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1. *Effects of Intervenient Boundaries on the Elastic Wave Propagation.*

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Abstract

Propagation of waves through various kinds of intervenient boundaries is systematically investigated in some simple models, namely the first and second order discontinuities and the perfectly continuous medium. The moduli of transmitted and reflected waves vary between maxima and minima with certain frequency intervals for the first order discontinuity and tend from one value to another for the second order discontinuity with increasing frequency. For the perfectly continuous medium, those vary monotonically. It is necessary to observe the wide band spectrum of the reflected waves to discuss the nature of the boundary.

The apparent phase velocity is determined from the phase of the complex transmission coefficient. It is faster in long period waves than in short ones.

1. Introduction

The seismic wave is a kind of progressive wave, and fields of their propagation seem in general to be aeolotropic and heterogeneous. Many wave phenomena, however, have satisfactorily been explained, even if aeolotropy is not taken into account. On the other hand, if a homogeneous medium is in contact with another homogeneous medium, a portion of waves makes to be reflected from the boundary. The theory of reflection and refraction (or transmission) of elastic waves at discontinuities has been studied by many researchers since the end of the last century[1]. Most of them have discussed wave phenomena in structures consisting of two media whose elastic properties vary discontinuously at the boundary,

1) C.G. KNOTT, "Earthquake and Earthquake Sound; as Illustration of General Theory of Elastic Vibrations," *Trans. Seism. Soc. Japan*, **12** (1888), 115-136.

L. M. BREKHOVSKIKH, *Waves in Layered Media* (Academic Press, 1960).

And also many textbooks on the wave phenomena.

and there are a few discussions on the difference of the transient nature of discontinuity.

The distribution of the elasticity across the transition can be classified as in Table 1. A discontinuity at which elasticity is discontinuous and its derivatives are continuous is here called a first order discontinuity, and a discontinuity at which it is continuous and its derivatives are discontinuous is called a second order discontinuity.

The resultant wave in a heterogeneous medium, in general, cannot be separated into progressive and other components[2], so that the wave phenomena in such a medium ought to be treated in what manner a heterogeneous medium with a finite thickness is inserted between two homogeneous media, in one of which an original wave is incident. The model in which the inserted medium is homogeneous corresponds to the first order discontinuity, and the model, in which it varies linearly, to the second order discontinuity.

The behaviour of waves in the case of the intermediate homogeneous medium has been discussed by several authors[3),4)]. It in the case of the transitional medium in which the velocity varies linearly has been discussed, too, and the comparison between them has been discussed little. The solutions in the latter case were written in terms of some Bessel functions[5] and of transcendental polynomials,[6] as well as the transitional medium by an ensemble of a large number of very thin homogeneous media.[7]

The behaviour of waves in such a medium where the property varies as a perfectly gradual manner has exactly been investigated in the case of electromagnetic field, using Kummer's solutions of the hypergeometric

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2) S. A. SCHELKUNOFF, "Remarks Concerning Wave Propagation in Stratified Media," *Comm. Pure and Appl. Math.*, **4** (1951), 117-128.

L. M. BREKHOVSKIKH, *loc. cit.*, p. 229.

3) K. SEZAWA and G. NISHIMURA, "Dispersion of a Shock in Echoing- and Dispersive-Elastic Bodies," *Bull. Earthq. Res. Inst.*, **8** (1930), 321-337.

4) K. SEZAWA and K. KANAI, "The Nature of Transverse Waves Transmitted through a Discontinuity Layer," *Bull. Earthq. Res. Inst.*, **14** (1936), 157-163.

5) K. SEZAWA and K. KANAI, "The Effect of Sharpness of Discontinuities on the Transmission and Reflection of Elastic Waves," *Bull. Earthq. Res. Inst.*, **13** (1935), 750-756.

6) A. WOLF, "The Reflection of Elastic Waves from Transition Layers of Variable Velocity," *Geophysics*, **2** (1937), 357-363.

7) T. MATUZAWA, "Reflexion und Refraktion der seismischen Wellen durch eine kontinuierlich verändernde Schicht: 1, SH Welle," *Bull. Earthq. Res. Inst.*, **33** (1955), 533-548.

