it turns out that:—

(1) The amplitude factors of the waves advancing through and reflected from the step of the bottom is the same as those derived from a consideration of flux of water which is introduced by Lamb.

(2) As far as the phase is concerned, the phase differences of $a_1^{(0)}h$ and $2\cdot a_1^{(0)}h$ take place for the advancing and reflected waves, where $a_1^{(0)}$ is the wave number of the incident waves and h the height of the step of the bottom.

(3) The damping terms (the terms of the disturbances in the nearby parts of the irregular boundary) are of the second order of the present approximation (the first approximation). Hence, these terms have no significance in the computation of the first order of the approximation.

Excepting the first result (1), the results (2) and (3) cannot be obtained by Lamb's consideration.

Although the approximation is confined to the first order in the present study, the further development of the theory will be made in the more generalized approximation in a future article.

1. Introduction

In the previous papers, we have introduced a new method which has conspicuous advantages in treating the boundary value problem with an abruptly varying boundary. The published papers relevant to this method are described in section 2.

The purpose of the present paper is a generalization of our method, i. e. the method of the buffer domain, to other cases. In the works studied so far, the variation of the boundary was restricted to a horizontal case, while, in this article, the case where the boundary changes abruptly in the vertical direction is considered.

The contents of this paper are as follows :—

In section 2, the method of the buffer domain is outlined.

In section 3, the general theory is developed for the case where the bottom of water varies abruptly with a step.

In section 4, the reduction of the general theory obtained in section 3 is carried out under the approximation of the first order.

2. The Method of the Buffer Domain

In the problems of the waves in the wave guides, which include the canal, the wave guide of the electromagnetic waves and the plate for the elastic waves, we have often met the mathematical difficulties caused by the irregularities of the boundaries. When the variation of the boundary is slight, the treatment of the problem is usually carried out by the methods of the perturbation or successive approximation. But these methods are at our disposal merely for the cases of the boundary problems with a relatively gradual variation. When the boundary changes abruptly, the methods mentioned above are not adaptable for the analysis of the problem.

On the contrary, the method developed by the author is very practicable for the treatment of the problem with abruptly varying boundaries. This method was first developed in the treatment of a tsunami in an L-shaped canal[1] to the first order of the approximation and, later on, the application of the method was extended to the cases in the approximations of the higher orders[2),3)]. Other examples of the applications of our method were demonstrated in the papers entitled "The Effects of the Coastlines on the Tsunami (2) and some Remarks on the Chilean Tsunami"[4], "Tsunami in a T-shaped Canal"[5], "Tsunami in a Canal of Varying Width"[6] and "The Effect of a Bottle-neck on Tsunami",[7] where the first paper mentioned above (denoted by the super-subscript 4)) is an article relevant to a long wave in an L-shaped closed canal, developed in the first approximation, and a more generalized theory of the last problem was presented in a paper named "Tsunami in an L-shaped Bay".[8]

Although the method of the buffer domain was detailed in the papers described above, the outline of the method is given hereunder :—

The most outstanding feature of our method is an establishment of the new domain named "buffer domain" in the neighbouring parts of the wave guide where the surface of the boundary changes abruptly. In solving the equation, we must firstly set up such domains in the irregular parts of the wave guide so that other parts of the domains have no irregular boundaries. This procedure makes possible a formation of the formal solutions with unknown factors in each domain which satisfy the boundary conditions in respective domain. And the solutions in the domains excepting the buffer ones are composed of an infinite number of the modes, which are resulted in from the boundary conditions and have, in general, the orthogonalities between the different modes. This characteristic of the solutions is very important for the reduction of the equations in our method. Using the conditions communicating the adjacent domains, we can connect the formal expressions of the waves in each domain and each term of the mode solutions can be expressed explicitly by use of the orthogonalities of the mode functions composed of the solutions of the series.

As a second reduction of our method, the expressions of the solutions

1) T. Momoi, *Bull, Earthq, Res. Inst.*, **40** (1962), 719.
2) T. Momoi, *Bull. Earthq. Res. Inst.*, **41** (1963), 581.
3) T. Momoi, *Bull. Earthq. Res. Inst.*, **42** (1964), 369.
4) T. Momoi, *Bull. Earthq. Res. Inst.*, **40** (1962), 733.
5) T. Momoi, *Bull. Earthq. Res. Inst.*, **41** (1963), 357.
6) T. Momoi, *Bull. Earthq. Res. Inst.*, **41** (1963), 375.
7) T. Momoi, *Bull. Earthq. Res. Inst.*, **41** (1963), 573.
8) T. Momoi, *Bull. Earthq. Res. Inst.*, **41** (1963), 705.

in the buffer domains are expanded in power series to retain the terms up to an appropriate order. The order of the terms to be retained are decided according to a required accuracy of the problem in question.

Here, in the above, although the expansion of the expressions in the buffer domains is first made before the explicit expression of the mode solutions by orthogonalities, the latter may be dealt with before the former. The preference of the anticipation of the above two reductions depends on actual situations of the treated problem. In the example treated in the present purview, the integrands are expanded beforehand.

At any rate, after the completion of the above two reductions, we proceed to eliminate the expressions of the buffer domains by simple algebraic reductions and reach the simultaneous equations which are relevant to the unknowns only in the domains excluding the buffer ones (these domains are called "non-buffer domains" in the subsequent discussions). Then, if necessary, some other relations are used. That is to say, when the problem under consideration is a two-dimensional one, the relations of the wave numbers derived from a separation of the variables of the equations become part of the above reductions. The works[1)–8)] referred to already are included in such cases. On the contrary, since the problem treated in this paper is a one-dimensional case, the relations of the wave numbers do not come explicitly into the reduction.

Now, the simultaneous equations with respect to the unknowns of the non-buffer domains are to be readily solved by elementary reductions, which are derived following the foregoing procedures. Here it should be noticed that, in reducing the equations, the approximations are applied to the expressions merely in the buffer domains. Therefore, the final expressions of the non-buffer domains have a physical meaning at most to the order of the approximations in the buffer domains, so that the discussion of the behaviors of the waves must be limited to the range of the order approximated in the buffer domains.

In the next sections, the problem of the water waves is treated in the case where there exists an abrupt change of the depth in water. This model is different from the models treated so far[1)–8)] in the respect that the boundary of the former varies vertically while the latter have horizontally varying boundaries.

