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21. *Effect of Pore for Deformation and Failure of Porous Media.*

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1. Introduction

The analysis of mechanical properties of porous media will take an important part in the field of rock mechanics which has been rapidly developed in recent years resulting from the increased interest in rock deformation in civil engineering, structural geology and physics of the earth crust. The existence of pores in rocks is the main cause for making the behaviour of rock deformation quite different from that of other materials. The theory of rock deformation is different from the simple elasticity or plasticity problem because of the existence of pores. Rock mechanics should have in future therefore a unique system of mathematical theory which will compete with the theory of elasticity or plasticity. Even though the models of porous media which have been developed by Biot[1)-8)], Gassmann[9),10)], and Geertsma[11),12)] are not always a satisfactory equivalent for material of the earth crust, they will certainly be better models for analyzing the mechanical phenomena which take place in the earth's crust. The purpose of this study is to extend the theory of porous media so as to account for the role of pores which seem

1) M. A. BIOT, *Jour. Appl. Phys.*, **12** (1941), 155, *ibid.*, 426, *ibid.*, 578, *ibid.*, **13** (1942), 35.
2) M. A. BIOT, *Jour. Appl. Phys.*, **26** (1955), 182.
3) M. A. BIOT, *Jour. Appl. Phys.*, **27** (1956), 459.
4) M. A. BIOT, *Jour. Acous. Soc. Ameri.*, **28** (1956), 168, *ibid.* 179.
5) M. A. BIOT, *Jour. Appl. Mech.*, **76** (1956), 91.
6) M. A. BIOT, and D. G. WILLIS, *Jour. Appl. Mech.*, **77** (1957), 594.
7) M. A. BIOT, *Jour. Appl. Phys.*, **33** (1962), 1482.
8) M. A. BIOT, *Jour. Acous. Soc. Ameri.*, **34** (1962), 1254.
9) F. GASSMANN, *Vierteljahrsschrift der Naturforschenden Gesselschaft in Zurich*, **96** (1951), 1.
10) F. GASSMANN, *Geophysics*, **16** (1951), 673.
11) J. GEERTSMA, *Trans. AIME*, **210** (1957), 331.
12) J. GEERTSMA, and D. C. SMIT, *Geophysics*, **26** (1961), 169.

to characterize the behaviour of rock deformation.

Biot[13] has presented a theory of porous media for the consolidation of liquid-filled surface soil. Later, he has developed the theory to a more elegant form by using 7–7 matrix representation for stress and strain relationship[14]. His treatment of porous media seems to describe the system of mechanics of porous media by using adequate descriptive quantities which correspond to elastic constants in elastic medium. For the determination of these descriptive quantities, he gives the method of measurement[15]. Recently he reformulated the theory in a more systematic form of representation, and has shown that the theory of porous media is another complete mechanical system[16]. He has also shown the application of the theory for the wave propagation[17], consolidation[18], and structural geology[19] etc.

On the other hand, Gassmann[20] presented independently a theory of porous media and has shown that the bulk modulus at liquid-filled state is expressed as functions of bulk modulus of framework, solid material, liquid and porosity. He successfully explained the increase of compressional wave velocity due to water saturation.

These two theories, Biot's and Gassmann's theories of porous media, seem to be independent at first sight. However, Geertsma[21] and De Witte and Warren[22] have shown that these two theories are equivalent. By the work of Geertsma and De Witte and Warren, the descriptive coefficients of deformation in Biot's theory were able to be expressed by the elastic constants of constituents of porous media.

In the above theory of porous media, the compressibility of the medium at dry condition remains still as a given quantity. The theory does not say anything about the compressibility of the framework. The elasticity of framework, however, the decrease of elastic constant of framework from that of constituent material is also another important problem for weathering and fracturing of rock mass. Therefore, it will be necessary to extend the theory so as to account for the framework

13) M. A. BIOT, *loc. cit.*, 1).
14) M. A. BIOT, *loc. cit.*, 2).
15) M. A. BIOT, and D. G. WILLIS, *loc. cit.*, 6).
16) M. A. BIOT, *loc. cit.*, 7) and 8).
17) M. A. BIOT, *loc. cit.*, 4) and 8).
18) M. A. BIOT, *loc. cit.*, 1) and 5).
19) M. A. BIOT, *Trans ASME*, June (1964), 194.
20) F. GASSMANN, *loc. cit.*, 9) and 10).
21) J. GEERTSMA, *loc. cit.*, 11).
22) A. J. DEWITTE and J. E. WARREN, *Trans. AIME*, **210** (1957), 339.

elasticity by the factors of constituents. In this paper, this will be done by introducing the concepts of pore compressibility and pore shear rigidity which will express deformation of pore.

As a natural consequence of introducing pore compressibility and pore shear rigidity, the effect of pore for the failure condition is studied. Because the failure condition of porous media is supposed to be controlled by the failure of pore.

In 2, terminology and basic concepts which represent the state of stress and strain in porous media are described.

In 3, stress-strain relationship in volumetric deformation is derived. Therein, pore compressibility is defined and introduced as an explicit quantity. The stress-strain relationship in shape deformation is also derived in 7, where pore shear rigidity is defined. From these stress-strain relationships, several physical quantities which express characteristics of deformation are derived. In 4, framework bulk compressibility is expressed as a function of porosity, pore compressibility, and material compressibility. In 5, partition of total mean stress into framework and liquid is derived. In 6, expression of total compressibility as functions of constituents is derived. In 8, the values of pore compressibility and pore shear rigidity are estimated by comparing the theory of porous media to the theory of deformation of elastic medium with dry holes. In 9, the failure of porous media is examined by assuming that the elastic limit is controlled by the limit of pore deformation.

