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5. *Tsunami in the Vicinity of a Wave Origin* [III].

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Abstract

In the numerical study of a tsunami, the most difficult question is the way in which the oscillatory factors $\cos\omega^*t^*$ and $J_0(k^*r^*)$, which are included in the integrand of the integration

$$a^*\int\frac{\cos\omega^*t^*}{\cosh k^*}J_0(k^*r^*)J_1(k^*a^*)dk^*\ ,$$

are to be transformed into a form suitable for numerical calculation. For this purpose, Filon's method is available, which makes the the above integration possible for large values of t^* and r^*. Since we have no available information with regard to the efficiency of this method, two examples of the integration are presented to check the efficiency of Filon's method in comparison with that devised by Simpson. In this paper, using Filon's method, the calculations of the above integration are carried out for values r^* in the range 25 to 50. Subsequently, the following conclusions have been arrived at:—

(1) notwithstanding an uniform elevation of a circular portion of the bottom, a saddle-shaped valley appears at the first intumescence of the waves,

(2) the toe of the waves appears to advance at a speed of the long wave velocity for a tsunami of near-field, while the intumescences of the later phase are propagated with less than the velocity of a long wave,

(3) the elevation of the first crest is in amount larger than the depression.

1. Introduction

This article is a continuation of the previous works[1),2)] under the same title. As described in the previous papers, the integration

$$\zeta_R=a^*\int_0^\infty\frac{\cos\omega^*t^*}{\cosh k^*}J_0(k^*r^*)J_1(k^*a^*)dk^* \tag{1}$$

1) T. MOMOI, *Bull. Earthq. Res. Inst.*, **42** (1964), 133-146.
2) T. MOMOI, *Bull. Earthq. Res. Inst.*, **42** (1964), 369-381.

(the notations and definitions are completely the same as those in the previous works[1),2)]) has vibrating factors $\cos \omega^* t^*$ and $J_0(k^* r^*)$ which make it impossible to integrate directly the above expression by use of Simpson's formula for large t^* and r^*. In order to avoid this difficulty, there is a method devised by Filon,[3)] by use of which the calculations of the integration (1) are continued in this paper. Since there is no available information with respect to the efficiency of Filon's method, the efficiency of the method is evaluated, in section 2, for two particular cases of integrands as compared with the results computed by Simpson's formula.

2. Filon's Method

Integrals of the form

$$\int_a^b \psi(x) \cdot \sin(kx) dx \,, \tag{2}$$

where $\psi(x)$ is a function with a limited number of turning points in the range of integration and k is a constant which may take up large values, frequently occur in investigations in mathematical physics, and their computation by quadratures is often desirable. As is well known, if $\psi(x)$ is continuous in the interval (a, b), including the end values, the limit of (2) when k increases without limit is 0. But the actual computation of the integral such as (2) by quaratures, when k is is large though not infinite, is, in practice, a matter of considerable difficulty. Because of the rapid oscillation of the function $\sin kx$, the ordinary quadrature formulae such as Simpson's require for their application the division of the range of integration into such minute steps that the labour of calculation is prohibitive.

In order to compute such integrals, Filon devised a special method for their numerical quadrature, which is outlined in the following.

In the calculation of the integration (2) by Simpson's formula, the *whole* integrand is, over the range $(x_r - h, x_r + h)$ (h is small interval of the numerical integration, and x_r the r-th point of the divisions) approximated by a quadratic formula of the type

$$\psi(x) \sin kx = A_1 + B_1(x - x_r) + C_1(x - x_r)^2 \,,$$

where A_1, B_1, C_1 are constants.

3) L.N.G. FILON, "On a Quadrature Formula for Trigonometric Integrals," *Proc. Roy. Soc. Edin.*, **49** (1928-29), 38-47.

But, in Filon's method, *a part* of the integrand, $\psi(x)$, except for $\sin kx$ is expressed as

$$\psi(x)=A_2+B_2(x-x_r)+C_2(x-x_r)^2 , \tag{3}$$

where A_2, B_2, C_2 are constants. Then the integration (2) becomes

$$\int_a^b \psi(x)\sin kx dx$$
$$=\sum_{r=1}^{2n}\int_{x_{2r-2}}^{x_{2r}}\{A_2+B_2(x-x_r)+C_2(x-x_r)^2\}\sin kx dx . \tag{4}$$

Following the same procedure as in Simpson's formula, the constants A_2, B_2, C_2 are expressed as

$$\left.\begin{aligned} A_2=\psi_r , \qquad & B_2=(\psi_{r+1}-\psi_{r-1})/2h , \\ C_2=(\psi_{r+1}+\psi_{r-1}-2\psi_r)/2h^2 , & \end{aligned}\right\} \tag{5}$$

where, for shortness, the notation

$$\psi(a+rh)=\psi(x_r)=\psi_r$$

is used.

Integrating by parts, (4) becomes as follows:

$$\int_a^b \psi(x)\sin kx dx$$
$$=\sum_{r=1}^{2n}\{-\psi_{r+1}(\cos kx_{r+1})/k+\psi_{r-1}(\cos kx_{r-1})/k$$
$$+\psi'_{r+1}(\sin kx_{r+1})/k^2-\psi'_{r-1}(\sin kx_{r-1})/k^2$$
$$+(2C_2/k^3)(\cos kx_{r+1}-\cos kx_{r+1})\} .$$

Substituting (5) into the above expression, and after a few reductions by use of trigonometric formulae, the following quadrature formula is obtained

$$\int_a^b \psi(x)\sin kx dx$$
$$=h[\alpha\{\psi(a)\cos ka-\psi(b)\cos kb\}+\beta S_{2r}+\gamma S_{2r-1}] , \tag{6}$$

where

$$\alpha=1/\theta+\cos\theta\sin\theta/\theta^2-2\sin^2\theta/\theta^3 .$$
$$\beta=2[(1+\cos^2\theta)/\theta^2-2\sin\theta\cos\theta/\theta^3] ,$$
$$\gamma=4(\sin\theta/\theta^3-\cos\theta/\theta^2) ,$$

$$\theta = kh \ ,$$

$$S_{2r} = \sum_{r=0}^{n} \psi(x_{2r}) \sin kx_{2r} - \{\psi(a) \sin ka - \psi(b) \sin kb\}/2 \ ,$$

$$S_{2r-1} = \sum_{r=1}^{n} \psi(x_{2r-1}) \sin kx_{2r-1} \ .$$

When k is small, i.e. θ tends to zero, the equation (6) reduces to Simpson's formula. The α, β, γ tend to 0, 2/3, 4/3 respectively and the right-hand side of (6) reduces to

$$\frac{1}{3} h[2S_{2r} + 4S_{2r-1}] \ ,$$

which is Simpson's formula. Hence, when the integration of the type given in (2) is made, Filon's method needs more time for computation of the factors α, β, γ and the term

$$\alpha\{\psi(a) \cos ka - \psi(b) \cos kb\}$$

than Simpson's method. But the weakness of the former is recovered by superiority of the convergence. It is of considerable significance to make an estimate of the superiority of the former to the latter in applying both formulae to the evaluation of the same integral in any given case. For two particular forms of $\psi(x)$, i.e. $\cos x$ and $\exp(-x)$, the efficiencies of Filon's method are examined, of which the results are shown in Tables 1 and 2.[4] The interval of the integrations is $(0, \pi/2)$ and the factor k the values in the range from 10 to 100.