Table 1. Classification of the boundary.

elasticity	derivatives of elasticity	nature of discontinuity
continuous	continuous	continuous
continuous	discontinuous	the second order discontinuous
discontinuous	continuous	the first order discontinuous
discontinuous	discontinuous	discontinuous

equation.[8] The effect on the continuous medium agrees approximately with that on a stratification of a lot of layers with weak reflections.[9),10]

In addition, the effect under which the movement is able to slide at the boundary surface has been discussed in detail by some authors.[11]

In this paper, propagation of waves through various kinds of intervenient boundaries is systematically discussed in some simple models. In sections 2, 3 and 4, the formulations for the reflection and transmission coefficients are expressed through the first and second discontinuities and continuously varying medium, respectively. In section 5, these for each discontinuity are compared with each other, and the apparent phase velocity through the intervenient medium is also discussed. In section 6, the concluding remarks and summary are stated.

2. First order discontinuity

Amplitudes of waves reflected from and passing through a contact plane of two different media are determined by the characters of each medium, and are independent of the wave length of incident waves. If velocities in the left- and right-hand side medium of the discontinuity are denoted by $c_1=c_0(1+\varepsilon)$ and $c_3=c_0(1-\varepsilon)$, respectively, as schematically shown in Fig. 1, and if the incident waves advance in the left-hand side medium, the reflection and transmission coefficients, R_0 and T_0, are given by the following formulae, respectively,

8) P. S. EPSTEIN, "Reflection of Waves in an Inhomogeneous Absorbing Medium," *Proc. Natl. Acad. Sci.*, **16** (1930), 627-637.

9) T. MATUZAWA, *loc. cit.*, 7).

10) H. JEFFREYS, "Elastic Waves in a Continuously Stratified Medium," *Mon. Not. Roy. Astr. Soc., Geophys. Suppl.*, **7** (1957), 332-337.

11) G. NISHIMURA and K. KANAI, "On the Effects of Discontinuity Surfaces upon the Propagation of Elastic Waves," *Bull. Earthq. Res. Inst.*, **11** (1933) 123-186, 595-631, **12** (1934) 277-316, 317-330, 331-367, **13** (1935) 519-539 and 540-554.

$$R_0=(1-c_{31})/(1+c_{31})=\varepsilon\ , \tag{1}$$

and $$T_0=2/(1+c_{31})=1+\varepsilon\ ,$$

where $$c_{ij}=c_i/c_j\ .$$

The velocity deviation ε is related to the velocity ratio c_{31} of each medium;

$$c_{31}=(1-\varepsilon)/(1+\varepsilon)\ , \tag{2}$$

and since c_{31} is always positive, ε is confined between -1 and 1.

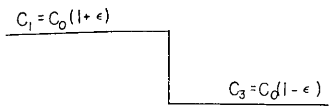

Fig. 1. Velocity distribution of sudden change.

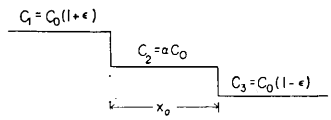

Fig. 2. Velocity distribution with an intermediate homogeneous medium.

Next, waves reflected from and passing through the first order discontinuity are discussed. The simplest model of this case shall be taken such that an intermediate homogeneous medium with velocity c_2 and thickness x_0 is inserted between both media, as schematically shown in Fig. 2, for the sake of comparison with the other cases. The reflection and transmission coefficients, R_1 and T_1, are defined as ratios of the amplitudes of waves reflected from and passing through the intermediate medium respectively to those of incident waves,

$$\begin{aligned} R_1&=\frac{(1-c_{31})\cos k_2x_0+i(c_{32}-c_{21})\sin k_2x_0}{(1+c_{31})\cos k_2x_0+i(c_{32}+c_{21})\sin k_2x_0}\\ &=R_0\sqrt{\frac{1-\sin^2 k_2x_0\{1-(1-\varepsilon^2-\alpha^2)/(2\alpha)^2\}}{1-\sin^2 k_2x_0\{1-(1-\varepsilon^2+\alpha^2)/(2\alpha)^2\}}}\exp\{i(\Phi_1'-\Phi_1)\}\ ,\\ T_1&=\frac{2}{(1+c_{31})\cos k_2x_0+i(c_{32}+c_{21})\sin k_2x_0}\\ &=T_0\frac{\exp(-i\Phi_1)}{\sqrt{1-\sin^2 k_2x_0\{1-(1-\varepsilon^2+\alpha^2)/(2\alpha)^2\}}}\ , \end{aligned} \tag{3}$$