2. Stress and Strain

The stress state of deformed porous media will be expressed by the total stress component, framework stress component, and liquid stress component as is described by Gassmann[23]. The total stress component is the one which represents the stress field of porous media as a whole. Framework stress component is a part of the total stress and is distributed in the framework. Liquid stress component is also a part of the total stress and is in liquid which fills the pore space. Thus the total stress component is composed of framework stress component and liquid stress component. Therefore, the total stress component is a function of framework stress and liquid stress. Often, the total stress component is supposed to be the sum of framework stress and liquid stress. This is, however, not always necessary for the formulation of the theory of

23) F. GASSMANN, *loc. cit.*, 9).

porous media. In the following treatment, the formulation starts with the general condition that the total stress is only a function of framework stress and liquid stress.

The total stress field will be devided into the mean stress field and deviatoric stress field as in the case of elastic media.

The strain field of deformed porous media will be expressed by the total bulk strain component, framework bulk strain component, material strain component, pore strain component, and liquid strain component. The total bulk strain component is the one which represents the strain field of the porous media as a whole. The framework bulk strain component is the one which represents the strain field of the solid framework as a whole. In the case of the closed system, that is, when the liquid does not flow in or out from the pore during deformation, these two strain components come into the same one. However, in the case of the open system, that is, when the liquid is allowed to flow in or out during deformation, these two strain components are unequal.

The material strain component represents the strain field of the solid material of the porous media. The pore strain component represents the strain field of the pore. The liquid strain component represents the strain field of liquid which fills the pore space in the porous media.

The total bulk strain component is a function of framework strain and liquid strain. The framework bulk strain component is a function of material strain and pore strain components.

The total strain field will be devided into mean strain field and deviatoric strain field as in the case of elastic media.

For simplicity, let us suppose in the following treatment that the medium is isotropic; the solid framework is elastic; rigidity of liquid is zero; liquid is sufficiently innert; there is no chemical reaction between solid part and liquid; deformation is differential. These restrictions are not essential in the theory of porous media. Both Biot and Gassmann have treated anisotropic, viscoelastic porous media. We will put the above restrictions only for the clarity of the representation of pore effect.

The assumption of isotropy enables us to make the separate formulation for the volumetric deformation and shape deformation. First, the volume change due to mean stress is considered. In a later section, shape deformation due to deviatoric stress is considered.

3. Stress-Strain Relation for Volumetric Deformation

First, let us consider the volumetric deformation of porous media saturated with liquid. Let us denote the bulk volume, volume of material, pore space, and liquid by V_b, V_s, V_p, and V_l respectively. Since the bulk volume of porous media is the sum of material volume and pore space volume, we have the relation

$$V_b = V_s + V_p\,. \tag{1}$$

Porosity n is defined by

$$n = V_p / V_b\,. \tag{2}$$

In order to obtain the stress strain relation for volumetric deformation, we will follow the line used by Geertsma[24].

The differential volume changes of bulk, material, and pore volume are related to the differential total stress change. Since the total stress is a function of framework stress and liquid stress,

$$d\sigma = f(d\bar{\sigma}, d\tilde{\sigma})\,, \tag{3}$$

the total differential of volume change is given by the following equations,

$$\left.\begin{aligned} dV_b &= \frac{\partial V_b}{\partial \bar{\sigma}} d\bar{\sigma} + \frac{\partial V_b}{\partial \tilde{\sigma}} d\tilde{\sigma} \\ dV_s &= \frac{\partial V_s}{\partial \bar{\sigma}} d\bar{\sigma} + \frac{\partial V_s}{\partial \tilde{\sigma}} d\tilde{\sigma} \\ dV_p &= \frac{\partial V_p}{\partial \bar{\sigma}} d\bar{\sigma} + \frac{\partial V_p}{\partial \tilde{\sigma}} d\tilde{\sigma}\,. \end{aligned}\right\} \tag{4}$$

As is clearly stated by Geertsma, the problem of deriving stress-strain relation in porous media is the determination of the values of these partial derivatives in the above formula (4).

Let us denote compressibility of framework, material, and pore by c_b, c_s, and c_p respectively, and define them by the relations

$$\left.\begin{aligned} c_b &= \frac{1}{V_b}\frac{\partial V_b}{\partial \bar{\sigma}} \\ c_p &= \frac{1}{V_p}\frac{\partial V_p}{\partial \bar{\sigma}} \\ c_s &= \frac{1}{V_s}\frac{\partial V_s}{\partial \tilde{\sigma}}\,. \end{aligned}\right\} \tag{5}$$

24) J. GEERTSMA, *loc. cit.*, 11).

Namely, we define c_b and c_p by the change of volume due to framework stress where liquid stress $\tilde{\sigma}$ is kept constant. The material compressibility c_s is defined by the change of material volume due to liquid stress $d\tilde{\sigma}$ where framework stress $\bar{\sigma}$ is kept constant. Neither Gassmann nor Geertsma explicitly defined pore compressibility as an independent deformation constant. However, the resistance of pore for the deformation of its shape will be different from those of solid material and framework. Pore compressibility should, therefore, be considered as an independent quantity. This will vary according to the geometrical shape, properties of the solid material within which the pore exists, the weathering of pore surface, contact mechanism of crack, and many other factors. Therefore, the introduction of pore compressibility will be essential to evaluate the effect of pore in the deformation and failure of porous media. The effect of pore will be expressed not only by porosity but also by pore compressibility. It will be shown in 4, that c_b is expressed as a function of c_p.