Since the values tabulated in Tables 1 and 2 are calculated by cut-off computation in a computer, they have, in the lower parts of the numerics, a few degrees of errors resulted in an accumulation of the cut-off operation.

As shown in Tables 1 and 2, in general, the convergence of Filon's method is much better than that of Simpson's method. When the factor k increases in magnitude, times necessary for computing the integrals in order to obtain the same accuracy are increased remarkably for Simpson's method as compared with the calculation by Filon's method. For example, when $k=100$, times needed for computations

$$\int_0^{\pi/2} \cos x \sin kx dx$$

or

$$\int_0^{\pi/2} e^{-x} \sin kx dx$$

Table 1. The integrated values of

$$\int_0^{\pi/2} \cos x \sin kx dx$$

for the factor k from 10 to 100. The symbols N, Simpson, and Filon stated in the table stand for a number of divisions of integration, the values integrated by use of Simpson's and Filon's method respectively.*

When $k=10$,

N	Simpson	Filon
6	.2541104665$_{10}$ 0	.1012070296$_{10}$ 0
10	.1060250994$_{10}$ 0	.1010103840$_{10}$ 0
16	.1016150297$_{10}$ 0	.1010099983$_{10}$ 0
20	.1012462306$_{10}$ 0	.1010100491$_{10}$ 0
26	.1010900298$_{10}$ 0	.1010100800$_{10}$ 0
30	.1010546577$_{10}$ 0	.1010100879$_{10}$ 0
36	.1010313541$_{10}$ 0	.1010100936$_{10}$ 0
40	.1010239798$_{10}$ 0	.1010100956$_{10}$ 0
46	.1010179972$_{10}$ 0	.1010100972$_{10}$ 0
50	.1010157435$_{10}$ 0	.1010100971$_{10}$ 0
60	.1010128096$_{10}$ 0	.1010100969$_{10}$ 0
70	.1010115579$_{10}$ 0	.1010100949$_{10}$ 0
80	.1010109540$_{10}$ 0	.1010100999$_{10}$ 0
90	.1010106299$_{10}$ 0	.1010100967$_{10}$ 0
100	.1010104473$_{10}$ 0	.1010100997$_{10}$ 0

When $k=20$,

N	Simpson	Filon
6	−.2690830219$_{10}$ 0	.5012479793$_{10}$− 1
10	.1503998842$_{10}$− 8	.4999999963$_{10}$− 1
16	.5759443745$_{10}$− 1	.5012546500$_{10}$− 1
20	.5252178446$_{10}$− 1	.5012532215$_{10}$− 1
26	.5084700457$_{10}$− 1	.5012530953$_{10}$− 1
30	.5051255173$_{10}$− 1	.5012530948$_{10}$− 1
36	.5030392607$_{10}$− 1	.5012531109$_{10}$− 1
40	.5024032383$_{10}$− 1	.5012531154$_{10}$− 1
46	.5018981948$_{10}$− 1	.5012531209$_{10}$− 1
50	.5017110919$_{10}$− 1	.5012531229$_{10}$− 1
60	.5014706711$_{10}$− 1	.5012531218$_{10}$− 1
70	.5013695042$_{10}$− 1	.5012531196$_{10}$− 1
80	.5013209453$_{10}$− 1	.5012531229$_{10}$− 1
90	.5012953017$_{10}$− 1	.5012531169$_{10}$− 1
100	.5012807185$_{10}$− 1	.5012531213$_{10}$− 1
150	.5012585428$_{10}$− 1	.5012530986$_{10}$− 1
200	.5012548371$_{10}$− 1	.5012531070$_{10}$− 1
300	.5012534719$_{10}$− 1	.5012531022$_{10}$− 1
400	.5012532454$_{10}$− 1	.5012531087$_{10}$− 1

(to be continued)

When $k=30$, (continued)

N	Simpson	Filon
6	.1806897811$_{10}$ 0	.3337145829$_{10}$ 1
10	−.1060251018$_{10}$ 0	.3337075489$_{10}$ 1
16	.2265326229$_{10}$ 0	.3338083348$_{10}$- 1
20	.4823368202$_{10}$- 1	.3337056081$_{10}$- 1
26	.3658434442$_{10}$- 1	.3337041959$_{10}$- 1
30	.3495448906$_{10}$- 1	.3337041227$_{10}$- 1
36	.3405290603$_{10}$- 1	.3337041017$_{10}$- 1
40	.3379818879$_{10}$- 1	.3337041035$_{10}$- 1
46	.3360403834$_{10}$- 1	.3337041035$_{10}$- 1
50	.3353425930$_{10}$- 1	.3337041041$_{10}$- 1
60	.3344670875$_{10}$- 1	.3337041091$_{10}$- 1
70	.3341075231$_{10}$- 1	.3337041070$_{10}$- 1
80	.3339374705$_{10}$- 1	.3337041086$_{10}$- 1
90	.3338484947$_{10}$- 1	.3337041066$_{10}$- 1
100	.3337982374$_{10}$- 1	.3337040986$_{10}$- 1
150	.3337224250$_{10}$- 1	.3337041032$_{10}$- 1
200	.3337098745$_{10}$- 1	.3337040936$_{10}$- 1
300	.3337052326$_{10}$- 1	.3337041383$_{10}$- 1
400	.3337044619$_{10}$- 1	.3337041296$_{10}$- 1
500	.3337042399$_{10}$- 1	.3337040601$_{10}$- 1

When $k=40$,

N	Simpson	Filon
6	−.1584335829$_{10}$ 0	.2501823563$_{10}$- 1
10	−.1180894131$_{10}$- 7	.2499999994$_{10}$- 1
16	−.6055294471$_{10}$- 1	.2501599209$_{10}$- 1
20	.6257905542$_{10}$- 8	.2499999965$_{10}$- 1
26	.3821254142$_{10}$- 1	.2501568265$_{10}$- 1
30	.3028535461$_{10}$- 1	.2501564424$_{10}$- 1
36	.2699784245$_{10}$- 1	.2501563603$_{10}$- 1
40	.2620013796$_{10}$- 1	.2501563515$_{10}$- 1
46	.2563443311$_{10}$- 1	.2501563477$_{10}$- 1
50	.2544120871$_{10}$- 1	.2501563471$_{10}$- 1
60	.2520778209$_{10}$- 1	.2501563448$_{10}$- 1
70	.2511546886$_{10}$- 1	.2501563436$_{10}$- 1
80	.2507275848$_{10}$- 1	.2501563460$_{10}$- 1
90	.2505071925$_{10}$- 1	.2501563443$_{10}$- 1
100	.2503838915$_{10}$- 1	.2501563440$_{10}$- 1
150	.2502000887$_{10}$- 1	.2501563346$_{10}$- 1
200	.2501700459$_{10}$- 1	.2501563322$_{10}$- 1
300	.2501590153$_{10}$- 1	.2501563119$_{10}$- 1
400	.2501571607$_{10}$- 1	.2501563196$_{10}$- 1
500	.2501566516$_{10}$- 1	.2501563272$_{10}$- 1
600	.2501564596$_{10}$- 1	.2501562999$_{10}$- 1

(to be continued)

* In the table, the expressions such as

.1234567891$_{10}$ 1

denote the conventional expressions

$$0.1234567891\times10^{1}.$$

Since the values typed out by an electronic computer are printed as they are, such expressions are employed in this table.