where

$$\begin{aligned} \tan\Phi_1&=\frac{c_{32}+c_{21}}{1+c_{31}}\tan k_2x_0=\frac{1-\varepsilon^2+\alpha^2}{2\alpha}\tan k_2x_0\ ,\\ \tan\Phi_1'&=\frac{c_{32}-c_{21}}{1-c_{31}}\tan k_2x_0=\frac{1-\varepsilon^2-\alpha^2}{2\alpha}\tan k_2x_0\ , \end{aligned} \tag{4}$$

and $$k_2x_0=\omega x_0/c_2\ ,\qquad c_2=\alpha c_0\ .$$

As the thickness or the frequency tends to zero, that is, the wave length becomes much larger than the thickness, R_1 and T_1 approach R_0 and T_0 respectively.

From formulae (3), the reflected wave is minimum for k_2x_0 equal to a half of odd times π, whereas the phemomena agree with those of very sharp discontinuity for k_2x_0 equal to integral times π. Particularly, the velocity c_2 that makes the reflection coefficient the largest is given by the relation

$$c_2=c_0\sqrt{1-\varepsilon^2}\ ,$$

and

$$R_1=\frac{R_0\cos k_2x_0}{\sqrt{1-\varepsilon^2\sin^2 k_2x_0}}\exp(-i\Phi_1)\ ,$$

$$T_1=\frac{T_0\exp(-i\Phi_1)}{\sqrt{1-\varepsilon^2\sin^2 k_2x_0}}\ , \tag{5}$$

where

$$\Phi_1=\tan^{-1}(\sqrt{1-\varepsilon^2}\tan k_2x_0)\ ,$$

and

$$k_2x_0=\frac{1}{\sqrt{1-\varepsilon^2}}\frac{\omega}{\omega_0}\ ,\qquad \omega_0=c_0/x_0\ .$$

3. Second order discountinuity

In this section, it is assumed that a heterogeneous medium in which the velocity increases or decreases linearly with distance x_0 is inserted between two media, velocities of which are $c_1=c_0(1+\varepsilon)$ and $c_3=c_0(1-\varepsilon)$ respectively, and that the velocity is continuous everywhere but its derivatives are discontinuous at each boundary, as schematically shown in Fig. 3. Such a layered model has been classified transition layers in the geophysical exploration.

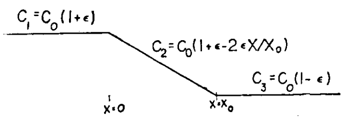

Fig. 3. Velocity distribution with a transient layer.

The wave motion in this medium has been obtained by several researchers. In this paper, the following solution is adopted[12]:

the wave equation
$$\left(\frac{\partial^2}{\partial\tau^2}+\omega^2-\varepsilon^2\omega_0^2\right)(\sqrt{\rho c_2}\,u)=0\ ,$$

travel time
$$\tau=\frac{1}{2\varepsilon\omega_0}\log\left(\frac{c_1}{c_2}\right)\ , \tag{6}$$

12) R. YOSHIYAMA, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction," *Bull. Earthq. Res. Inst.*, **38** (1960), 467-478.

displacement
$$u \propto \frac{1}{\sqrt{\rho c_2}} \exp\left\{ \pm i \sqrt{\left(\frac{\omega}{\omega_0}\right)^2 - \varepsilon^2}\ \omega_0 \tau \right\},$$
for $0 \leq x \leq x_0$,

where $\omega_0 = c_0/x_0$.

The average velocity defined as $x_0/\tau(x_0)$ is determined by (6),

$$\bar{c}_2 = 2c_0\varepsilon / \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right). \tag{7}$$

The reflection and transmission coefficients R_2 and T_2 are given as in the preceding section by the formulae

$$R_2 = -R_0 \frac{\sin\varphi}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^2 - \varepsilon^2}} \frac{\exp(-i\Phi_2)}{\sqrt{1+\varepsilon^2 \sin^2\varphi \Big/ \left\{\left(\frac{\omega}{\omega_0}\right)^2 - \varepsilon^2\right\}}},$$

and

$$T_2 = T_0 \frac{\exp(-i\Phi_2)}{\sqrt{1-\varepsilon^2} \sqrt{1+\varepsilon^2 \sin^2\varphi \Big/ \left\{\left(\frac{\omega}{\omega_0}\right)^2 - \varepsilon^2\right\}}}, \tag{8}$$

where
$$\varphi = \frac{1}{2\varepsilon} \sqrt{\left(\frac{\omega}{\omega_0}\right)^2 - \varepsilon^2} \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right),$$

and
$$\Phi_2 = \tan^{-1} \frac{\omega}{\omega_0} \frac{\tan\varphi}{\sqrt{(\omega/\omega_0)^2 - \varepsilon^2}}.$$