In order to evaluate the other three partial derivatives in formula (4), let us suppose the special cases of deformation under the condition of $d\tilde{\sigma}=0$ or $d\bar{\sigma}=0$. First, let us take the case $d\tilde{\sigma}=0$. The case of $d\tilde{\sigma}=0$ is called the open system, where liquid is allowed to flow out or in freely during deformation. This is also the case of dry state.

From equations (1) and (2), we have the relations

$$dV_b = dV_s + dV_p \,, \tag{6}$$

$$dV_b/V_b = (1-n)dV_s/V_s + n\,dV_p/V_p \,. \tag{7}$$

By substituting equations (4) and (5) in equation (7), we get

$$\frac{1}{V_s}\frac{\partial V_s}{\partial \bar{\sigma}} = \frac{1}{(1-n)}(c_b - nc_p) \qquad (d\tilde{\sigma}=0)\,. \tag{8}$$

Next, let us take the case $d\bar{\sigma}=0$, where the only stress working in the media is liquid stress. In equation (4) under this condition, there are two unknown partial derivatives, $\partial V_b/\partial\tilde{\sigma}$ and $\partial V_p/\partial\tilde{\sigma}$. In order to determine these values we must have one more relation among volume changes in addition to the relation (7). Here we take the assumption of similarity for the deformation due to liquid stress. Namely, we assume that the relation

$$\frac{dV_b}{V_b} = \frac{dV_s}{V_s} = \frac{dV_p}{V_p} \qquad (d\bar{\sigma}=0) \tag{9}$$

holds for the deformation due to liquid stress where the frame stress is kept constant. Then we have from equations (4) and (9),

$$\frac{1}{V_b}\frac{\partial V_b}{\partial \tilde{\sigma}}=\frac{1}{V_s}\frac{\partial V_s}{\partial \tilde{\sigma}}=\frac{1}{V_p}\frac{\partial V_p}{\partial \tilde{\sigma}}=c_s\,. \tag{10}$$

Since the middle term is defined as c_s in formula (5), the values $(1/V_b)\partial V_b/\partial\tilde{\sigma}$ and $(1/V_p)\partial V_p/\partial\tilde{\sigma}$ are equal to c_s. Thus we have determined all values of partial derivatives in equation (4).

Therefore, from equation (4), we finally get the stress-strain relation for the volumetric change in porous media,

$$\left.\begin{aligned}\Theta_b&=dV_b/V_b=c_b d\bar{\sigma}+c_s d\tilde{\sigma}\\ \Theta_s&=dV_s/V_s=\frac{1}{(1-n)}(c_b-nc_p)d\bar{\sigma}+c_s d\tilde{\sigma}\\ \Theta_p&=dV_p/V_p=c_p d\bar{\sigma}+c_s d\tilde{\sigma}\,.\end{aligned}\right\} \tag{11}$$

From this stress-strain relationship for volumetric deformation, several interesting physical properties of porous media will be derived. Let us consider them in the following sections.

4. Framework Bulk Compressibility

In equation (11), the volumetric deformation of porous media is described by four independent quantities, c_b, c_s, c_p, and n. It will be natural, however, to expect that there is some relation between framework bulk compressibility c_b and other quantities. Experimental results of ultrasonic wave velocity measurement show that the values of velocity of the rock sample have a wide range even though the rock samples belong to the same classification of rock kind such as granite, sandstone and etc. There seems to be some relations between value of velocity and their degree of weathering, lithification, cementation, and many other factors. Now let us see how this relation will be derived in the theory of porous media.

Gassmann has taken up the special case of deformation under the condition of $d\tilde{\sigma}=0$, namely the deformation of dry specimen. He set up an independent relation for volumetric change of solid material by assuming that the integral law of elasticity[25] holds for this kind of deformation. The volumetric change of material solid due to framework

25) H. Love, *Mathematical theory of Elasticity, section 123～125*, (1944).

stress is given by the integral law in the form of

$$dV_s = V_b c_s d\bar{\sigma} \qquad (d\tilde{\sigma}=0). \tag{12}$$

Since $V_s=(1-n)V_b$, we get the relation

$$\frac{dV_s}{V_s} = \frac{1}{(1-n)} c_s d\bar{\sigma} \qquad (d\tilde{\sigma}=0), \tag{13}$$

under the condition of $d\tilde{\sigma}=0$. Therefore, by comparing equations (13) and (11·2), we get the relation

$$c_b = c_s + nc_p. \tag{14}$$

This is the relation which expresses the framework bulk compressibility as a function of c_s, c_p, and n.

Instead of using the assumption of integral law for the deformation of dry porous media, Geertsma has assumed a reciprocity theorem to hold for framework bulk volume change due to total stress and pore volume change due to framework stress. This assumption also leads to the same relation (14). Both Gassmann and Geertsma have used the relation (14) during their derivation of stress-strain relationship without defining the pore compressibility c_p. The relation (14), however, will be very important for considering the effect of pore in the deformation of porous media.

The introduction of the concept of pore compressibility c_p shows explicitly that the c_b is a function of porosity n. In the treatment of porous media by Biot, Gassmann, and Geertsma, framework bulk compressibility c_b seems to be thought of as a given quantity. However, in fact, c_b is a function of n, c_p, and c_s, as is seen in the formula (14). The framework bulk compressibility c_b is a linear function of porosity n, if pore compressibility c_p, material compressibility c_s are kept constant. However, pore compressibility c_p may also be a function of porosity. Generally speaking, both pore compressibility and porosity seem to be closely related to the degree of weathering, lithification, and cementation. Therefore, the explict introduction of c_p will give a place where such factors are quantitatively taken into consideration.

For a special geometrical shape of pore, the pore compressibility c_p will be calculated. This will be examined in 8.