When $k=50$, (continued)

N	Simpson	Filon
6	.6670598070$_{10}$ 0	.2001469284$_{10}-$ 1
10	.1060250973$_{10}$ 0	.2000808299$_{10}-$ 1
16	−.7365578067$_{10}-$ 1	.2000801630$_{10}-$ 1
20	−.4823368141$_{10}-$ 1	.2000811961$_{10}-$ 1
26	.2238482544$_{10}$ 0	.2001041466$_{10}-$ 1
30	.3992493183$_{10}-$ 1	.2000805075$_{10}-$ 1
36	.2536491116$_{10}-$ 1	.2000800780$_{10}-$ 1
40	.2293448654$_{10}-$ 1	.2000800445$_{10}-$ 1
46	.2142674405$_{10}-$ 1	.2000800307$_{10}-$ 1
50	.2095429078$_{10}-$ 1	.2000800281$_{10}-$ 1
60	.2041607836$_{10}-$ 1	.2000800246$_{11}-$ 1
70	.2021483545$_{10}-$ 1	.2000800271$_{10}-$ 1
80	.2012457723$_{10}-$ 1	.2000800265$_{10}-$ 1
90	.2007889564$_{10}-$ 1	.2000800266$_{10}-$ 1
100	.2005366553$_{10}-$ 1	.2000800278$_{10}-$ 1
150	.2001664631$_{10}-$ 1	.2000800264$_{10}-$ 1
200	.2001069835$_{10}-$ 1	.2000800305$_{10}-$ 1
300	.2000853021$_{10}-$ 1	.2000800279$_{10}-$ 1
400	.2000816978$_{10}-$ 1	.2000800477$_{10}-$ 1
500	.2000807114$_{10}-$ 1	.2000800437$_{10}-$ 1
600	.2000803674$_{10}-$ 1	.2000800457$_{10}-$ 1
700	.2000802117$_{10}-$ 1	.2000800478$_{10}-$ 1

When $k=60$,

N	Simpson	Filon
6	.1004057290$_{10}-$ 7	.1666666666$_{10}-$ 1
10	.5950921917$_{10}-$ 8	.1666666659$_{10}-$ 1
16	−.2667069072$_{10}$ 0	.1667142388$_{10}-$ 1
20	−.5252178411$_{10}-$ 1	.1667130816$_{10}-$ 1
26	−.4896632676$_{10}-$ 1	.1667139545$_{10}-$ 1
30	.1985894981$_{10}-$ 7	.1666666644$_{10}-$ 1
36	.3318315551$_{10}-$ 1	.1667131649$_{10}-$ 1
40	.2397943024$_{10}-$ 1	.1667130207$_{10}-$ 1
46	.1975604471$_{10}-$ 1	.1667129884$_{10}-$ 1
50	.1862867540$_{10}-$ 1	.1667129795$_{10}-$ 1
60	.1745927512$_{10}-$ 1	.1667129791$_{10}-$ 1
70	.1705711675$_{10}-$ 1	.1667129759$_{10}-$ 1
80	.1688439601$_{10}-$ 1	.1667129770$_{10}-$ 1
90	.1679921420$_{10}-$ 1	.1667129739$_{10}-$ 1
100	.1675295882$_{10}-$ 1	.1667129762$_{10}-$ 1
150	.1668644914$_{10}-$ 1	.1667129716$_{10}-$ 1
200	.1667598972$_{10}-$ 1	.1667129655$_{10}-$ 1
300	.1667220888$_{10}-$ 1	.1667129578$_{10}-$ 1
400	.1667158177$_{10}-$ 1	.1667129526$_{10}-$ 1
500	.1667141164$_{10}-$ 1	.1667129458$_{10}-$ 1
600	.1667134949$_{10}-$ 1	.1667129402$_{10}-$ 1
700	.1667132277$_{10}-$ 1	.1667129295$_{10}-$ 1
800	.1667131053$_{10}-$ 1	.1667129498$_{10}-$ 1

(to be continued)

When $k=70$, (continued)

N	Simpson	Filon
6	$-.6670597985_{10}\ 0$	$.1428739779_{10}-1$
10	$-.1060251025_{10}\ 0$	$.1428866000_{10}-1$
16	$.1715555223_{10}\ 0$	$.1428878091_{10}-1$
20	$-.1012462344_{10}\ 0$	$.1428863042_{10}-1$
26	$-.3499738950_{10}-1$	$.1428864300_{10}-1$
30	$-.3992492772_{10}-1$	$.1428867021_{10}-1$
36	$.2230694092_{10}\ 0$	$.1428953519_{10}-1$
40	$.3714772960_{10}-1$	$.1428865073_{10}-1$
46	$.2119878081_{10}-1$	$.1428863275_{10}-1$
50	$.1829461575_{10}-1$	$.1428863119_{10}-1$
60	$.1573604518_{10}-1$	$.1428862974_{10}-1$
70	$.1496373234_{10}-1$	$.1428862998_{10}-1$
80	$.1465159350_{10}-1$	$.1428862973_{10}-1$
90	$.1450288028_{10}-1$	$.1428862983_{10}-1$
100	$.1442387423_{10}-1$	$.1428862948_{10}-1$
150	$.1431311670_{10}-1$	$.1428862979_{10}-1$
200	$.1429615496_{10}-1$	$.1428862971_{10}-1$
300	$.1429008635_{10}-1$	$.1428863025_{10}-1$
400	$.1428908724_{10}-1$	$.1428862999_{10}-1$
500	$.1428881611_{10}-1$	$.1428863039_{10}-1$
600	$.1428871973_{10}-1$	$.1428863194_{10}-1$
700	$.1428867792_{10}-1$	$.1428863218_{10}-1$
800	$.1428865655_{10}-1$	$.1428862923_{10}-1$
900	$.1428864557_{10}-1$	$.1428862996_{10}-1$