Though φ is imaginary for ω/ω_0 smaller than $|\varepsilon|$, both $\sin\varphi/\varphi$ and $\cos\varphi$ remain real. If the displacement in the linearly varying medium could be explicitly stated by the exact solution (6): $(\rho c_2)^{-1/2} \exp\{i\omega t - i\sqrt{(\omega/\omega_0)^2 - \varepsilon^2}\omega_0 \tau\}$, waves advancing to $+x$ direction would disappear for frequencies lower than $\varepsilon\omega_0$. This is unacceptable from the physical point of view. Therefore, the stability of the propagation of long period waves should be discussed in the transmission coefficient defined above, and this treatment is one of the practical proofs of Schelkunoff's remarks.

Such waves approach R_0 and T_0 respectively, as the thickness or wave frequency tends to zero, whereas are transmitted without reflection for short period incident waves. The greatest of the reflection maxima appears for the very long incident wave, and the subsequent maxima decrease with the wave length of incident waves. For frequencies

$$\omega_n = \varepsilon\omega_0 \sqrt{1 + \left(2\pi n / \log\frac{1+\varepsilon}{1-\varepsilon}\right)^2} \tag{9}$$

corresponding to the cut-off frequencies of Wolf when n is unity, the transmission modulus is equivalent to that of maxima in the intermediate medium, and reflection disappears.

4. Continuous transition

The differential equation with respect to the stress S can be written as

$$\frac{d^2S}{dx^2}+\frac{\omega^2}{c^2}S=0\ , \tag{10}$$

where c is the velocity, ω the frequency, and the density ρ is assumed to be constant throughout the whole medium. The elasticity is proportional to the square of velocity, and the stress to the elasticity, wave number and displacement in a homogeneous medium; i.e., the displacement is proportional to the product of the stress and the inverse of the velocity. The reflection and transmission coefficients, therefore, are written as follows: If A_1 and B_1 are moduli of the stress of progressive and retrogressive waves at $x=-\infty$ respectively, and A_3 is one of the progressive waves at $x=\infty$,

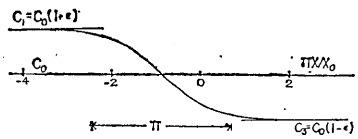

Fig. 4. Velocity distribution of continuous transition for $\varepsilon=0.5$: $\frac{1}{c^2}=\frac{1}{c_1{}^2}+\frac{\exp(\pi x/x_0)}{\{1+\exp(\pi x/x_0)\}}\left(\frac{1}{c_3{}^2}-\frac{1}{c_1{}^2}\right)$.

$$R_3=B_1/A_1\ ,$$

and

$$T_3=(c_1A_3)/(c_3A_1)\ , \tag{11}$$

where c_1 and c_3 are velocities at $x=-\infty$ and $x=+\infty$, respectively.

If the velocity varies continuously from c_1 at a position distant from a transition to c_3 at another distant position, the velocity distribution can be expressed as

$$\frac{1}{c^2}=\frac{1}{c_1^2}+\frac{\exp(\gamma x)}{\{1+\exp(\gamma x)\}}\left(\frac{1}{c_3^2}-\frac{1}{c_1^2}\right), \tag{12}$$

for which Equation (10) becomes the same form as that which Epstein obtained.

A solution converging at $x\to+\infty$ with a factor $\exp(-ik_3x)$ is combined with solutions converging at $x\to-\infty$ with factors $\exp(\pm ik_1x)$ as follows;

$$
\begin{aligned}
&e^{-ik_3x}F(i(\kappa_3-\kappa_1),\, i(\kappa_3+\kappa_1);\, 2i\kappa_3+1;\, -e^{-\gamma x}) \\
&\quad = A(i(\kappa_3-\kappa_1),\, i(\kappa_3+\kappa_1))e^{-ik_1x}F(-i(\kappa_3+\kappa_1),\, i(\kappa_3-\kappa_1);\, -2i\kappa_3+1;\, -e^{\gamma x}) \\
&\qquad + A(i(\kappa_3+\kappa_1),\, i(\kappa_3-\kappa_1))e^{+ik_1x}F(i(\kappa_3-\kappa_1),\, i(\kappa_3+\kappa_1);\, 2i\kappa_3+1;\, -e^{\gamma x})\,, \qquad (13)
\end{aligned}
$$