5. Partition of Stress in Closed System

When a liquid-filled porous medium is deformed, the total stress is partitioned to framework stress and liquid stress. Let us consider the ratio of this partition. For simplicity, first, we take the case of closed system for the estimation of the maximum amount of liquid pressure which will be caused by the deformation of porous medium. The closed system is defined by the case where liquid is not able to flow in or out from the pore space. Therefore, the pore volume change is equal to the liquid volume change. Let us denote the liquid compressibility by c_l. Then, by the definition of c_l, we have

$$\frac{dV_p}{V_p}=\frac{dV_l}{V_l}=c_l d\tilde{\sigma}\,, \tag{15}$$

where V_l denotes the liquid volume. From the equations (15) and (11·3) we have the relation in the closed system

$$\frac{dV_p}{V_p}=c_l d\tilde{\sigma}=c_p d\bar{\sigma}+c_s d\tilde{\sigma}\,. \tag{16}$$

From this equation and formula (14), we have

$$\frac{d\bar{\sigma}}{d\tilde{\sigma}}=\frac{c_l-c_s}{c_p}=\frac{n(c_l-c_s)}{(c_b-c_s)}\,. \tag{17}$$

This is the relation which gives the partition ratio of stress into framework and liquid.

If the total stress is given by the sum of the framework stress and the liquid stress,

$$d\sigma=d\bar{\sigma}+d\tilde{\sigma}\,, \tag{18}$$

the partition of total stress into the framework and liquid is given by the relation

$$\left.\begin{aligned} d\tilde{\sigma}&=\beta d\sigma \\ d\bar{\sigma}&=(1-\beta)d\sigma\,, \end{aligned}\right\} \tag{19}$$

where

$$\beta=\frac{c_p}{c_l-c_s+c_p}=\frac{c_b-c_s}{n(c_l-c_s)+(c_b-c_s)}\,. \tag{20}$$

β may be called the coefficient of partition to liquid stress in the closed system. The relations (19), and (20) will be used for the evaluation of the effect of liquid pressure in the porous medium during deformation.

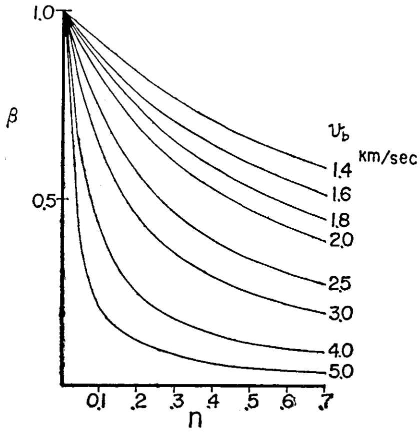

Fig. 1. An example of coefficient β of stress partition into liquid in the closed system as a function of porosity n.
v_b: elastic wave velocity of the framework
v_s=6.0 km/sec, elastic wave velocity of solid material
p_s=2.7 gr/cm³, density of solid material

As a numerical example of partition of stress, the value of the coefficient of partition β is computed and illustrated in Fig. 1 as a function of porosity. The values c_b, c_s, c_l, and their corresponding physical constants are tabulated in Table 1.

Table 1. Elastic constants of solid material, framework and liquid in porous media.

	Velocity (km/sec)	Density (gr/cm³)	Poisson's Ratio	Compressibility (cm²/dyne)	Bulk Modulus (dyne/cm²)
v_b	1.40	2.10	0.25	4.36×10^{-11}	0.229×10^{-11}
	1.60	2.10	〃	3.34 〃	0.299 〃
	1.80	2.10	〃	2.64 〃	0.379 〃
	2.00	2.10	〃	2.13 〃	0.469 〃
	2.50	2.30	〃	1.31 〃	0.763 〃
	3.00	2.30	〃	0.902 〃	1.11 〃
	4.00	2.40	〃	0.469 〃	2.13 〃
	5.00	2.40	〃	0.300 〃	3.33 〃
v_s	6.00	2.70	〃	0.185 〃	5.40 〃
v_l	1.48	1.00	0.50	4.57 〃	0.219 〃

6. Total Compressibility in Closed System

The total compressibility c of porous medium in closed system is defined by the bulk volume change due to the total stress change,

$$\frac{dV_b}{V_b} = cd\sigma \,. \tag{21}$$

Then, from equations (11·1) and (21), we have the relation

$$cd\sigma = c_b d\bar{\sigma} + c_s d\tilde{\sigma} \,. \tag{22}$$

We also have the other equation for $cd\sigma$ from equations (7), (11·2), and (21),

$$cd\sigma = (c_b - nc_p)d\bar{\sigma} + \{(1-n)c_s + nc_l\}d\tilde{\sigma} \,. \tag{23}$$

When the total stress is given by the sum of the framework stress and liquid stress, the equations (22) and (23) are seen as simultaneous equations for $d\bar{\sigma}$ and $d\tilde{\sigma}$. In order that the solution of $d\bar{\sigma}$ and $d\tilde{\sigma}$ exists, the determinant of the coefficients of these equations should be zero. By this condition, we have the relation

$$c = \frac{nc_p(c_l - c_s) + c_s - nc_p}{nc_p + n(c_l - c_s)} \,, \tag{24}$$

or, by using the relation of $c_b = c_s + nc_p$,

$$c = \frac{nc_b + c_s\delta}{n + \delta} \,, \qquad \delta = \frac{c_b - c_s}{c_l - c_s} \,. \tag{25}$$

These are the relations which show that total bulk compressibility of liquid-filled porous medium is a function of porosity, liquid, material, pore and framework bulk compressibility. The formula (25) is the same one as obtained by Gassmann[26]. A nomograph is prepared by the writer[27] in an early paper for computing c from c_b for various values of porosity n.