When $k=80$,

N	Simpson	Filon
6	$.1584335922_{10}\ 0$	$.1250162049_{10}-1$
10	$-.2406186598_{10}-7$	$.1249999996_{10}-1$
16	$.6576653012_{10}-1$	$.1250196105_{10}-1$
20	$-.3663603884_{10}-7$	$.1250000007_{10}-1$
26	$-.4310332582_{10}-1$	$.1250195516_{10}-1$
30	$-.3028535544_{10}-1$	$.1250196038_{10}-1$
36	$-.4575281041_{10}-1$	$.1250199225_{10}-1$
40	$.3241515060_{10}-7$	$.1249999964_{10}-1$
46	$.3113129089_{10}-1$	$.1250196190_{10}-1$
50	$.2126167372_{10}-1$	$.1250195621_{10}-1$
60	$.1512190468_{10}-1$	$.1250195363_{10}-1$
70	$.1364079350_{10}-1$	$.1250195380_{10}-1$
80	$.1309249216_{10}-1$	$.1250195359_{10}-1$
90	$.1284308712_{10}-1$	$.1250195323_{10}-1$
100	$.1271427061_{10}-1$	$.1250195396_{10}-1$
150	$.1253926566_{10}-1$	$.1250195357_{10}-1$
200	$.1251331260_{10}-1$	$.1250195341_{10}-1$
300	$.1250413597_{10}-1$	$.1250195209_{10}-1$
400	$.1250263644_{10}-1$	$.1250195335_{10}-1$
500	$.1250223129_{10}-1$	$.1250195128_{10}-1$
600	$.1250208529_{10}-1$	$.1250195011_{10}-1$
700	$.1250202303_{10}-1$	$.1250195362_{10}-1$
800	$.1250199304_{10}-1$	$.1250195313_{10}-1$
900	$.1250197626_{10}-1$	$.1250195261_{10}-1$
1000	$.1250196670_{10}-1$	$.1250195181_{10}-1$

(to be continued)

When $k=90$, (continued)

N	Simpson	Filon
6	$-.1806897812_{10}\ \ 0$	$.1111252314_{10}\ \ 1$
10	$.1060250943_{10}\ \ 0$	$.1111249707_{10}-\ 1$
16	$.7045606098_{10}-\ 1$	$.1111247390_{10}-\ 1$
20	$.1012462236_{10}\ \ 0$	$.1111251010_{10}-\ 1$
26	$-.7202081800_{10}\ \ 1$	$.1111248267_{10}-\ 1$
30	$-.3495448820_{10}-\ 1$	$.1111248437_{10}-\ 1$
36	$-.2665567369_{10}-\ 1$	$.1111248913_{10}-\ 1$
40	$-.3714772506_{10}-\ 1$	$.1111250075_{10}-\ 1$
46	$.2227408544_{10}\ \ 0$	$.1111291533_{10}\ \ 1$
50	$.3588394510_{10}-\ 1$	$.1111249350_{10}-\ 1$
60	$.1596946339_{10}-\ 1$	$.1111248373_{10}-\ 1$
70	$.1300602365_{10}-\ 1$	$.1111248304_{10}-\ 1$
80	$.1204679369_{10}-\ 1$	$.1111248262_{10}-\ 1$
90	$.1163730027_{10}-\ 1$	$.1111248276_{10}-\ 1$
100	$.1143339247_{10}-\ 1$	$.1111248252_{10}-\ 1$
150	$.1116689126_{10}-\ 1$	$.1111248233_{10}-\ 1$
200	$.1112886717_{10}-\ 1$	$.1111248237_{10}-\ 1$
300	$.1111560908_{10}-\ 1$	$.1111248207_{10}-\ 1$
400	$.1111345886_{10}-\ 1$	$.1111248232_{10}-\ 1$
500	$.1111287871_{10}-\ 1$	$.1111248240_{10}-\ 1$
600	$.1111267130_{10}-\ 1$	$.1111248293_{10}-\ 1$
700	$.1111258294_{10}-\ 1$	$.1111248329_{10}-\ 1$
800	$.1111254001_{10}-\ 1$	$.1111248161_{10}-\ 1$
900	$.1111251746_{10}-\ 1$	$.1111248306_{10}-\ 1$
1000	$.1111250388_{10}-\ 1$	$.1111248142_{10}-\ 1$
1100	$.1111249486_{00}-\ 1$	$.1111248364_{10}-\ 1$

When $k=100$,

N	Simpson	Filon
6	$.2690830229_{10}\ \ 0$	$.1000133577_{10}-\ 1$
10	$.1408276897_{10}-\ 7$	$.9999999993_{10}-\ 2$
16	$-.9760327145_{10}-\ 1$	$.1000112505_{10}-\ 1$
20	$.5251178516_{10}-\ 1$	$.1000100259_{10}-\ 1$
26	$-.2666723361_{10}\ \ 0$	$.1000104174_{10}-\ 1$
30	$-.5051255048_{10}\ \ 1$	$.1000100008_{10}-\ 1$
36	$-.2568783199_{10}-\ 1$	$.1000100180_{10}-\ 1$
40	$-.2397943300_{10}-\ 1$	$.1000100360_{10}-\ 1$
46	$-.4441602939_{10}-\ 1$	$.1000101882_{10}-\ 1$
50	$.3859446869_{10}-\ 7$	$.9999999536_{10}-\ 2$
60	$.1983399114_{10}-\ 1$	$.1000100216_{10}-\ 1$
70	$.1317546011_{10}-\ 1$	$.1000099974_{10}-\ 1$
80	$.1145990258_{10}-\ 1$	$.1000099948_{10}-\ 1$
90	$.1079078507_{10}-\ 1$	$.1000100010_{10}-\ 1$
100	$.1047326722_{10}-\ 1$	$.1000100031_{10}-\ 1$
150	$.1007769460_{10}-\ 1$	$.1000100114_{10}-\ 1$
200	$.1002380671_{10}-\ 1$	$.1000100082_{10}-\ 1$
300	$.1000531680_{10}-\ 1$	$.1000100039_{10}-\ 1$
400	$.1000234599_{10}-\ 1$	$.1000100071_{10}-\ 1$
500	$.1000154752_{10}-\ 1$	$.1000100034_{10}-\ 1$
600	$.1000126275_{10}-\ 1$	$.1000100084_{10}-\ 1$
700	$.1000114098_{10}-\ 1$	$.1000100049_{10}-\ 1$
800	$.1000108189_{10}-\ 1$	$.1000100054_{10}-\ 1$
900	$.1000104931_{10}-\ 1$	$.1000099935_{10}-\ 1$
1000	$.1000103200_{10}\ \ 1$	$.1000100022_{10}-\ 1$
1100	$.1000102162_{10}-\ 1$	$.1000100112_{10}-\ 1$
1200	$.1000101581_{10}-\ 1$	$.1000100178_{10}-\ 1$

Table 2. The integrated values of

$$\int_0^{\pi/2} e^{-x} \sin kx dx$$

for the factor k from 10 to 100. The symbols N, Simpson, and Filon stated in the table are a number of divisions of integration, the values integrated by use of Simpson's and Filon's methods respectively.*