where $k_j=\omega/c_j\,,\qquad \kappa_j=k_j/\gamma\,,\qquad j=1,\,3,$

$$
A(\alpha,\,\beta)=\frac{\Gamma(\alpha+\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta+1)\Gamma(\beta)}\,,
$$

and $F(a,\, b;\, c;\, z)$ is a hypergeometric function which tends to unity as z vanishes. From the relation (13), the reflection and transmission coefficients can be determined by means of Equation (11),

$$
R_3=\frac{(\kappa_3+\kappa_1)\Gamma(-2i\kappa_1)}{(\kappa_3-\kappa_1)\Gamma(2i\kappa_1)}\left[\frac{\Gamma(i(\kappa_3+\kappa_1))}{\Gamma(i(\kappa_3-\kappa_1))}\right]^2 e^{2ik_1x_1} \quad \text{at } x_1(<0)\,,
$$

and

$$
T_3=\frac{c_1}{c_3}\,\frac{\kappa_3+\kappa_1}{2\kappa_3}\,\frac{[\Gamma(i(\kappa_3+\kappa_1))]^2}{\Gamma(2i\kappa_1)\Gamma(2i\kappa_3)}e^{-ik_3x_3+ik_1x_1} \quad \text{at } x_3(>0)\,, \qquad (14)
$$

where both positions x_1 and x_3 are remote from the transient zone; these moduli are simplified as

$$
|R_3|=\left|\frac{\sinh(k_3-k_1)x_0}{\sinh(k_3+k_1)x_0}\right|\,,
$$

and

$$
|T_3|=\frac{T_0}{\sqrt{1-\varepsilon^2}}\sqrt{1-|R_3|^2}\,, \qquad (15)
$$

which tend respectively to R_0 and T_0 as the period of waves becomes very long.

If γ is π/x_0, x_0 can regarded as equivalent to the effective thickness of velocity variation, as shown in Fig. 4.

5. Comparison of reflected and transmission coefficients and the apparent phase velocity through intervenient media

Reflection and transmission coefficients for various kinds of discontinuity are obtained in the preceding sections. Those for the first order discontinuity R_1 and T_1 seem to be undulating with respect to the wave frequency, and undulations of those for the second order discontinuity R_2 and T_2 diminish with increasing frequency. Those for the perfectly continuous medium R_3 and T_3 vary monotonically. The larger the velocity contrast, the more remarkable are the characters of those coefficients. Fig. 5 shows moduli of these coefficients for a very sharp discontinuity of $|\varepsilon|=0.5$, for which the velocity contrast is thrice or one-third.

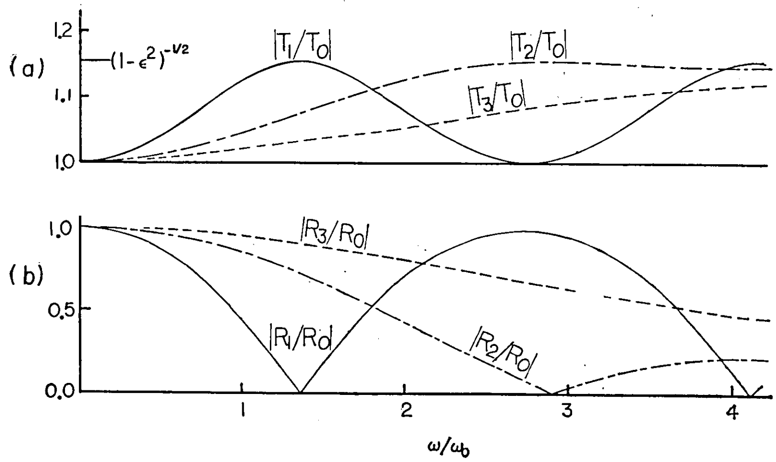

Fig. 5. Moduli of transmission and reflection coefficients (a) and (b), respectively: $\omega_0 = c_0/x_0$.