7. Stress-Strain Relation in Shape Deformation

So far, we have treated the volumetric deformation of porous medium, and examined the compressibility of the medium as functions of pore compressibility and porosity. There is another independent elastic modulus,

26) F. GASSMANN, *loc. cit.*, 9).

27) S. NAGUMO, *Bull. Geological Survey of Japan*, **8** (1957), 505 and 523.

rigidity, for describing the deformation of the isotropic porous medium. Let us see now how the rigidity of the medium is controlled by the existence of pore.

We will take up the pure shear deformation. Let us denote the deviatoric strain component of framework, material, and pore by e^b_{ij}, e^s_{ij}, and $e^p_{ij}\,(ij=xy, yz, zx)$ respectively. Also let us denote the deviatoric stress component of bulk volume and framework by $d\tau_{ij}$ and $d\bar{\tau}_{ij}$ respectively. Since liquid does not bear shearing stress in the shape deformation, the framework bulk deviatoric stress is equal to the total bulk deviatoric stress,

$$d\tau_{ij}=d\bar{\tau}_{ij}\,. \tag{26}$$

Then bulk shear compliance γ_b and pore shear compliance γ_p are defined by the relation

$$\left.\begin{aligned} e^b_{ij}&=\gamma_b d\bar{\tau}_{ij} \\ e^p_{ij}&=\gamma_p d\bar{\tau}_{ij}\,. \end{aligned}\right\} \tag{27}$$

This is the similar relation as eq. (5·1) (5·2) in the volumetric deformation. Shear compliance of the material solid γ_s is not defined by the similar relation of (5·3). It is given by the elastic constant of its own substance. The relation between material deviatoric strain e^s_{ij} and the framework deviatoric stress $d\bar{\tau}_{ij}$ is obtained as follows. Since the shear strain component is generally expressed by the variation of angle due to deformation, it will be supposed that the addition law weighted by porosity holds for the framework, material, and pore deviatoric strain component. Namely, we suppose that the relation

$$e^b_{ij}=(1-n)e^s_{ij}+ne^p_{ij}\,, \tag{28}$$

holds in shape deformation. Then, from equations (27) and (28), we have

$$e^s_{ij}=\frac{1}{(1-n)}(\gamma_b-n\gamma_p)d\bar{\tau}_{ij}\,. \tag{29}$$

In order to find the relation among γ_b, γ_p, and γ_s, we assume that the integral law of elastic deformation also holds for the pure shear deformation as in the case of volumetric deformation. Namely, we assume

$$(1-n)e^s_{ij}=\gamma_s d\bar{\tau}_{ij}\,. \tag{30}$$

Then, from equations (29) and (30), we have the relation

$$\gamma_b=\gamma_s+n\gamma_p\,. \tag{31}$$

This is the relation which expresses framework shear compliance as functions of porosity, pore, material shear compliance. The dependence of shear compliance to the porosity, material, and pore is the same as in the case of volumetric deformation.

The introduction of pore shear compliance will play an important role in the failure of porous medium, which will be discussed in § 9.

8. Value of Pore Compressibility and Pore Shear Compliance

The pore compressibility c_p and pore shear compliance γ_p were introduced as independent deformation coefficients in the porous medium. However, since these deformation coefficients depend upon the way of pore deformation, it will be supposed that the values of c_p and γ_p should be calculated if the deformation of pore is controlled only by the special geometrical conditions.

Elastic constants of media with small holes are calculated by Y. Satô[28], following the methods of Machenzie[29] and Fröhlich and Sack[30]. In their results, the elastic constants of equivalent elastic media are expressed by the relation

$$\left.\begin{aligned} \frac{1}{k}&=\frac{1}{k_0\rho}+\frac{3(1-\rho)}{4\mu_0\rho}+0[(1-\rho)^3] \\ \frac{\mu_0-\mu}{\mu_0}&=5(1-\rho)\frac{3k_0+4\mu_0}{9k_0+8\mu_0}+0[(1-\rho)^2]. \end{aligned}\right\} \tag{32}$$

where

k, k_0 : bulk modulus of actual material and real material
μ, μ_0 : rigidity of actual material and real material
ρ : relative density
$(1-\rho)$: porosity.

From the equation (32), we have other expressions,

$$\left.\begin{aligned} \frac{1}{k}&=\frac{1}{k_0}+(1-\rho)\left(\frac{1}{k_0}+\frac{3}{4\mu_0\rho}\right) \\ \frac{1}{\mu}&=\frac{1}{\mu_0}+(1-\rho)\frac{5(3k_0+4\mu_0)}{\mu_0(9k_0+8\mu_0)}\,. \end{aligned}\right\} \tag{33}$$

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28) Y. Sotô, *Bull. Earthq. Res. Inst.*, **30** (1952), 179.
29) J. K. Mackenzie, *Proc. Phys. Soc.*, **B 63** (1950), 2.
30) Frohlich and Sack, *Proc. Roy. Soc.*, A, **185** (1946), 415.

By comparing this equation (33) with equations (14) and (31), we have

$$\left.\begin{aligned} c_p &= c_s + \frac{3}{4(1-n)}\gamma_s \\ \gamma_p &= 5\gamma_s\left(\frac{3\gamma_s+4c_s}{9\gamma_s+8c_s}\right) . \end{aligned}\right\} \tag{34}$$

These are the expressions of pore compressibility and pore shear compliance of elastic medium with small holes as functions of elasticities of material solid.

As stated in 3 the way of pore deformation seems to be controlled not only by geometrical shape but also the weathering condition at the pore surface, contacting mechanism of crack, and many other factors. Therefore the value of c_p and γ_p will not be a simple function of constituents in a general porous medium.