When $k=10$,

N	Simpson	Filon
6	$.1980436807_{10}\ 0$	$.1197324211_{10}\ 0$
10	$.1249442430_{10}\ 0$	$.1195923726_{10}\ 0$
16	$.1202673674_{10}\ 0$	$.1195919177_{10}\ 0$
20	$.1198576125_{10}\ 0$	$.1195919774_{10}\ 0$
26	$.1196823744_{10}\ 0$	$.1195920138_{10}\ 0$
30	$.1196424805_{10}\ 0$	$.1195920239_{10}\ 0$
36	$.1196161348_{10}\ 0$	$.1195920300_{10}\ 0$
40	$.1196077835_{10}\ 0$	$.1195920321_{10}\ 0$
46	$.1196010018_{10}\ 0$	$.1195920338_{10}\ 0$
50	$.1195984459_{10}\ 0$	$.1195920350_{10}\ 0$
60	$.1195951166_{10}\ 0$	$.1195920360_{10}\ 0$
70	$.1195936942_{10}\ 0$	$.1195920327_{10}\ 0$
80	$.1195930068_{10}\ 0$	$.1195920366_{10}\ 0$
90	$.1195926404_{10}\ 0$	$.1195920349_{10}\ 0$
100	$.1195924316_{10}\ 0$	$.1195920372_{10}\ 0$

When $k=20$,

N	Simpson	Filon
6	$-.1875187678_{10}\ 0$	$.3950707045_{10}-\ 1$
10	$.1648242193_{10}-\ 8$	$.3960602098_{10}-\ 1$
16	$.4513775102_{10}-\ 1$	$.3950737136_{10}-\ 1$
20	$.4134773585_{10}-\ 1$	$.3950725919_{10}-\ 1$
26	$.4006606423_{10}-\ 1$	$.3950724837_{10}-\ 1$
30	$.3980778198_{10}-\ 1$	$.3950724967_{10}-\ 1$
36	$.3964613145_{10}-\ 1$	$.3950724936_{10}-\ 1$
40	$.3959674511_{10}-\ 1$	$.3950725035_{10}-\ 1$
46	$.3955748387_{10}-\ 1$	$.3950725023_{10}-\ 1$
50	$.3954292612_{10}-\ 1$	$.3950725195_{10}-\ 1$
60	$.3952420767_{10}-\ 1$	$.3950725012_{10}-\ 1$
70	$.3951632558_{10}-\ 1$	$.3950725202_{10}-\ 1$
80	$.3951254109_{10}-\ 1$	$.3950725136_{10}-\ 1$
90	$.3951054147_{10}-\ 1$	$.3950725186_{10}-\ 1$
100	$.3950940452_{10}-\ 1$	$.3950724967_{10}-\ 1$
150	$.3950767437_{10}-\ 1$	$.3950725080_{10}-\ 1$
200	$.3950738490_{10}-\ 1$	$.3950725047_{10}-\ 1$
300	$.3950727751_{10}-\ 1$	$.3950724940_{10}-\ 1$
400	$.3950725889_{10}-\ 1$	$.3950724952_{10}-\ 1$

* See the footnote of Table 1.

(to be continued)

When $k=30$, (continued)

N	Simpson	Filon
6	.2037909708$_{10}$ 0	.4021915913$_{10}-$ 1
10	−.1249442438$_{10}$ 0	.4021837152$_{10}-$ 1
16	.1662292656$_{10}$ 0	.4022550321$_{10}-$ 1
20	.5738549151$_{10}-$ 1	.4021814255$_{10}-$ 1
26	.4403080198$_{10}-$ 1	.4021797579$_{10}-$ 1
30	.4210522112$_{10}-$ 1	.4021796792$_{10}-$ 1
36	.4103340104$_{10}-$ 1	.4021796661$_{10}-$ 1
40	.4072958166$_{10}-$ 1	.4021796606$_{10}-$ 1
46	.4049763983$_{10}-$ 1	.4021796699$_{10}-$ 1
50	.4041418721$_{10}-$ 1	.4021796688$_{10}-$ 1
60	.4030939623$_{10}-$ 1	.4021796640$_{10}-$ 1
70	.4026632544$_{10}-$ 1	.4021796744$_{10}-$ 1
80	.4024594635$_{10}-$ 1	.4021796644$_{10}-$ 1
90	.4023528033$_{10}-$ 1	.4021796737$_{10}-$ 1
100	.4022925445$_{10}-$ 1	.4021796800$_{10}-$ 1
150	.4022016263$_{10}-$ 1	.4021796715$_{10}-$ 1
200	.4021865736$_{10}-$ 1	.4021796792$_{10}-$ 1
300	.4021810127$_{10}-$ 1	.4021796778$_{10}-$ 1
400	.4021800893$_{10}-$ 1	.4021796783$_{10}-$ 1
500	.4021798259$_{10}-$ 1	.4021796534$_{10}-$ 1

When $k=40$,

N	Simpson	Filon
6	−.1145239123$_{10}$ 0	.1979242081$_{10}-$ 1
10	−.1276804029$_{10}-$ 7	.1980301061$_{10}-$ 1
16	−.4684434928$_{10}-$ 1	.1979091518$_{10}-$ 1
20	−.5708492713$_{10}-$ 8	.1980301058$_{10}-$ 1
26	.2994474142$_{10}-$ 1	.1979067893$_{10}-$ 1
30	.2390211592$_{10}-$ 1	.1979064877$_{10}-$ 1
36	.2134620286$_{10}-$ 1	.1979064218$_{10}-$ 1
40	.2072168428$_{10}-$ 1	.1979064082$_{10}-$ 1
46	.2027766824$_{10}-$ 1	.1979063953$_{10}-$ 1
50	.2012576747$_{10}-$ 1	.1979064091$_{10}-$ 1
60	.1994206880$_{10}-$ 1	.1979064083$_{10}-$ 1
70	.1986935078$_{10}-$ 1	.1979064076$_{10}-$ 1
80	.1983568893$_{10}-$ 1	.1979063984$_{10}-$ 1
90	.1981831312$_{10}-$ 1	.1979064087$_{10}-$ 1
100	.1980859012$_{10}-$ 1	.1979064078$_{10}-$ 1
150	.1979409232$_{10}-$ 1	.1979064039$_{10}-$ 1
200	.1979172174$_{10}-$ 1	.1979063786$_{10}-$ 1
300	.1979085088$_{10}-$ 1	.1979073962$_{10}-$ 1
400	.1979070426$_{10}-$ 1	.1979063914$_{10}-$ 1
500	.1979066397$_{10}-$ 1	.1979063753$_{10}-$ 1
600	.1979064781$_{10}-$ 1	.1979063714$_{10}-$ 1

(to be continued)

When $k=50$, (continued)