Next, the phases are discussed. The phase of a complex transmission coefficient may be interpreted due to propagation of waves in distance x_0 with an apparent phase velocity c^*. No phase shift exists in the very sharp discontinuity. In the first order discontinuity, the phase expression can be expanded as

$$\Phi_1 = k_2 x_0 \left\{ 1 - 2q \frac{\sin 2k_2 x_0}{2k_2 x_0} + 2q^2 \frac{\sin 4k_2 x_0}{4k_2 x_0} - \cdots \right\}, \tag{16}$$

where $$q = -\frac{(1-\alpha)^2 - \varepsilon^2}{(1+\alpha)^2 - \varepsilon^2} \leq \frac{1-\sqrt{1-\varepsilon^2}}{1+\sqrt{1-\varepsilon^2}}.$$

The apparent velocity c_1^*, as the phase Φ_1 can be written by $\omega x_0/c_1^*$, is approximated by the form

$$c_1^* = c_2 \left\{ 1 + 2q \frac{\sin 2k_2 x_0}{2k_2 x_0} + O(q^2) \right\}. \tag{17}$$

This relation shows that the apparent phase velocity is dispersive at the low frequency, if ε^2 is not neglected, that is, the velocity contrast is large, and it tends apparently from velocity $c_2(1+q)/(1-q)$ to the average velocity c_2 with increasing frequency.

In the second order discontinuity the phase angle can be expressed as follows:

$$\Phi_2=\frac{\omega}{\omega_0}\left\{1+\frac{\varepsilon^2}{15}\left(\frac{\omega}{\omega_0}\right)^2+O\left(\varepsilon^4\frac{\omega^4}{\omega_0^4}\right)\right\}, \qquad \text{for } \frac{\omega}{\omega_0}<\frac{\pi}{2} \tag{18}$$

and $$\Phi_2=\varphi\left\{1+2q'\frac{\sin 2\varphi}{2\varphi}+O(q'^2)\right\}, \qquad \text{for } \frac{\omega}{\omega_0}>\frac{\pi}{2}$$

where $$q'=4\left(\varepsilon\frac{\omega_0}{\omega}\right)^2\left\{1-\frac{1}{2}\left(\varepsilon\frac{\omega_0}{\omega}\right)^2+O\left(\varepsilon^4\frac{\omega_0^4}{\omega^4}\right)\right\}.$$

Therefore, the apparent phase velocity c_2^* is given, since Φ_2 is taken equal to $\omega x_0/c_2^*$, by the form

$$c_2^*=c_0\left\{1-\frac{\varepsilon^2}{15}\left(\frac{\omega}{\omega_0}\right)^2+O\left(\varepsilon^4\frac{\omega^4}{\omega_0^4}\right)\right\}, \qquad \text{for } \frac{\omega}{\omega_0}<\frac{\pi}{2}$$

and

$$c_2^*=2\varepsilon\left\{\log\frac{1+\varepsilon}{1-\varepsilon}\right\}^{-1}\cdot c_0\left\{1+\frac{1}{2}\left(\varepsilon\frac{\omega_0}{\omega}\right)^2+O\left(\varepsilon^4\frac{\omega_0^4}{\omega^4}\right)\right\},$$

$$\text{for } \frac{\omega}{\omega_0}>\frac{\pi}{2}. \tag{19}$$

The factor of this expression is expanded as

$$2\varepsilon\left\{\log\frac{1+\varepsilon}{1-\varepsilon}\right\}^{-1}=1-\frac{\varepsilon^2}{3}-\frac{4}{45}\varepsilon^4-O(\varepsilon^6), \tag{20}$$

so that the apparent velocity at low frequencies is larger than that at higher frequencies, and its variation is monotonic similarly as in the first order discontinuity. In addition, it is noticeable that this dispersion is independent of the sign of ε.

In each relation, the apparent phase velocity for high frequencies tends to the average velocity, *(distance)/(travel time)*.