3. General Theory

Referring to Fig. 1, the Cartesian co-ordinates x and z are used, x being measured at the undisturbed free surface of water and z vertically upward. Let D_j $(j=1, 2)$ denote the domains $(h<x,\ 0>z>-H_1)$ and $(x<0,\ 0>z>-H_2)$ respectively, where $H_1=H_2+h$, and B_j $(j=1, 2)$ the domains $(0<x<h,\ -H_2>z>-H_1)$ and $(0<x<h,\ 0>z>-H_2)$, where the domains B_j imply the "buffer domain" mentioned in the preceding section. Then the velocity potentials ϕ_j in four domains satisfy the equations of the continuity:

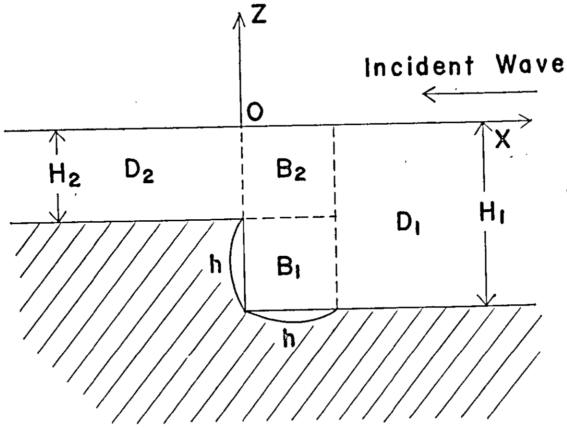

Fig. 1.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\phi_j=0 \quad (j=1,\ 2,\ B1,\ B2), \tag{1}$$

where the velocity potentials in the domains D_1, D_2, B_1 and B_2 are expressed by ϕ_1, ϕ_2, ϕ_{B1} and ϕ_{B2}.

In like manner, the physical quantities relevant to the domains D_1, D_2, B_1 and B_2 are described by subscriptions of 1, 2, $B1$ and $B2$ respectively in the following discussion, unless otherwise stated.

Confining the problem to the linear case of the theory, the surface conditions $(z=0)$ are:

$$\frac{\partial\phi_j}{\partial t}=-g\zeta_j,\quad \frac{\partial\zeta_j}{\partial t}=\frac{\partial\phi_j}{\partial z}\ , \tag{2}$$

$$(j=1,\ 2,\ B2)$$

or

$$\frac{\partial^2\phi_j}{\partial t^2}+g\frac{\partial\phi_j}{\partial z}=0\ , \tag{3}$$

where ζ_j is the elevation of the water from the undisturbed free surface of water, g the acceleration of gravity and t a variable of time.
The bottom conditions are:

$$\frac{\partial \phi_1}{\partial z}=0 \quad (z=-H_1)\ , \tag{4}$$

$$\left.\begin{aligned}&\frac{\partial \phi_{B1}}{\partial z}=0 \quad (z=-H_1)\ ,\\ &\frac{\partial \phi_{B1}}{\partial x}=0 \quad (x=0)\ ,\end{aligned}\right\} \tag{5}$$

$$\frac{\partial \phi_2}{\partial z}=0 \quad (z=-H_2)\ , \tag{6}$$

where H_1 and H_2 and the depths of the deep and shallow waters respectively, as shown in Fig. 1.

For the case of a train of the periodic waves, the equation (1) and the conditions (3)—(6) are reduced to the following:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\phi_j'=0 \quad (j=1,\ 2,\ B1,\ B2) \tag{1'}$$

$$\left(-\omega^2+g\frac{\partial}{\partial z}\right)\phi_j'=0 \quad (j=1,\ 2,\ B2\ ;\ z=0) \tag{3'}$$

$$\frac{\partial \phi_1'}{\partial z}=0 \quad (z=-H_1)\ , \tag{4'}$$

$$\left.\begin{aligned}&\frac{\partial \phi_{B1}'}{\partial z}=0 \quad (z=-H_1)\ ,\\ &\frac{\partial \phi_{B1}'}{\partial x}=0 \quad (x=0)\ ,\end{aligned}\right\} \tag{5'}$$

$$\frac{\partial \phi_2'}{\partial z}=0 \quad (z=-H_2)\ , \tag{6'}$$

where ϕ_j' is the velocity potential eliminated the time factor $\exp(-i\omega t)$ (ω: the angular frequency of the incident waves). Hereafter the prime ($'$) of ϕ_j' is omitted for simplicity.

Suppose that the incident waves are expressed by

$$\left.\begin{aligned}&\phi_{in}=\phi_0 e^{-i\omega t-ia_1^{(0)}x}\cosh a_1^{(0)}(H_1+z)\\ \text{or}\quad &\zeta_{in}=\zeta_0 e^{-i\omega t-ia_1^{(0)}x}\end{aligned}\right\}, \tag{7}$$

where ϕ_0 and ζ_0 are the amplitude factors which are related with each other by the relation (2), i.e.

$$\zeta_0 = i\cdot\frac{\omega}{g}\cosh a_1^{(0)} H\cdot\phi_0 \,, \tag{8}$$

the solutions of the equation (1′) satisfying the conditions (3′) to (6′) become as follows :—

in the domain D_1,

$$\phi_1 = (\phi_0 e^{-ia_1^{(0)}x} + A_1^{(0)} e^{+ia_1^{(0)}x}) \cosh a_1^{(0)}(H_1+z) + \sum_{s=1}^{\infty} A_1^{(s)} e^{-a_1^{(s)}x} \cos a_1^{(s)}(H_1+z) \,; \tag{9}$$

in the domain D_2,

$$\phi_2 = A_2^{(0)} e^{-ia_2^{(0)}x} \cosh a_2^{(0)}(H_2+z) + \sum_{s=1}^{\infty} A_2^{(s)} e^{+a_2^{(s)}x} \cos a_2^{(s)}(H_2+z) \,; \tag{10}$$

in the domain B_1,

$$\phi_{B1} = \sum_{B1} A_{B1} \cos a_{B1} x \cosh a_{B1}(H_1+z) \,; \tag{11}$$

in the domain B_2,

$$\phi_{B2} = \sum_{B2} (A_{B2} \sin a_{B2}x + B_{B2} \cos a_{B2}x) \times (\omega^2 \sinh a_{B2}z + g a_{B2} \cosh a_{B2}z) \,; \tag{12}$$

where $A_1^{(r)}$, $A_2^{(r)}$ $(r=0, 1, 2, 3, \ldots)$ are the arbitrary constants to be determined by the conditions communicating the neighbouring domains ; A_{B1}, A_{B2} denote the coefficients relevant to the wave numbers a_{B1}, a_{B2} in the domains B_1, B_2 ; $\sum_{B1}$, $\sum_{B2}$ the integrations with respect to a_{B1} and a_{B2} (even the complex values are permissible for a_{B1} and a_{B2}) ; $a_1^{(r)}$, $a_2^{(r)}$ $(r=0, 1, 2, 3, \ldots)$ are the *eigenvalues* of the equation (1′) under the conditions (3′), (4′) and (6′), i.e. the solutions of $\omega^2 = a_j^{(0)} g\cdot\tanh a_j^{(0)} H_j = -a_j^{(s)} g\cdot\tan a_j^{(s)} H$ $(j=1, 2\,;\ s=1, 2, 3, \ldots)$.