N	Simpson	Filon
6	.4832045850$_{10}$ 0	.2415270564$_{10}-$ 1
10	.1249442425$_{10}$ 0	.2414802678$_{10}-$ 1
16	−.8797136367$_{10}-$ 1	.2414794823$_{10}-$ 1
20	−.5738549143$_{10}-$ 1	.2414806993$_{10}-$ 1
26	.1630107068$_{10}-$ 0	.2414967812$_{10}-$ 1
30	.4740698318$_{10}-$ 1	.2414798804$_{10}-$ 1
36	.3054582383$_{10}-$ 1	.2414793785$_{10}-$ 1
40	.2765486714$_{10}-$ 1	.2414793416$_{10}-$ 1
46	.2585182853$_{10}-$ 1	.2414793260$_{10}-$ 1
50	.2528529182$_{10}-$ 1	.2414793308$_{10}-$ 1
60	.2463895771$_{10}-$ 1	.2414793378$_{10}-$ 1
70	.2439690524$_{10}-$ 1	.2414793378$_{10}-$ 1
80	.2428824713$_{10}-$ 1	.2414793335$_{10}-$ 1
90	.2423330775$_{10}-$ 1	.2414793401$_{10}-$ 1
100	.2420292983$_{10}-$ 1	.2414793331$_{10}-$ 1
150	.2415834500$_{10}-$ 1	.2414793415$_{10}-$ 1
200	.2415117954$_{10}-$ 1	.2414793248$_{10}-$ 1
300	.2414856734$_{10}-$ 1	.2414793503$_{10}-$ 1
400	.2414813300$_{10}-$ 1	.2414793258$_{10}-$ 1
500	.2414801434$_{10}-$ 1	.2414793324$_{10}-$ 1
600	.2414797255$_{10}-$ 1	.2414793284$_{10}-$ 1
700	.2414795402$_{10}-$ 1	.2414793634$_{10}-$ 1

When $k=60$,

N	Simpson	Filon
6	.7274459480$_{10}-$ 8	.1320200702$_{10}-$ 1
10	.5576000867$_{10}-$ 8	.1320200699$_{10}-$ 1
16	−.1864027251$_{10}$ 0	.1319842964$_{10}-$ 1
20	−.4134773606$_{10}-$ 1	.1319834916$_{10}-$ 1
26	−.3774438964$_{10}-$ 1	.1319841589$_{10}-$ 1
30	.1870880443$_{10}-$ 7	.1320200706$_{10}-$ 1
36	.2597481223$_{10}-$ 1	.1319835622$_{10}-$ 1
40	.1892287646$_{10}-$ 1	.1319834437$_{10}-$ 1
46	.1562658774$_{10}-$ 1	.1319834178$_{10}-$ 1
50	.1474118443$_{10}-$ 1	.1319834090$_{10}-$ 1
60	.1382037269$_{10}-$ 1	.1319834005$_{10}-$ 1
70	.1350309930$_{10}-$ 1	.1319834042$_{10}-$ 1
80	.1336672326$_{10}-$ 1	.1319833944$_{10}-$ 1
90	.1329943597$_{10}-$ 1	.1319834030$_{10}-$ 1
100	.1326288800$_{10}-$ 1	.1319833875$_{10}-$ 1
150	.1321032006$_{10}-$ 1	.1319834041$_{10}-$ 1
200	.1320205058$_{10}-$ 1	.1319833641$_{10}-$ 1
300	.1319906073$_{10}-$ 1	.1319833722$_{10}-$ 1
400	.1319856450$_{10}-$ 1	.1319833494$_{10}-$ 1
500	.1319842986$_{10}-$ 1	.1319833423$_{10}-$ 1
600	.1319838036$_{10}-$ 1	.1319833827$_{10}-$ 1
700	.1319835923$_{10}-$ 1	.1319833584$_{10}-$ 1
800	.1319834967$_{10}-$ 1	.1319833689$_{10}-$ 1

(to be continued)

When $k=70$, (continued)

N	Simpson	Filon
6	−.4832045703$_{10}$ 0	.1719539221$_{10}-$ 1
10	−.1249442423$_{10}$ 0	.1725193683$_{10}-$ 1
16	.1959999285$_{10}$ 0	.1725207348$_{10}-$ 1
20	−.1198576116$_{10}$ 1	.1740353670$_{10}-$ 1
26	−.4208047494$_{10}-$ 1	.1725191692$_{10}-$ 1
30	−.4740698249$_{10}-$ 1	.1725194855$_{10}-$ 1
36	.1620728357$_{10}$ 0	.1725255628$_{10}-$ 1
40	.4406094703$_{10}-$ 1	.1725192473$_{10}-$ 1
46	.2552457474$_{10}-$ 1	.1725190343$_{10}-$ 1
50	.2206196443$_{10}-$ 1	.1725190218$_{10}-$ 1
60	.1899429006$_{10}-$ 1	.1725190156$_{10}-$ 1
70	.1806528755$_{10}-$ 1	.1725190261$_{10}-$ 1
80	.1768939308$_{10}-$ 1	.1725190411$_{10}-$ 1
90	.1751020522$_{10}-$ 1	.1725190506$_{10}-$ 1
100	.1741497820$_{10}-$ 1	.1725190417$_{10}-$ 1
150	.1728143636$_{10}-$ 1	.1725190745$_{10}-$ 1
200	.1726097897$_{10}-$ 1	.1725190344$_{10}-$ 1
300	.1725365821$_{10}-$ 1	.1725190470$_{10}-$ 1
400	.1725245315$_{10}-$ 1	.1725190747$_{10}-$ 1
500	.1725212609$_{10}-$ 1	.1725190970$_{10}-$ 1
600	.1725200996$_{10}-$ 1	.1725190760$_{10}-$ 1
700	.1725195948$_{10}-$ 1	.1725190564$_{10}-$ 1
800	.1725193391$_{10}-$ 1	.1725191130$_{10}-$ 1
900	.1725192045$_{10}-$ 1	.1725190926$_{10}-$ 1

When $k=80$,

N	Simpson	Filon
6	.1145239175$_{10}$ 0	.9899736805$_{10}-$ 2
10	−.2864416817$_{10}-$ 7	.9901505282$_{10}-$ 2
16	.5159531420$_{10}-$ 1	.9899964290$_{10}-$ 2
20	−.3686075172$_{10}-$ 7	.9901505363$_{10}-$ 2
26	−.3400791297$_{10}-$ 1	.9899959467$_{10}-$ 2
30	−.2390211649$_{10}-$ 1	.9899963835$_{10}-$ 2
36	−.3520714777$_{10}-$ 1	.9899988089$_{10}-$ 2
40	.2912010732$_{10}-$ 7	.9901506085$_{10}-$ 2
46	.2435244292$_{10}-$ 1	.9899967799$_{10}-$ 2
50	.1677363427$_{10}-$ 1	.9899961211$_{10}-$ 2
60	.1196742837$_{10}-$ 1	.9899958327$_{10}-$ 2
70	.1079982568$_{10}-$ 1	.9899958397$_{10}-$ 2
80	.1036683301$_{10}-$ 1	.9899957124$_{10}-$ 2
90	.1016973247$_{10}-$ 1	.9899957656$_{10}-$ 2
100	.1006789191$_{10}-$ 1	.9899955199$_{10}-$ 2
150	.9929480075$_{10}-$ 2	.9899954881$_{10}-$ 2
200	.9908946435$_{10}-$ 2	.9899955207$_{10}-$ 2
300	.9901685098$_{10}-$ 2	.9899955101$_{10}-$ 2
400	.9900497986$_{10}-$ 2	.9899948781$_{10}-$ 2
500	.9900177300$_{10}-$ 2	.9899954796$_{10}-$ 2
600	.9900061384$_{10}-$ 2	.9899955628$_{10}-$ 2
700	.9900012027$_{10}-$ 2	.9899949010$_{10}-$ 2
800	.9899988589$_{10}-$ 2	.9899953053$_{10}-$ 2
900	.9899975380$_{10}-$ 2	.9899952574$_{10}-$ 2
1000	.9899967562$_{10}-$ 2	.9899954083$_{10}-$ 2