9. Failure Criteria

Now let us consider the failure of porous media. It is well known that strength of volcanic rocks decreases as the weathering process proceeds, and that the strength of sedimentary rocks increases as the degree of lithification and compaction increases. There should be some certain correlation between strength of rocks and some physical parameters of rocks. However, we do not know yet the quantitative relation for the dependence of strength to such macroscopic elements as porosity, weathering of material solid, elastic wave velocity, non-linearity of stress-strain curve, and amount of hysteresis strain, etc. In order to look for such relationships, and also to design the experimental procedure which would describe the mechanism of the failure of rock, it will be worthwhile to examine the failure criteria which will be derived or expected in the theory of porous media. Since the behaviour of failure of natural rock is very complex, agreement will not be expected. Instead, it will give a preliminary viewpoint for further consideration as a thought experiment.

Terzaghi[31] has pointed out in soil mechanics that the consolidation of soil is controlled by effective stress, that is, total stress minus liquid stress. Hubbert and Rubey[32] have also presented a view that Coulomb-

31) K. TERZAGHI, *Theoretical Soil Mechanics* (John Wiley, 1943).
32) M. K. HUBBERT and W. W. RUBEY, *Bull. Geol. Soc. Amer.*, **70** (1959), 115.
33) J. HANDIN, *Bull. Amer. Assoc. Petroleum Geologist*, **47** (1963), 717.

Mohr's internal friction criterion with respect to effective stress is useful for the failure of porous material. This view is confirmed by Handin's[33] laboratory experiments.

Since the effective stress which has been used by Terzaghi, and Hubbert and Rubey is nothing but the framework bulk stress in the theory of porous media, it will be natural to suppose that the failure of porous media is controlled by framework. Since the elastic constant of framework is expressed as functions of elastic constants of pore and solid materials, as is shown in the preceeding sections, let us first suppose that (1) the elastic limit of porous media will be controlled by the elastic limit of pore deformation. Next, let us assume further, for simplicity, that (2) the failure of media occurs when the failure of pore takes place. As regards the second assumption, there will be much room for modification which will be left for further consideration.

As for the elastic limit of pore deformation, let us take up the following criteria for the failure of rock which have been examined by Robertson[34]. They are (1) Maximum principal stress criterion, (2) Maximum shear stress criterion, (3) Maximum strain criterion, (4) Coulomb-Mohr's internal friction criterion, (5) Griffith's criterion. Using these criteria for the elastic limit of pore deformation, let us examine how the critical value of total stress is controlled by the elements of porous media, To make the argument simple, we will take the case of open system or dry state. Namely, we consider the case where liquid pressure is kept constant or zero. A simple method of estimating the effect of liquid will be reported in a coming paper. Let us start with the maximum principal stress criterion.

9.1 Maximum Principal Stress Criterion

This criterion states that the failure of porous medium takes place when the maximum principal stress reaches a certain value. In order to apply this criterion to pore stress, we have to define pore stress. The framework bulk stress is supposed to be partitioned into material solid and pore framework. When the porous medium is deformed, the microscopic stress distribution in framework will be very complex and heterogeneous as is expected from the photo-elastic experiment for material with holes. However, as a macroscopic effect of this heterogeneous stress distribution, the stress field would be equated by the

34) E. C. ROBERTSON, *Bull. Geol. Soc. Amer.*, **66** (1955), 1275.

superposition of two homogeneous stress fields of total solid stress $d\bar{\sigma}_s$ and total pore stress $d\bar{\sigma}_p$. The partition of the mean stress will be defined by the partition of bulk dilatation into solid dilatation and pore dilatation. Then, from equations (11) and (7), we get the relations

$$d\bar{\sigma}=\frac{1}{c_b}\Theta_b=\frac{1}{c_b}\{(1-n)\Theta_s+n\Theta_p\}\ , \tag{35}$$

$$d\bar{\sigma}=d\bar{\sigma}_s+d\bar{\sigma}_p\ , \tag{36}$$

where $d\bar{\sigma}_s$ and $d\bar{\sigma}_p$ are defined by

$$\left.\begin{aligned} d\bar{\sigma}_s&=\frac{1}{c_b}(1-n)\Theta_s \\ d\bar{\sigma}_p&=\frac{n}{c_b}\Theta_p\ . \end{aligned}\right\} \tag{37}$$

Then, by substituting the stress-strain relation (11·1) and (11·2) at dry condition into (37), we have the relations

$$\left.\begin{aligned} d\bar{\sigma}_s&=\frac{c_b-nc_p}{c_b}d\bar{\sigma} \\ d\bar{\sigma}_p&=n\cdot\frac{c_p}{c_b}d\bar{\sigma}\ . \end{aligned}\right\} \tag{38}$$

Now, let us apply maximum principal stress criterion for pore stress. Let us denote the value of critical stress by K_I. Then

$$K_I=d\bar{\sigma}_{pc}=n\cdot\frac{c_p}{c_b}d\bar{\sigma}_c\ , \tag{39}$$

where suffix c denotes the critical value. Then we have

$$d\bar{\sigma}_c=K_I\cdot\frac{1}{n}\cdot\frac{c_b}{c_p}\ . \tag{40}$$

If we use the relation $c_b=c_s+nc_p$, we have another expression of

$$d\bar{\sigma}_c=K_I\cdot\frac{c_b}{c_b-c_s}=K_I\left\{1+\frac{c_s}{nc_p}\right\}\ . \tag{41}$$

When c_s and c_b are expressed by the elastic wave velocities and densities, (41) becomes

$$d\bar{\sigma}_c=K_I\cdot\frac{1}{1-(\alpha_s\rho_b v_b^2/\alpha_b\rho_s v_s^2)}\ , \tag{42}$$

where v_b and v_s are compressional wave velocity of framework and material solid respectively, and are given by

$$v_b = \sqrt{\frac{\alpha_b}{\rho_b c_b}}\ , \qquad v_s = \sqrt{\frac{\alpha_s}{\rho_s c_s}}\ , \tag{43}$$

where α_b, α_s are constants given by Poisson's ratio. The equations (40), (41) and (42) are the expressions of critical framework stress. From these equations, it is clearly seen that if c_b/c_p were kept constant during the process of weathering or lithification, critical stress would be inversely proportional to the porosity.