In order to determine the arbitrary constants $A_1^{(r)}$ and $A_2^{(r)}$ $(r=0, 1, 2, 3, \ldots)$, the following conditions are available :

at $x=0$ $(0>z>-H_2)$,

$$\left.\begin{aligned} \phi_2 &= \phi_{B2} \\ \frac{\partial\phi_2}{\partial x} &= \frac{\partial\phi_{B2}}{\partial x} \end{aligned}\right\} \,; \tag{13}$$

at $x=h$ $(0>z>-H_2)$,

$$\left.\begin{aligned} \phi_{B2}&=\phi_1 \\ \frac{\partial\phi_{B2}}{\partial x}&=\frac{\partial\phi_1}{\partial x} \end{aligned}\right\} ; \tag{14}$$

at $x=h$ $(-H_2>z>-H_1)$,

$$\left.\begin{aligned} \phi_{B1}&=\phi_1 \\ \frac{\partial\phi_{B2}}{\partial x}&=\frac{\partial\phi_1}{\partial x} \end{aligned}\right\} ; \tag{15}$$

at $z=-H_2$ $(h>x>0)$,

$$\left.\begin{aligned} \phi_{B2}&=\phi_{B1} \\ \frac{\partial\phi_{B2}}{\partial z}&=\frac{\partial\phi_{B1}}{\partial z} \end{aligned}\right\} . \tag{16}$$

In this stage, the reduction of our method is separated into two ways, that is to say, one is firstly to approximate the expressions of the buffer domains before the operations of the orthogonal functions and the other to carry it out in inverse order, which has already been made mentioned of in section 2. In this paper, the way of the former is taken. Hence before substituting (9)—(12) into the conditions (13)—(16), the approximation is given to (9)—(12).

Let the approximated functions be designated by the super-subscript *ap*, and the approximated functions are substituted into (13)—(16). Then the expressions (13)—(16) become as follows :—

at $x=0$ $(0>z>-H_2)$,

$$\left.\begin{aligned} \phi_2&=\phi_{B2}^{ap} \\ \frac{\partial\phi_2}{\partial x}&=\frac{\partial\phi_{B2}^{ap}}{\partial x} \end{aligned}\right\} ; \tag{13'}$$

at $x=h$ $(0>z>-H_2)$,

$$\left.\begin{aligned} \phi_{B2}^{ap}&=\phi_1 \\ \frac{\partial\phi_{B2}^{ap}}{\partial x}&=\frac{\partial\phi_1}{\partial x} \end{aligned}\right\} ; \tag{14'}$$

at $x=h$ $(-H_2>z>-H_1)$,

$$\left.\begin{aligned}\phi_{B1}^{ap}&=\phi_1\\ \frac{\partial\phi_{B1}^{ap}}{\partial x}&=\frac{\partial\phi_1}{\partial x}\end{aligned}\right\} ; \tag{15'}$$

at $z=-H_2$ $(h>x>0)$,

$$\left.\begin{aligned}\phi_{B2}^{ap}&=\phi_{B1}^{ap}\\ \frac{\partial\phi_{B2}^{ap}}{\partial z}&=\frac{\partial\phi_{B1}^{ap}}{\partial z}\end{aligned}\right\} . \tag{16'}$$

Allowing for the orthogonalities of the functional series

$$\{\cosh a_j^{(0)}(H_j+z),\ \cos a_j^{(s)}(H_j+z)\ ;\ s=1,\ 2,\ 3,\ \ldots\}$$

in the range $0>z>-H_j$ for respective $j(=1, 2)$, the following integrations are made :—

at $x=0$;

$$\int_{-H_2}^{0}\left\{\begin{matrix}\phi_2\\ \dfrac{\partial\phi_2}{\partial x}\end{matrix}\right\}\cosh a_2^{(0)}(H_2+z)dz=\int_{-H_2}^{0}\left\{\begin{matrix}\phi_{B2}^{ap}\\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x}\end{matrix}\right\}\cosh a_2^{(0)}(H_2+z)dz\ , \tag{17}$$

$$\int_{-H_2}^{0}\left\{\begin{matrix}\phi_2\\ \dfrac{\partial\phi_2}{\partial x}\end{matrix}\right\}\cos a_2^{(s)}(H_2+z)dz=\int_{-H_2}^{0}\left\{\begin{matrix}\phi_{B2}^{ap}\\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x}\end{matrix}\right\}\cos a_2^{(s)}(H_2+z)dz : \tag{18}$$

at $x=h$;

$$\begin{aligned}\int_{-H_1}^{0}\left\{\begin{matrix}\phi_1\\ \dfrac{\partial\phi_1}{\partial x}\end{matrix}\right\}\cosh a_1^{(0)}(H_1+z)dz=&\int_{-H_2}^{0}\left\{\begin{matrix}\phi_{B2}^{ap}\\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x}\end{matrix}\right\}\cosh a_1^{(0)}(H_1+z)dz\\ &+\int_{-H_1}^{-H_2}\left\{\begin{matrix}\phi_{B1}^{ap}\\ \dfrac{\partial\phi_{B1}^{ap}}{\partial x}\end{matrix}\right\}\cosh a_1^{(0)}(H_1+z)dz\ ,\end{aligned} \tag{19}$$

$$\begin{aligned}\int_{-H_1}^{0}\left\{\begin{matrix}\phi_1\\ \dfrac{\partial\phi_1}{\partial x}\end{matrix}\right\}\cos a_1^{(s)}(H_1+z)dz=&\int_{-H_2}^{0}\left\{\begin{matrix}\phi_{B2}^{ap}\\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x}\end{matrix}\right\}\cos a_1^{(s)}(H_1+z)dz\\ &+\int_{-H_1}^{-H_2}\left\{\begin{matrix}\phi_{B1}^{ap}\\ \dfrac{\partial\phi_{B1}^{ap}}{\partial x}\end{matrix}\right\}\cos a_1^{(s)}(H_1+z)dz\ ,\end{aligned} \tag{20}$$

where the expressions in brackets are taken in the same order.