A similar dispersion is also found in the medium varying continuously: From (14) and the definition of apparent phase velocity c_3^*,

$$-\arg T_3=\frac{\omega}{c_3^*}x_0'=\frac{\omega}{c_3}x_3-\frac{\omega}{c_1}x_1-\arg\frac{\Gamma(i\alpha)\Gamma(i\alpha)}{\Gamma(i(\alpha+\beta))\Gamma(i(\alpha-\beta))},$$

where $$x_0'=x_3-x_1, \qquad \alpha=\kappa_3+\kappa_1=\frac{2}{\pi(1-\varepsilon^2)}\frac{\omega}{\omega_0},$$

and $$\beta=\kappa_3-\kappa_1=\frac{2\varepsilon}{\pi(1-\varepsilon^2)}\frac{\omega}{\omega_0}.$$

If a virtual velocity $\tilde{c}$ is defined as

$$x_0'/\tilde{c}=x_3/c_3-x_1/c_1,$$

and by means of the infinite product formula of gamma functions, the apparent velocity is written as

$$c_3^* = \tilde{c} \Big/ \left[1 + \frac{\tilde{c}}{\omega x_0'} \sum_{n=1}^{\infty} \tan^{-1} \frac{4n\alpha\beta^2}{(n^2+\alpha^2)^2+\beta^2(n^2-\alpha^2)} \right],$$

$$\fallingdotseq \tilde{c} \Big/ \left[1 + \frac{\tilde{c}}{\omega_0 x_0'} \frac{32\varepsilon^2}{\pi^3(1-\varepsilon^2)^3} \left(\frac{\omega}{\omega_0} \right)^2 \sum_{n=1}^{\infty} \frac{n}{(n^2+\alpha^2)^2+\beta^2(n^2-\alpha^2)} \right].$$

Therefore, it tends to $\tilde{c}$ in a limit of zero frequency, and becomes slower with increasing frequency. On the other hand, $4n\alpha\beta^2/\{(n^2+\alpha^2)^2+\beta^2(n^2-\alpha^2)\}$ is smaller than $6\sqrt{3}\,\varepsilon^2/\{(8+\varepsilon^2)\sqrt{1-\varepsilon^2}\}$, and its maximum is for the integer n nearest to $\sqrt{(\alpha^2-\beta^2)/3}$ or $(2/\pi)(\omega/\omega_0)\{3(1-\varepsilon^2)\}^{-1/2}$, at which the summation term can be approximated by integration. The result is

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{4n\alpha\beta^2}{(n^2+\alpha^2)^2+\beta^2(n^2-\alpha^2)}$$

$$\fallingdotseq \frac{\omega}{\omega_0} \frac{2\varepsilon}{\pi(1-\varepsilon^2)\sqrt{2+\varepsilon^2/4}} \log \frac{1+\dfrac{\varepsilon^2}{2}+\varepsilon\sqrt{2+\dfrac{\varepsilon^2}{4}}}{1+\dfrac{\varepsilon^2}{2}-\varepsilon\sqrt{2+\dfrac{\varepsilon^2}{4}}},$$

which is proportional to the frequency. Therefore, it tends to a constant, for high frequency,

$$c_3^* = \tilde{c} \Big/ \left[1 + \frac{\tilde{c}}{\omega_0 x_0'} \frac{4\varepsilon^2}{\pi(1-\varepsilon^2)(1+\varepsilon^2/2)} \right],$$

which is smaller than $\tilde{c}$ and is independent of the frequency.

This dispersion corresponds to such as surface waves, so that it is suggested that the transmitted pulse becomes blunt.

6. Concluding Remarks

Propagation of elastic waves in a heterogeneous medium with periodic structures was previouly investigated.[13] In that study, distribution of the velocity and its derivative was continuous everywhere. In this paper, the effect of an intervenient transition layer is discussed in connection

13) I. ONDA, "Propagation and Apparent Attenuation of Elastic Waves in a Heterogeneous Medium with Certain Periodic Structures," *Bull. Earthq. Res. Inst.*, **42** (1964), 427-447.

with the stability to transmitted waves. Schelkunoff's remarks[14] are ascertained in propagation of long period wave through the medium varying linearly.

If a medium is in contact with another, amplitudes of the transmitted waves are deduced from the theory of reflection and refraction (or transmission), which are independent of the frequency. If a medium is inserted between these two media, the reflected and transmitted waves depend upon the frequency. Then, to compare a type of transition medium between these two media with another, a very simple model is considered: an inserted medium with a finite thickness is taken into account. If it is homogeneous, the model corresponds to the first order discontinuity, and if the velocity in the inserted medium varies linearly, the model corresponds to the second order discontinuity. The transmission and reflection coefficients T and R are defined as ratios of the amplitudes of the transmitted and reflected waves to those of the incident wave, respectively. In this paper, the transmission and reflection coefficients are discussed by taking their ratios to the coefficients for the model of sudden change without an intervenient layer. By doing so, it is possible to discuss, by means of the same expression, the case for the given velocity ratio c_1/c_3 between the right- and left-hand side media and another case for the velocity ratio to be equal to c_3/c_1: for example, the relative coefficients for $c_1/c_3=2$ are equal to those for $c_3/c_1=2$.