(to be continued)

When $k=90$, (continued)

N	Simpson	Filon
6	$-.2037909653_{10}\ 0$	$.1341927331_{10}-\ 1$
10	$.1249442420_{10}\ 0$	$.1341924414_{10}-\ 1$
16	$.8127880270_{10}-\ 1$	$.1341921719_{10}-\ 1$
20	$.1198576102_{10}\ 0$	$.1341925943_{10}-\ 1$
26	$-.8610204750_{10}-\ 1$	$.1341922717_{10}-\ 1$
30	$-.4210521987_{10}-\ 1$	$.1341922958_{10}-\ 1$
36	$-.3204682500_{10}-\ 1$	$.1341923467_{10}-\ 1$
40	$-.4406094640_{10}-\ 1$	$.1341924765_{10}-\ 1$
46	$.1616766510_{10}\ 0$	$.1341953876_{10}-\ 1$
50	$.4253665817_{10}-\ 1$	$.1341923731_{10}-\ 1$
60	$.1925678463_{10}-\ 1$	$.1341922663_{10}-\ 1$
70	$.1570032583_{10}-\ 1$	$.1341922781_{10}-\ 1$
80	$.1454562964_{10}-\ 1$	$.1341922904_{10}-\ 1$
90	$.1405217629_{10}-\ 1$	$.1341922900_{10}-\ 1$
100	$.1380633630_{10}-\ 1$	$.1341923323_{10}-\ 1$
150	$.1348488396_{10}-\ 1$	$.1341923917_{10}-\ 1$
200	$.1343900064_{10}-\ 1$	$.1341923536_{10}-\ 1$
300	$.1342300130_{10}-\ 1$	$.1341924410_{10}-\ 1$
400	$.1342040646_{10}-\ 1$	$.1341923977_{10}-\ 1$
500	$.1341970643_{10}-\ 1$	$.1341923573_{10}-\ 1$
600	$.1341945663_{10}-\ 1$	$.1341923795_{10}-\ 1$
700	$.1341934995_{10}-\ 1$	$.1341923418_{10}-\ 1$
800	$.1341929778_{10}-\ 1$	$.1341924406_{10}-\ 1$
900	$.1341927024_{10}-\ 1$	$.1341923409_{10}-\ 1$
1000	$.1341925420_{10}-\ 1$	$.1341924422_{10}-\ 1$
1100	$.1341924426_{10}-\ 1$	$.1341924950_{10}-\ 1$

When $k=100$,

N	Simpson	Filon
6	$.1875187601_{10}\ 0$	$.2920640326_{10}-\ 2$
10	$.1253048976_{10}-\ 7$	$.7921204306_{10}-\ 2$
16	$-.6899201041_{10}-\ 1$	$.7920499483_{10}-\ 2$
20	$.4134773591_{10}-\ 1$	$.7920414051_{10}-\ 2$
26	$-.1863843418_{10}\ 0$	$.7920441531_{10}-\ 2$
30	$-.3980778307_{10}-\ 1$	$.7920412066_{10}-\ 2$
36	$-.2030688557_{10}-\ 1$	$.7920413320_{10}-\ 2$
40	$-.1892287828_{10}-\ 1$	$.7920415016_{10}-\ 2$
46	$-.3415010291_{10}-\ 1$	$.7920427099_{10}-\ 2$
50	$.4038614002_{10}-\ 7$	$.7921205560_{10}-\ 2$
60	$.1564390154_{10}-\ 1$	$.7920416723_{10}-\ 2$
70	$.1042699610_{10}-\ 1$	$.7920412547_{10}-\ 2$
80	$.9073751104_{10}-\ 2$	$.7920411325_{10}-\ 2$
90	$.8545083577_{10}-\ 2$	$.7920410491_{10}-\ 2$
100	$.8294042048_{10}-\ 2$	$.7920410222_{10}-\ 2$
150	$.7981113368_{10}-\ 2$	$.7920406098_{10}-\ 2$
200	$.7938464537_{10}-\ 2$	$.7920401878_{10}-\ 2$
300	$.7923828688_{10}-\ 2$	$.7920396661_{10}-\ 2$
400	$.7921476929_{10}-\ 2$	$.7920396139_{10}-\ 2$
500	$.7920844623_{10}-\ 2$	$.7920397699_{10}-\ 2$
600	$.7920618903_{10}-\ 2$	$.7920392166_{10}-\ 2$
700	$.7920522442_{10}-\ 2$	$.7920395089_{10}-\ 2$
800	$.7920475926_{10}-\ 2$	$.7920403485_{10}-\ 2$
900	$.7920450438_{10}-\ 2$	$.7920404489_{10}-\ 2$
1000	$.7920436205_{10}-\ 2$	$.7920404257_{10}-\ 2$
1100	$.7920427426_{10}-\ 2$	$.7920396710_{10}-\ 2$
1200	$.7920422038_{10}-\ 2$	$.7920391651_{10}-\ 2$

are, to the accurracy of six digits, about 4 minutes by Simpson's method and about 20 seconds by Filon's method.[4] When $k=50$, times are needed about 1.40 minutes for Simpson's and about 17 seconds for Filon's methods.[4]

Inasmuch as the integrand of the integration (1) can be transformed into the coupled form of $\cos k^*$, $\exp(-k^*)$, and $\sin k^*$, the evaluation of the efficiencies of the above two particular examples are of considerable significance for the present study.