As far as the equations (16′) are concerned, these relations are treated, case by case, in convenient forms for the reductions. In section 4, these relations are used as they are. The integrations of these equations over the range $0<x<h$ may be taken, if the case treated is of a form convenient for these reductions.

The actual relations of the orthogonalities of the functional series

$$\{\cosh a_j^{(0)}(H_j+z),\ \cos a_j^{(s)}(H_j+z)\ ;\ s=1,\ 2,\ 3\ \ldots\}$$

are expressed as follows :—[9)–11)]

$$I_j^{(0)}=\int_{-H_j}^{0}\cosh^2 a_j^{(0)}(H_j+z)dz$$
$$=\frac{1}{2}\left(\frac{1}{2a_j^{(0)}}\sinh 2a_j^{(0)}H_j+H_j\right)\ ; \tag{21}$$

$$I_j^{(s)}=\int_{-H_j}^{0}\cos^2 a_j^{(s)}(H_j+z)dz$$
$$=\frac{1}{2}\left(\frac{1}{2a_j^{(s)}}\sin 2a_j^{(s)}H_j+H_j\right)$$
$$(s=1,\ 2,\ 3,\ \ldots)\ ; \tag{22}$$

$$\int_{-H_j}^{0}\cosh a_j^{(0)}(H_j+z)\cos a_j^{(s)}(H_j+z)dz=0$$
$$(s=1,\ 2,\ 3,\ \ldots)\ ; \tag{23}$$

$$\int_{-H_j}^{0}\cos a_j^{(s)}(H_j+z)\cos a_j^{(r)}(H_j+z)dz=0$$
$$(s,\ r=1,\ 2,\ 3,\ \ldots\ ;\ s\neq r)\ , \tag{24}$$

for respective $j(=1,\ 2)$.

Using the relations (21)—(24), the equations (17)—(20) are reduced to the following :—

$$\left.\begin{matrix} A_2^{(0)}\cdot I_2^{(0)} \\ A_2^{(0)}\cdot(-ia_2^{(0)})\cdot I_2^{(0)} \end{matrix}\right\}=\int_{-H_2}^{0}\left\{\begin{matrix} \phi_{B2}^{ap} \\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x} \end{matrix}\right\}_{x=0}\cosh a_2^{(0)}(H_2+z)dz\ , \tag{25}$$

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9) T. H. HAVELOCK, "Forced Surface-Waves on Water," *Phil. Mag.*, **8** (1929), 569.

10) K. TAKANO, "Effects d'un obstacle parallelepedique sur la propagation de la houle," *La Houille Blanche* (Mai 1960), 247.

11) T. MOMOI, *Bull. Earthq. Res. Inst.*, **41** (1963), 9.

$$\left.\begin{array}{l} A_2^{(s)}\cdot I_2^{(s)} \\ A_2^{(s)}\cdot a_2^{(s)}\cdot I_2^{(s)} \end{array}\right\}=\int_{-H_2}^{0}\left\{\begin{array}{c}\phi_{B2}^{ap} \\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x}\end{array}\right\}_{x=0}\cos a_2^{(s)}(H_2+z)dz\,; \tag{26}$$

$$\left.\begin{array}{l} (\phi_0 e^{-ia_1^{(0)}h}+A_1^{(0)}e^{+ia_1^{(0)}h})\cdot I_1^{(0)} \\ (\phi_0 e^{-ia_1^{(0)}h}-A_1^{(0)}e^{+ia_1^{(0)}h})\cdot(-ia_1^{(0)})\cdot I_1^{(0)} \end{array}\right\}$$

$$=\int_{-H_2}^{0}\left\{\begin{array}{c}\phi_{B2}^{ap} \\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x}\end{array}\right\}_{x=h}\cosh a_1^{(0)}(H_1+z)dz+\int_{-H_1}^{-H_2}\left\{\begin{array}{c}\phi_{B1}^{ap} \\ \dfrac{\partial\phi_{B1}^{ap}}{\partial x}\end{array}\right\}_{x=h}\cosh a_1^{(0)}(H_1+z)dz\,, \tag{27}$$

$$\left.\begin{array}{l} A_1^{(s)}\cdot e^{-a_1^{(s)}h}\cdot I_1^{(s)} \\ A_1^{(s)}\cdot(-a_1^{(s)})\cdot e^{-a_1^{(s)}h}\cdot I_1^{(s)} \end{array}\right\}$$

$$=\int_{-H_2}^{0}\left\{\begin{array}{c}\phi_{B2}^{ap} \\ \dfrac{\partial\phi_{B2}^{ap}}{\partial x}\end{array}\right\}_{x=h}\cos a_1^{(s)}(H_1+z)dz+\int_{-H_1}^{-H_2}\left\{\begin{array}{c}\phi_{B1}^{ap} \\ \dfrac{\partial\phi_{B1}^{ap}}{\partial x}\end{array}\right\}_{x=h}\cos a_1^{(s)}(H_1+z)dz\,; \tag{28}$$

where $s=1, 2, 3, \ldots$, and $I_j^{(s)}$ $(j=1, 2)$ are given in (21) and (22).

Now we proceed to the actual reductions of the equations (25)—(28) under suitable approximations.

Firstly, in the next section, the above equations are solved in the first order of the approximation.

4. The First Approximation

In this section, further reductions of the general theory developed in the foregoing section are made under the approximation :—

$$\left.\begin{array}{ll} \cos a_j x\simeq 1, & \sin a_j x\simeq a_j x, \\ \cosh a_j y\simeq 1, & \sinh a_j y\simeq a_j y, \\ \multicolumn{2}{c}{(j=B1,\ B2),} \end{array}\right\} \tag{1, 1}$$

where

$$y=z \quad \text{or} \quad y=H_1+z\,.$$

Substituting (1, 1) into (11) and (12), the approximated functions of ϕ_{B1} and ϕ_{B2} become

$$\left.\begin{aligned} \phi^{ap}_{B1} &= \sum_{B1} A_{B1} , \\ \phi^{ap}_{B2} &= \sum_{B2} (A_{B2}a_{B2}x + B_{B2}) \\ &\quad \times (\omega^2 a_{B2} z + g a_{B2}) , \end{aligned}\right\} \tag{1, 2}$$

where ω^2 and g are made dimensionless with characteristic time scale.