In addition, the apparent velocity is obtained from phases of the complex transmission coefficient.

Amplitude of transmitted waves: For the first order discontinuity, it varies between T_0 and $T_0/\sqrt{1-\varepsilon^2}$ with some frequency intervals. For the second order discontinuity, it increases from T_0 to $T_0/\sqrt{1-\varepsilon^2}$ and after slightly diminishing, converges to $T_0/\sqrt{1-\varepsilon^2}$ with increasing frequency. On the other hand, it increases monotonically from T_0 to $T_0/\sqrt{1-\varepsilon^2}$, in the model with a perfectly continuous variation. At low frequencies, its gradient to the frequency for the second order discontinuity is smaller than for the first order discontinuity and larger than for the perfectly continuous medium. This behaviour is shown in Fig. 5, as an example.

Amplitude of reflected waves: As the conservation of energy flux holds within the incident, reflected and transmitted waves, it has the property similar to that of the transmitted waves. When their discrimination is marked, the spectrum of reflected waves should be used to determine

14) S. A. SCHELKUNOFF, *loc. cit.*, 2).

the nature of the boundary.

Apparent phase velocity: The apparent phase velocity in the very short period waves agrees with the average phase velocity, while it is faster in the long period waves than in the short period waves. This dispersion is qualitatively similar to it of surface waves. The amount of this dispersion is of the order of magnitude ε^2, but it cannot be neglected in the case where the velocity deviation ε is not small in the path of the wave propagation.

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1. 境界の性質が弾性波の伝搬に及ぼす影響

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地震波は一種の進行波であり，それを伝える場はたとえ等方と考えられるにしても不均質である．不均質媒質における波の表現から進行波だけを分離することはできないので，進行波の安定性を議論するためには，不均質媒質を挾んで両側に均質媒質を置いて，それらの中の特定方向に伝わる波を調べるという手段をとらなくてはならない．(このことについて，速度が一様に変化している場合について一つの証明を与えた．) 今回はこのような立場から，簡単な問題を扱つた．

違つた媒質がお互いに接しているときには反射屈折の理論に従つて透過波や反射波が決められる．そこで，この接続の違いに関して比較を行なうために，できるだけ簡単なそして系統的なモデルを考えた．すなわち，一定の厚さの媒質を中間層と考えて，第一種不連続に相当するものは均質媒質で，また第二種不連続に対しては速度が一様に変化している不均質媒質で表わした．透過波および反射波の振幅を入射波の振幅で除したものをそれぞれ透過係数 T および反射係数 R として定義して，中間層のない場合のそれら T_0 と R_0 に対する比によつて議論を行なつた．このようにすれば，左右の均質媒質の速度比 c_1/c_3 が n 倍の場合と $1/n$ 倍の場合のそれらは共に同じ式で与えられる．

次に透過係数の位相の部分から見掛けの速度が求められた．

透過波の振幅：第一種不連続に対するものは周波数の増加につれて T_0 と $T_0/\sqrt{1-\varepsilon^2}$ の間を大きくなつたり小さくなつたりする．第二種不連続に対するものは T_0 から大きくなつて一度 $T_0/\sqrt{1-\varepsilon^2}$ まで達し少し減少して，更に周波数を増すと極大値 $T_0/\sqrt{1-\varepsilon^2}$ に収束する．また完全に (数学的に) 連続な場合には一様に T_0 から $T_0/\sqrt{1-\varepsilon^2}$ まで増加する．周波数の小さい場合には周波数に対する勾配は第一種＞第二種＞完全連続となつている (第5図参照).

反射波の振幅：エネルギー束の保存則が成り立つているので，透過波のそれと符号を別として傾

向は同じである.

境界の性質を判定するためには反射波のスペクトル構造によつた方が容易である.

見掛けの速度：短い波の速度は平均の速度に一致しているが，長い波ほど見掛け上速度が早い．その分散の大きさは $O(\varepsilon^2)$(ここで ε は速度の変化量）である．これは速度変化の大きな場合には無視され得ない量である.