9.2 Maximum Shear Stress Criterion

This criterion states that the failure takes place when the maximum shear stress reaches a certain value. In order to apply this criterion to pore stress, we have to define the partition of shear stress as is the case of mean stress. The framework shear stress is supposed to be partitioned according to the partition of shear strain. From equations (27) and (28), we have

$$d\bar{\tau}^b_{ij} = \frac{e^b_{ij}}{\gamma_b} = \frac{1}{\gamma_b}\{(1-n)e^s_{ij} + ne^p_{ij}\}\ , \tag{44}$$

$$d\bar{\tau}^b_{ij} = d\bar{\tau}^s_{ji} + d\bar{\tau}^p_{ij}\ , \tag{45}$$

where pore shear stress $d\bar{\tau}^p_{ij}$ and material shear stress $d\bar{\tau}^s_{ij}$ are defined by

$$\left.\begin{aligned} d\bar{\tau}^s_{ij} &= \frac{1}{\gamma_b}(1-n)e^s_{ij} \\ d\bar{\tau}^p_{ij} &= \frac{1}{\gamma_b} ne^p_{ij}\ . \end{aligned}\right\} \tag{46}$$

By substituting eq. (27), (29) into (46), we have

$$\left.\begin{aligned} d\bar{\tau}^s_{ij} &= \frac{\gamma_b - n\gamma_p}{\gamma_b} d\bar{\tau}^b_{ij} \\ d\bar{\tau}^p_{ij} &= n\cdot\frac{\gamma_p}{\gamma_b} d\bar{\tau}^b_{ij}\ . \end{aligned}\right\} \tag{47}$$

Now applying the maximum shear stress criterion for the pore framework, the critical pore shear stress is given by

$$d\bar{\tau}_c^p = K_{II} = n\frac{\gamma_p}{\gamma_b} d\bar{\tau}_c\,, \tag{48}$$

where $d\bar{\tau}_c^p$, $d\bar{\tau}_c$, and K_{II} denote the critical value of pore shear stress, framework shear stress and constant of critical value respectively. Therefore we have expressions for critical framework stress in various forms of

$$d\bar{\tau}_c = \frac{\gamma_b}{n\gamma_p} K_{II}\,, \tag{49}$$

$$d\bar{\tau}_c = K_{II} \cdot \frac{1}{1-\gamma_s/\gamma_b}\,, \tag{50}$$

$$d\bar{\tau}_c = K_{II} \cdot \frac{1}{1-(\rho_b V_b^2/\rho_s V_s^2)}\,, \tag{51}$$

where V_b, V_s denote the shear wave velocity of framework and material solid respectively. These relations also show that critical framework stress is inversely proportional to porosity. The dependency of critical value is the same as in the case of the first criterion.

9.3 Maximum Shear Strain Criterion

The partition of strain into solid material and pore frame is given in equations (7) and (28). The mean and deviatoric strains of pore are $n\Theta_p$ and ne_{ij}^p respectively. The maximum strain criterion for pore states that failure of the whole medium occurs as either ne_{ij}^p or $n\Theta_p$ reaches a certain value, L. From the relations of $\Theta_p = c_p d\bar{\sigma}_c$, and $L = n\Theta_p$, the critical framework stress, $d\bar{\sigma}_c$, is given by

$$d\bar{\sigma}_c = \frac{1}{nc_p} L\,. \tag{52}$$

If we use the relation $c_b = c_s + nc_p$, it becomes

$$d\bar{\sigma}_c = L \frac{1}{c_b - c_s}\,. \tag{53}$$

9.4 Strain Energy Criterion

The strain energy W_p of pore deformation is defined by the product of total pore strain and total pore stress, i.e.

$$W_p = \sum ne_{ij}^p \cdot d\tau_{ij}^p\,. \tag{54}$$

By substituting equations (27) and (46) into (54), it becomes

$$W_p = n^2 \cdot \frac{\gamma_p^2}{\gamma_b} \sum (d\bar{\tau}_{ij})^2 . \tag{55}$$

If we denote the critical value of pore strain energy by S,

$$W_{pc} = S , \tag{56}$$

the critical strain energy of bulk framework $\gamma_b \sum (d\bar{\tau}_{ij})^2$ is given by the relation

$$\gamma_b \sum (d\bar{\tau}_{ij})^2 = \frac{\gamma_b^2}{n^2 \gamma_p^2} S . \tag{57}$$

Therefore, the critical strain energy of bulk framework is proportional to $1/n^2$ if γ_b^2/γ_p^2 is kept constant.