Using the above expressions (1, 2), the equations (25)—(28) are reduced to the following :—

$$\left.\begin{aligned} A_2^{(0)} \cdot I_2^{(0)} &= (\omega^2 K_1^{(0)} + g K_2^{(0)}) \cdot \sum_{B2} B_{B2} a_{B2} , \\ A_2^{(0)} \cdot (-i a_2^{(0)}) \cdot I_2^{(0)} &= (\omega^2 K_1^{(0)} + g K_2^{(0)}) \cdot \sum_{B2} A_{B2} a_{B2}^2 , \end{aligned}\right\} \tag{1, 3}$$

where

$$\left.\begin{aligned} K_1^{(0)} &= \int_{-H_2}^{0} z \cosh a_2^{(0)} (H_2 + z) dz , \\ K_2^{(0)} &= \int_{-H_2}^{0} \cosh a_2^{(0)} (H_2 + z) dz \end{aligned}\right\} ; \tag{1, 4}$$

$$\left.\begin{aligned} A_2^{(s)} \cdot I_2^{(s)} &= (\omega^2 K_1^{(s)} + g K_2^{(s)}) \cdot \sum_{B2} B_{B2} a_{B2} , \\ A_2^{(s)} \cdot a_2^{(s)} \cdot I_2^{(s)} &= (\omega^2 K_1^{(s)} + g K_2^{(s)}) \cdot \sum_{B2} A_{B2} a_{B2}^2 , \end{aligned}\right\} \tag{1, 5}$$

where

$$\left.\begin{aligned} K_1^{(s)} &= \int_{-H_2}^{0} z \cos a_2^{(s)} (H_2 + z) dz ; \\ K_2^{(s)} &= \int_{-H_2}^{0} \cos a_2^{(s)} (H_2 + z) dz \end{aligned}\right\} ; \tag{1, 6}$$

$$\left.\begin{aligned} &(\phi_0 e^{-i a_1^{(0)} h} + A_1^{(0)} e^{+i a_1^{(0)} h}) \cdot I_1^{(0)} \\ &\quad = (\omega^2 K_3^{(0)} + g K_4^{(0)}) \cdot \sum_{B2} (A_{B2} a_{B2}^2 h + B_{B2} a_{B2}) + K_5^{(0)} \cdot \sum_{B1} A_{B1} , \\ &(\phi_0 e^{-i a_1^{(0)} h} - A_1^{(0)} e^{+i a_1^{(0)} h}) \cdot (-i a_1^{(0)}) \cdot I_1^{(0)} \\ &\quad = (\omega^2 K_3^{(0)} + g K_4^{(0)}) \cdot \sum_{B2} A_{B2} a_{B2}^2 , \end{aligned}\right\} \tag{1, 7}$$

where

$$\left.\begin{aligned} K_3^{(0)} &= \int_{-H_2}^{0} z \cosh a_1^{(0)} (H_1 + z) dz , \\ K_4^{(0)} &= \int_{-H_2}^{0} \cosh a_1^{(0)} (H_1 + z) dz , \\ K_5^{(0)} &= \int_{-H_1}^{-H_2} \cosh a_1^{(0)} (H_1 + z) dz \end{aligned}\right\} ; \tag{1, 8}$$

$$\left.\begin{aligned} &A_1^{(s)}\cdot e^{-a_1^{(s)}h}\cdot I_1^{(s)} \\ &\quad=(\omega^2K_3^{(s)}+gK_4^{(s)})\cdot\sum_{B2}(A_{B2}a_{B2}^2h+B_{B2}a_{B2})+K_5^{(s)}\cdot\sum_{B1}A_{B1} \\ &A_1^{(s)}\cdot(-a_1^{(s)})\cdot e^{-a_1^{(s)}h}\cdot I_1^{(s)} \\ &\quad=(\omega^2K_3^{(s)}+gK_4^{(s)})\cdot\sum_{B2}A_{B2}a_{B2}^2\ , \end{aligned}\right\} \qquad (1,9)$$

where

$$\left.\begin{aligned} K_3^{(s)}&=\int_{-H_2}^{0} z\cos a_1^{(s)}(H_1+z)dz\ , \\ K_4^{(s)}&=\int_{-H_2}^{0} \cos a_1^{(s)}(H_1+z)dz\ , \\ K_5^{(s)}&=\int_{-H_1}^{-H_2} \cos a_1^{(s)}(H_1+z)dz\ . \end{aligned}\right\} \qquad (1,10)$$

And also substituting (1, 2) into (16′) in section 3, we have :—

$$\left.\begin{aligned} &(-\omega^2H_2+g)\cdot\sum_{B2}(A_{B2}a_{B2}^2x+B_{B2}a_{B2})=\sum_{B1}A_{B1}\ , \\ &\omega^2\cdot\sum_{B2}(A_{B2}a_{B2}^2x+B_{B2}a_{B2})=0\ , \quad \text{for} \quad h>x>0\ . \end{aligned}\right\} \qquad (1,11)$$

In the above, the second relation has a validity merely under the existence of the factor ω^2, of which the order is $(a_j^{(0)}H_j)^2$ from the expression $\omega^2=a_j^{(0)}g\tanh a_j^{(0)}H_j$ $(j=1,\ 2)$. The meaning of this equation is that the product of ω^2 and $\sum_{B2}(A_{B2}a_{B2}^2x+B_{B2}a_{B2})$ is negligible in the present approximation described in (1, 1), instead of rigorously zero. Hence the identity

$$\sum_{B2}(A_{B2}a_{B2}^2x+B_{B2}a_{B2})=0 \qquad (1,12)$$

which is derived from dividing the second equation by ω^2, has no significance in the reduction of the equations in the present study. If one needs this expression, the relation of the product form (1, 11), instead of (1, 12), must be employed in the reduction of our method. A similar problem has already been experienced in the previous work.[12]

Putting the second equation into the first one of (1, 11), we have

$$\sum_{B2}(A_{B2}a_{B2}^2x+B_{B2}a_{B2})=\frac{1}{g}\sum_{B1}A_{B1} \quad \text{for} \quad h>x>0\ .$$

12) T. MOMOI, *Bull. Earthq. Res. Inst.*, **42** (1964) 449.

Since the above equation holds in the range $h>x>0$, if one sets x equal to 0 or h, the following equations are obtained :—
when $x=0$,

$$\sum_{B2} B_{B2}a_{B2}=\frac{1}{g}\sum_{B1} A_{B1}\,; \tag{1, 13}$$

when $x=h$,

$$\sum_{B2}(A_{B2}a_{B2}^2h+B_{B2}a_{B2})=\frac{1}{g}\sum_{B1} A_{B1}\,. \tag{1, 14}$$

Substituting the left-hand side of (1, 14) into the right-hand members of the first equations of (1, 7) and (1, 9), we have :—
from (1, 7),

$$(\phi_0 e^{-ia_1^{(0)}h}+A_1^{(0)}e^{+ia_1^{(0)}h})\cdot I_1^{(0)}=P_1^{(0)}\cdot\sum_{B1} A_{B1}\,; \tag{1, 15}$$

from (1, 9),

$$A_1^{(s)}e^{-a_1^{(s)}h}\cdot I_1^{(s)}=P_1^{(s)}\cdot\sum_{B1} A_{B1}\quad (s=1,\ 2,\ 3,\ \cdots)\,; \tag{1, 16}$$

where

$$P_1^{(r)}=\frac{\omega^2}{g}K_3^{(r)}+K_4^{(r)}+K_5^{(r)}\quad (r=0,\ 1,\ 2,\ \cdots)\,; \tag{1, 17}$$

Likewise, substituting (1, 13) into the first relations of (1, 3) and (1, 5), the following expressions are obtained :—
from (1, 3),

$$A_2^{(0)}\cdot I_2^{(0)}=P_2^{(0)}\cdot\sum_{B1} A_{B1}\,; \tag{1, 18}$$

from (1, 5),

$$A_2^{(s)}\cdot I_2^{(s)}=P_2^{(s)}\cdot\sum_{B1} A_{B1}\quad (s=1,\ 2,\ 3,\ \cdots)\,; \tag{1, 19}$$

where

$$P_2^{(r)}=\frac{\omega^2}{g}K_1^{(r)}+K_2^{(r)}\quad (r=0,\ 1,\ 2,\ \cdots)\,. \tag{1, 20}$$

Eliminating the expression $\sum_{B1} A_{B1}$ from (1, 15) and (1, 18), the equation with respect to the unknowns in the non-buffer domain is obtained, i.e.,

$$I_2^{(0)}\cdot P_1^{(0)}\cdot A_2^{(0)}-I_1^{(0)}\cdot P_2^{(0)}\cdot e^{+ia_1^{(0)}h}\cdot A_1^{(0)}$$
$$=\phi_0\cdot I_1^{(0)}\cdot P_2^{(0)}\cdot e^{-ia_1^{(0)}h} \qquad (1,21)$$

In order to solve the equation in terms of the unknowns $A_j^{(0)}(j=1, 2)$, one more relation is needed.

From the second relations of (1, 3) and (1, 7), the elimination of the expression $\sum_{B2} A_{B2}a_{B2}^2$ yields

$$I_2^{(0)}\cdot(P_1^{(0)}-K_5^{(0)})\cdot A_2^{(0)}+I_1^{(0)}\cdot P_2^{(0)}\cdot e^{+ia_1^{(0)}h}\cdot A_1^{(0)}$$
$$=\phi_0\cdot I_1^{(0)}\cdot P_2^{(0)}\cdot e^{-ia_1^{(0)}h}\ , \qquad (1,22)$$

where the expressions (1, 17) and (1, 20) are used.

Solving the simultaneous equations (1, 21) and (1, 22), we have :—

$$\left.\begin{aligned} A_1^{(0)}&=\phi_0\cdot\frac{K_5^{(0)}}{2P_1^{(0)}-K_5^{(0)}}\cdot e^{-i\cdot 2a_1^{(0)}h}\ , \\ A_2^{(0)}&=\phi_0\cdot\frac{I_1^{(0)}}{I_2^{(0)}}\cdot\frac{2P_2^{(0)}}{2P_1^{(0)}-K_5^{(0)}}\cdot e^{-ia_1^{(0)}h}\ . \end{aligned}\right\} \qquad (1,23)$$

Calculating the integrations (1, 4), (1, 6), (1, 8) and (1, 10), these integrations become as follows :—

$$\left.\begin{aligned} K_1^{(0)}&=\frac{1}{(a_1^{(0)})^2}\cdot(1-\cosh a_2^{(0)}H_2)\ , \\ K_2^{(0)}&=\frac{1}{a_2^{(0)}}\cdot\sinh a_2^{(0)}H_2\ , \\ K_3^{(0)}&=\frac{1}{a_1^{(0)}}\cdot H_2\cdot\sinh a_1^{(0)}h \\ &\quad-\frac{1}{(a_1^{(0)})^2}\cdot(\cosh a_1^{(0)}H_1-\cosh a_1^{(0)}h)\ , \\ K_4^{(0)}&=\frac{1}{a_1^{(0)}}\cdot(\sinh a_1^{(0)}H_1-\sinh a_1^{(0)}h)\ , \\ K_5^{(0)}&=\frac{1}{a_1^{(0)}}\cdot\sinh a_1^{(0)}h\ , \end{aligned}\right\} \qquad (1,24)$$

$$K_1^{(s)}=\frac{1}{(a_2^{(s)})^2}\cdot(-1+\cos a_2^{(s)}H_2)\ ,$$

$$\left.\begin{aligned}
K_2^{(s)} &= \frac{1}{a_2^{(s)}}\cdot\sin a_2^{(s)}H_2\ ,\\
K_3^{(s)} &= \frac{1}{a_1^{(s)}}\cdot H_2\cdot\sin a_1^{(s)}h\\
&\quad+\frac{1}{(a_1^{(s)})^2}\cdot(\cos a_1^{(s)}H_1-\cos a_1^{(s)}h)\ ,\\
K_4^{(s)} &= \frac{1}{a_1^{(s)}}\cdot(\sin a_1^{(s)}H_1-\sin a_1^{(s)}h)\ ,\\
K_5^{(s)} &= \frac{1}{a_1^{(s)}}\cdot\sin a_1^{(s)}h\ .
\end{aligned}\right\}\qquad(1,25)$$

Since the theory is developed under the first order of the approximation, the expression $\cosh a_j^{(0)}H_j$ and $\sinh a_j^{(0)}H_j$ have physical meaning at most to the first order of $a_j^{(0)}H_j$. Therefore, the integrated results (1, 24) are reduced to the following :—

$$\left.\begin{aligned}
K_1^{(0)} &= 0\ ,\\
K_2^{(0)} &= H_2\ ,\\
K_3^{(0)} &= H_2h\ ,\\
K_4^{(0)} &= H_1-h\ ,\\
K_5^{(0)} &= h\ .
\end{aligned}\right\}\qquad(1,26)$$