If we define the generalized shear stress $d\tau_n$ by the relation

$$d\tau_n = \sqrt{\sum (d\tau_{ij})^2} , \tag{58}$$

the relation (57) becomes

$$d\tau_n = \frac{\sqrt{\gamma_b}}{n\gamma_p} \sqrt{S} . \tag{59}$$

9.5 Coulomb-Mohr's Internal Friction Criterion

The internal friction criterion for pore deformation will be expressed by

$$d\tau_n^p = c + d\sigma_n^p \tan \theta , \tag{60}$$

where $d\sigma_n^p$ is the mean pore stress and $d\tau_n^p$ is the tangential pore stress. These are given by the equations

$$d\sigma_n^p = (1/3) \sum d\sigma_{ii}^p , \tag{61}$$

$$d\tau_n^p = \sqrt{\sum (d\tau_{ij}^p)^2} . \tag{62}$$

Therefore, by expressing $d\tau_n^p$ and $d\sigma_n^p$ with framework stress, we get the relation

$$d\bar{\tau}_n = c' + d\bar{\sigma} \tan \theta' , \tag{63}$$

where

$$\left.\begin{aligned} c' &= c\frac{\gamma_b}{n\gamma_p} \\ \tan\theta' &= \frac{c_p\gamma_b}{c_b\gamma_p}\tan\theta\ , \end{aligned}\right\} \tag{64}$$

c' and θ' are cohesive force and angle of internal friction of framework. Equations (63) and (64) show that θ' and c' change as functions of n, c_p, c_b, γ_b, and γ_p.

So far we have examined several criteria, and seen how the limit of elastic deformation is controlled by porosity and constants of constituents of porous media. The condition of failure seems to be independent of elastic constant as far as perfect elastic media is concerned. The condition of failure is not derived in the theory of elasticity. Instead, the criterion comes into the theory as an assumption so as to account for the experimental results.

On the other hand, there are interesting experimental phenomena which show close correlation between elastic constant and strength of materials. As Nadai[35] pointed out, the derivation of failure criterion from the theory of deformation is one of the basic problems of the mechanics of failure. As is seen in the above treatment, the critical stress is expressed as functions of porosity and elastic constants of constituents of porous media. This result will provide a certain key to further development to find the relation between strength and deformation coefficient of porous rock.

10. Summary and Conclusions

The theory of porous media which has been developed by Biot, Gassmann, and Geertsma is extended so as to clarify the effect of pore for the deformation and failure of porous media. Pore compressibility and pore shear compliance, the reciprocal of pore shear rigidity, are introduced as independent deformation coefficients. Stress-strain relationships expressed by porosity and elastic constants of constituents are derived both for volumetric deformation and shape deformation.

Elastic constant of porous medium at dry condition is controlled not only by porosity but also by the elastic constants of constituents. The dependency is expressed by the equations, $c_b = c_s + nc_p$, and $\gamma_b = \gamma_s + n\gamma_p$. This relation will relate to increase of elasticity due to lithification of

35) A. NADAI, *Theory of Flow and Fracture of Solid Vol. I*, (McGraw-Hill, 1950).

sedimentary rock, and decrease of elasticity due to weathering.

The partition of total stress into framework stress and liquid stress is controlled by the compressibility of liquid, pore, and solid material.

The relation between the limit of elastic deformation and porosity, elastic constants of constituents, are derived for various failure criteria, under the assumption of elastic limit being controlled by the limit of pore deformation. As a general tendency, the critical stress seems to be proportional to the reciprocal of porosity.

Analysis of pore effect in stress field of porous media will be reported in another paper.

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Table of Notations

	Stress*		Strain		Volume	Elastic Coefficient	
	mean	deviatoric	mean	deviatoric		compressibility	shear-compliance
Total	σ	τ_{ij}	Θ	e_{ij}	V	c	γ
Framework	$\bar{\sigma}$	$\bar{\tau}_{ij}$	Θ_b	e^b_{ij}	V_b	c_b	γ_b
Solid Material	σ^s	τ^s_{ij}	Θ_s	e^s_{ij}	V_s	c_s	γ_s
Pore	σ^p	τ^p_{ij}	Θ_p	e^p_{ij}	V_p	c_p	γ_p
Liquid	$\tilde{\sigma}$		Θ_l		V_l	c_l	

porosity: n

* positive is taken for dilatative stress

21. 多孔性媒質の変形・破壊におよぼす孔隙の影響

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この報告は, Biot, Gassmann, Geertsma 等によつて発展させられてきた多孔性媒質の力学理論を発展させ, 多孔性媒質の変形・破壊におよぼす孔隙の影響を明かにしようとしたものである. このため, 孔隙の変形を表わすために, 孔隙圧縮率, 孔隙シャコンプライアンス (剛性率の逆数) を独立の変形係数として導入した. 応力歪関係式を孔隙率および媒質の構成要素の弾性係数で表現し, 体積変形, 形状変形のそれぞれについて求めた. その応力歪関係式から, (1) 乾燥状態における多孔性媒質の弾性常数は孔隙率によつて支配されるのみならず, 構成要素の弾性常数によつても支配され, 関係式 $c_b=c_s+nc_p$, $\gamma_b=\gamma_s+n\gamma_p$ で表わされること, (2) 多孔性媒質にかかる全応力が framework

応力と流体応力とに分配される割合は，流体，孔隙，構成物質の圧縮率によつて支配されることなどが分つた．

次に孔隙圧縮率，孔隙シャコンプライアンスを導入した必然的結果として，孔隙の変形あるいは孔隙の受持つ応力がある一定の限界値に達した場合に媒質全体としての弾性限界が表われるという仮定の上に立つて，媒質全体としての限界応力が孔隙率や構成要素の弾性常数とどのように関係するか調べてみた．孔隙に対する限界条件として最大応力条件，最大歪条件，**Coulomb-Mohr** の内部摩擦条件等々を課した場合の媒質の限界応力を求めた．その結果，いろいろな限界条件に共通する一般的傾向として，限界応力が孔隙率に逆比例するという性質があることが分つた．弾性限界を構成要素から導こうとする試みの一つの手懸りになるであろう．

応力場における孔隙の影響，流体圧の影響については別に報告する予定である．