Likewise, applying the first approximation to (21) in section 3, (21) is reduced to

$$I_j^{(0)}=H_j\quad(j=1,\ 2)\ .\qquad(1,27)$$

Using (1, 26), (1, 17) and (1, 20) become

$$P_1^{(0)}=\frac{\omega^2}{g}\cdot H_2h+H_2+h\ ,\qquad(1,28)$$

$$P_2^{(0)}=H_2\qquad(1,29)$$

Taking into account that ω^2 is the second order of $a_j^{(0)}H_j$, i. e. $(a_j^{(0)})^2gH_j(\doteqdot\sqrt{a_j^{(0)}g\tanh a_j^{(0)}H_j})$ (g: dimensionless value), (1, 28) is further reduced to

$$P_1^{(0)}=H_2+h\ .\qquad(1,30)$$

Now substituting (1, 26), (1, 27), (1, 29) and (1,30) into (1, 23), we have the final expressions for $A_j^{(0)}$ as follows :—

$$\left.\begin{aligned} A_1^{(0)} &= \phi_0 \cdot \frac{H_1 - H_2}{H_1 + H_2} \cdot e^{-i \cdot 2a_1^{(0)} h} \;, \\ A_2^{(0)} &= \phi_0 \cdot \frac{2H_1}{H_1 + H_2} \cdot e^{-i \cdot a_1^{(0)} h} \;. \end{aligned}\right\} \tag{1, 31}$$

In the above expressions, as far as the amplitude factors are concerned, these factors are completely the same as those derived from a consideration of flux devised by Lamb. On the contrary, the phase factors appear in the solutions obtained by our method, while Lamb's method cannot yield these factors.

In this paper, our primary concern is to examine the applicability of the method of the buffer domain to the problem in which there exists an abruptly changing vertical boundary. Hence, the discussion of the phases is postponed until the theory is developed in the second approximation.

Next, let us consider the higher modes of the waves.

Substituting the expression $\sum_{B1} A_{B1}$ in (1, 18) into (1, 16) and (1, 19), we have :—

$$\left.\begin{aligned} A_1^{(s)} &= \frac{P_1^{(s)}}{P_2^{(0)}} \cdot \frac{I_2^{(0)}}{I_1^{(s)}} \cdot e^{+a_1^{(s)} h} \cdot A_2^{(0)} \;, \\ A_2^{(s)} &= \frac{P_2^{(s)}}{P_2^{(0)}} \cdot \frac{I_2^{(0)}}{I_2^{(s)}} \cdot A_2^{(0)} \end{aligned}\right\} \tag{1, 32}$$

And $A_1^{(s)}$ and $A_2^{(s)}$ ($s=1,\ 2,\ 3,\ \cdots$) are proved to be the second order of $a_j^{(0)} H_j (j=1,\ 2)$. Therefore, in the range of the approximation used in this paper, the terms of the higher modes of the waves have no significance. The verification is carried out in the following way.

Since ω^2 is of the order of $(a_j^{(0)})^2 \cdot g \cdot H_j^{(0)}$, the Airy's relation $\omega^2 = -a_j^{(s)} g \tan a_j^{(s)} H_j$ ($s=1,\ 2,\ 3,\ \cdots$), which is derived from the equation under the surface and bottom conditions, becomes

$$(a_j^{(0)})^2 H_j = -a_j^{(s)} \tan a_j^{(s)} H_j \;. \tag{1, 33}$$

Introducing a parameter y, (1, 33) is separated into two equations

$$\left.\begin{aligned} & y = \tan a_j^{(s)} H_j^{(s)} \\ & (a_j^{(s)} H_j) \cdot y = -(a_j^{(0)} H_j^{(0)})^2 \;. \end{aligned}\right\} \tag{1, 34}$$

And the solutions of (1, 33) are given as the values of $a_j^{(s)} H_j$ at the intersecting points of the two equations (1, 34), of which the curves are shown in Fig. 2.

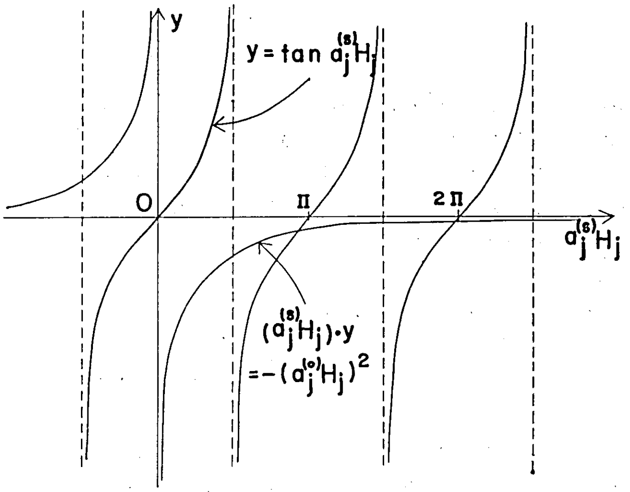

Fig. 2.

The second equation of (1, 34) denotes the hyperbola, of which the asymptotes are the $a_j^{(s)}H_j$- and y- axes. Since $(a_j^{(0)}H_j)^2$ is very small, this hyperbola runs so close to these asymptotes that the solutions of (1, 33) can be approximated by

$$a_j^{(s)}H_j = s\pi - \delta(a_j^{(s)}H_j) \ , \tag{1, 35}$$

where $\delta(a_j^{(s)}H_j)$ is very small in amount.

Substituting (1, 35) into (1, 33), (1, 33) becomes

$$(a_j^{(0)}H_j)^2 = -\{s\pi - \delta(a_j^{(s)}H_j)\}\cdot \tan \delta(a_j^{(s)}H_j) \ . \tag{1, 36}$$

Using $\tan \delta(a_j^{(s)}H_j) \simeq \delta(a_j^{(s)}H_j)$, the expression (1, 36) is reduced to

$$\{\delta(a_j^{(s)}H_j)\}^2 - s\pi\cdot\delta(a_j^{(s)}H_j) + (a_j^{(0)}H_j)^2 = 0 \ . \tag{1, 37}$$

Solving the equation (1, 37), we have

$$\delta(a_j^{(s)}H_j) = \frac{1}{2}\cdot\left\{s\pi \pm s\pi\cdot\sqrt{1-4\left(\frac{a_j^{(0)}H_j}{s\pi}\right)^2}\right\} \ .$$

Expanding the expression of a square root by power series and allowing for a requirement that $\delta(a_j^{(s)}H_j)$ must be positive, the above expression is reduced to

$$\delta(a_j^{(s)} H_j) = \frac{1}{s\pi} \cdot (a_j^{(0)} H_j)^2 \,. \qquad (1, 38)$$

Now we find that $\delta(a_j^{(s)} H_j)$ is of the second order of $(a_j^{(0)} H_j)$. Therefore, neglecting the term of the second order of $(a_j^{(0)} H_j)$, the value of $(a_j^{(s)} H_j)$ becomes, from (1, 35)

$$a_j^{(s)} H_j = s\pi \quad (s=1,\ 2,\ 3,\ \cdots) \qquad (1, 39)$$

By use of (1, 39), the expressions (1, 25) are further reduced to the following forms :—

$$\left.\begin{aligned}
K_1^{(s)} &\begin{cases} =0 & (s:\text{even}) \\ =-2\cdot\left(\dfrac{H_2}{s\pi}\right)^2 & (s:\text{odd})\,, \end{cases} \\
K_2^{(s)} &=0\,, \\
K_3^{(s)} &=\frac{H_1\cdot H_2}{s\pi}\cdot\sin a_1^{(s)} h \\
&\quad +\left(\frac{H_1}{s\pi}\right)^2\{(-1)^s-\cos a_1^{(s)} h\}, \\
K_4^{(s)} &=-\frac{H_1}{s\pi}\cdot\sin a_1^{(s)} h\,, \\
K_5^{(s)} &=\frac{H_1}{s\pi}\cdot\sin a_1^{(s)} h\,.
\end{aligned}\right\} \qquad (1, 40)$$

From (1, 17), (1, 20) and (1, 40), the following is obtained :—

$$\left.\begin{aligned}
P_1^{(s)} &=\frac{\omega^2}{g}\cdot\left[\frac{H_1\cdot H_2}{s\pi}\cdot\sin a_1^{(s)} h\right. \\
&\quad \left.+\left(\frac{H_1}{s\pi}\right)^2\cdot\{(-1)^s-\cos a_1^{(s)} h\}\right], \\
P_2^{(s)} &\begin{cases} =0 & (s:\text{even}) \\ =-\dfrac{\omega^2}{g}\cdot 2\cdot\left(\dfrac{H_2}{s\pi}\right)^2 & (s:\text{odd})\,. \end{cases}
\end{aligned}\right\} \qquad (1, 41)$$

Likewise, substituting (1,39) into (22) in section 3, we have

$$I_j^{(s)} = \frac{1}{2} H_j \quad (j=1, 2)\,. \qquad (1, 42)$$

Finally, putting (1, 27), (1, 29), (1, 41) and (1, 42) into (1, 32), the higher modes of the waves are expressed as

$$\left.\begin{aligned} A_1^{(s)} &= 2\cdot\frac{\omega^2}{g}\cdot\left[\frac{H_2}{s\pi}\cdot\sin a_1^{(s)}h \right.\\ &\quad \left. +\frac{H_1}{(s\pi)^2}\cdot\{(-1)^s-\cos a_1^{(s)}h\}\right]\cdot e^{+a_1^{(s)}h}\cdot A_2^{(0)}\ , \\ A_2^{(s)} &\begin{cases} =0 & (s:\text{even}) \\ =-4\cdot\dfrac{\omega^2}{g}\cdot\dfrac{H_2}{(s\pi)^2}\cdot A_2^{(0)} & (s:\text{odd})\ . \end{cases} \end{aligned}\right\} \quad (1, 43)$$

The first expression in the above has a factor $\exp(a_1^{(s)}h)$. Since the expression of the wave height in the domain D_1 is defined in the range $x>h$, the term $\exp(a_1^{(s)}h)$ in the first expression of (1, 43) does not actually contribute to the magnitude of $A_1^{(s)}$. That is to say, if one describes the higher modes of the waves in complete form, these terms are expressed as

$$\begin{aligned} &A_1^{(s)}e^{-a_1^{(s)}x}\cos a_1^{(s)}(H_1+z) \\ &=2\cdot\frac{\omega^2}{g}\cdot\left[\frac{H_2}{s\pi}\cdot\sin a_1^{(s)}h+\frac{H_1}{(s\pi)^2}\cdot\{(-1)^s-\cos a_1^{(s)}h\}\right] \\ &\times e^{-a_1^{(s)}(x-h)}\cdot A_2^{(0)}\cdot\cos a_1^{(s)}(H_1+z)\ . \end{aligned}$$

The above expression has a significance only for the range $x>h$, so that the value of the factor $\exp\{-a_1^{(s)}(x-h)\}$ is smaller than 1.

As already mentioned, the following approximation is valid, i. e.,

$$\omega^2\simeq(a_j^{(0)})^2gH\ .$$

By virtue of the above approximation, it turns out, from (1, 43), that the higher modes of the waves in the advancing and reflected waves are of the second order of $(a_j^{(0)}H_j)$.

In the next paper, the theory will be developed in the second order of the approximation. And also the discussion will be detailed for the behaviors of all modes of the waves advancing through and reflected at the step of the bottom.

19. ステップ状の水底をもつ水中における Buffer Domain の方法について [I]

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先に筆者は，境界が急激に変るような場における波を解析的にとり扱うに便利な方法を導入した(例えば，L字水路における津波など)．この方法の特長は急激に変る境界部近傍に buffer domain と称される新しい領域を作り上げるところである．それ故に，この方法は buffer domain の方法と言われる．

今迄の取り扱いは境界が水平方向に変る場合のみであつたが，本論説において，筆者はこの方法が，境界が垂直方向に急激に変る場合にも適用し得ることを証明せんと試み，実際に適用できることを知つた．

本論説の主な目的は， buffer domain の方法が，境界が垂直方向に急に変化する場合の問題に適用可能であることを証することであり，二次的な目的として，不規則な境界の近傍での波の状態を知ることである．そしてステップ近傍での波の様相として次の結果を得た．

(1) ステップを通つて進む波，およびステップの所から反射する波の振幅因子は，ラムの流量概念によつて得られる式と同じである．

(2) 位相に関しては，ステップを通過する波に対して $a_1^{(0)}h$，ステップより反射する波に対して $2a_1^{(0)}h$ の位相のズレが起きている．ただし $a_1^{(0)}$ は進入波の波数，h はステップの高さである．

(3) ステップ近傍の波の減衰項は二次の位の微小量である．すなわち $O\{(a_j^{(0)}H_j)^2\}$ (H_j は水深を示す) である．

上に得られた結果のうち，(1) を除いた (2), (3) の結果はラムの流量概念によつては得られない．