3. Transformation of Integrand

Using the asymptotic expression, the Bessel function $J_0(z)$ can be expressed as

$$J_0(z)=\sqrt{\frac{2}{\pi z}}\left[A_n(z)\cos\left(z-\frac{1}{4}\pi\right)-B_n(z)\sin\left(z-\frac{1}{4}\pi\right)\right] \tag{7}$$

where

$$A_n(z)=1+\sum_{r=1}^{[n/2]}(-1)^r\frac{(-1^2)(-3^2)\cdots[-(4r-1)^2]}{(2r)!\,(8z)^{2r}}\,,$$

$$B_n(z)=\sum_{r=0}^{[(n-1)/2]}(-1)^r\frac{(-1^2)(-3^2)\cdots[-(4r+1)^2]}{(2r+1)!\,(8z)^{2r+1}}\,.$$

The above expression is valid for only large z. On the other hand, $J_0(z)$ is, in general, expressible, as ascending series, i.e.

$$J_0(z)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{(r!)^2}\left(\frac{1}{2}z\right)^{2r}. \tag{8}$$

The use of the expression (7) is possible only for the large value z and Filon's method is applied through an asymptotic from (7), so that the lower part of the integration (1) must be computed, in a form of the the expression (8), by means of Simpson's method. Hence the integration (1) is separated into three parts

$$\int_0^\infty=\int_0^{k_1^*}+\int_{k_1^*}^{k_2^*}+R\,. \tag{9}$$

In the above expression, k_1^* is a critical value, of which the lower side

4) The calculation of this part was made by use of the OKI-TAC 5090 at the Computation Centre of Tokyo University.

is integrated by Simpson's formula and the upper side, up to k_2^*, by Filon's method. R is the remainder of the integration when the infinity (∞) is taken as k_2^*. If $k_2^*=10$ (for $a^*=10$), R is estimated as the order of 0.0001 in the previous paper.[5)]

When the Bessel function expressed as ascending series such as (8) is calculated by an electronic computer, the rounding or cut-off errors, as z increases, are accumulated as a result of so heavy an increase of the computed terms that there exists no significance of the computed values.

The critical value k_1^* must be determined appropriately on the basis of the facts mentioned above.

Using the ascending series (8), the first integral in the right-hand side of (9) is computed without modification of the integrand, i.e.

$$\int_0^{k_1^*}=\int_0^{k_1^*}a^*\frac{\cos\omega^*t^*}{\cosh k^*}J_0(k^*r^*)J_1(k^*a^*)dk^* . \tag{10}$$

On the other hand, the second integral must be transformed by use of the asymptotic expansion (7) for convenience of an application of Filon's method. Substituting (7) into the second integral of (9), we have

$$\int_{k_1^*}^{k_2^*}=\int_{k_1^*}^{k_2^*}dk^*\left\{\psi_A\cos\left(k^*r^*-\frac{\pi}{4}\right)-\psi_B\sin\left(k^*r^*-\frac{\pi}{4}\right)\right\} , \tag{11}$$

where

$$\psi_A=a^*\sqrt{\frac{2}{\pi r^*}}\frac{\cos\omega^*t^*}{\cosh k^*}\frac{A_n(k^*r^*)}{\sqrt{k^*}}J_1(k^*a^*) ,$$

$$\psi_B=a^*\sqrt{\frac{2}{\pi r^*}}\frac{\cos\omega^*t^*}{\cosh k^*}\frac{B_n(k^*r^*)}{\sqrt{k^*}}J_1(k^*a^*) .$$

As already made by Ichiye[6)] for a one-dimensional case, setting down ω^* $(=\sqrt{k^*\tanh k^*})$ equal to $a_j+b_j\cdot k^*$ in the j-th interval $m_{j+1}\geqq k^*>m_j$, (11) becomes

$$\int_{k_1^*}^{k_2^*}=\sum_j\int_{m_j}^{m_{j+1}}dk^*\left\{\psi_A^{(1)}\cos(a_j+b_jk^*)t^*\cdot\cos\left(k^*r^*-\frac{\pi}{4}\right)\right.$$
$$\left.-\psi_B^{(1)}\cos(a_j+b_jk^*)t^*\cdot\sin\left(k^*r^*-\frac{\pi}{4}\right)\right\} , \tag{12}$$

where

5) T. MOMOI, *loc. cit.*, 1)

6) T. ICHIYE, *Jour. Ocean, Soc. Japan*, **14** (1958), 35.

$$\psi_A^{(1)}=a^*\sqrt{\frac{2}{\pi r^*}}\frac{A_n(k^*r^*)}{\sqrt{k^*}\cosh k^*}J_1(k^*a^*)\ ,$$
$$\psi_B^{(1)}=a^*\sqrt{\frac{2}{\pi r^*}}\frac{B_n(k^*r^*)}{\sqrt{k^*}\cosh k^*}J_1(k^*a^*)\ .$$

In the above expression, the following reductions can be made for trigonometric factors:-

$$\left.\begin{aligned}
&\cos(a_j+b_jk^*)t^*\cdot\cos\left(k^*r^*-\frac{\pi}{4}\right)\\
&=\frac{1}{2}\left[\sin\left\{(r^*+b_jt^*)k^*+a_jt^*+\frac{\pi}{4}\right\}\right.\\
&\quad\left.+\sin\left\{(r^*-b_jt^*)k^*-a_jt^*+\frac{\pi}{4}\right\}\right]\ ,\\
&\cos(a_j+b_jk^*)t^*\cdot\sin\left(k^*r^*-\frac{\pi}{4}\right)\\
&=\frac{1}{2}\left[\sin\left\{(r^*+b_jt^*)k^*+a_jt^*-\frac{\pi}{4}\right\}\right.\\
&\quad\left.+\sin\left\{(r^*-b_jt^*)k^*-a_jt^*-\frac{\pi}{4}\right\}\right]\ .
\end{aligned}\right\}\qquad(13)$$

Substituting (13) into (12) and after a few reductions, the integration (12) becomes

$$\begin{aligned}
\int_{k_1^*}^{k_2^*}=\sum_j\Big\{&\int_{m_j}^{m_{j+1}}\psi_A^{(2)}(k^*)\sin(c_j^{(+)}k^*+d_j^{(+)})dk^*\\
&+\int_{m_j}^{m_{j+1}}\psi_A^{(2)}(k^*)\sin(c_j^{(-)}k^*-d_j^{(-)})dk^*\\
&+\int_{m_j}^{m_{j+1}}\psi_B^{(2)}(k^*)\sin(c_j^{(+)}k^*+d_j^{(-)})dk^*\\
&+\int_{m_j}^{m_{j+1}}\psi_B^{(2)}(k^*)\sin(c_j^{(-)}k^*-d_j^{(+)})dk^*\Big\}\ ,
\end{aligned}\qquad(14)$$

where

$$\left.\begin{aligned}
&\psi_A^{(2)}(k^*)=\frac{+a^*}{\sqrt{2\pi k^*r^*}}\frac{J_1(k^*a^*)}{\cosh k^*}A_n(k^*r^*)\ ,\\
&\psi_B^{(2)}(k^*)=\frac{-a^*}{\sqrt{2\pi k^*r^*}}\frac{J_1(k^*a^*)}{\cosh k^*}B_n(k^*r^*)\ ,\\
&\qquad c_j^{(\pm)}=r^*\pm b_jt^*\ ,\\
&\qquad d_j^{(\pm)}=a_jt^*\pm\frac{\pi}{4}\ ,
\end{aligned}\right\}\qquad(14')$$

(the double signs ($\pm$) must be taken in the same order).

The integration (14) is a formal expression which enables us to calculate the integration of the left-hand side by use of Filon's method.

Although the formula of (6) has been established for the integral

$$\int_a^b \psi(x)\sin kx dx\ ,$$

it can be extended to cover the case of the integral

$$\int_a^b \psi(x)\sin(kx+\epsilon)dx \qquad (\epsilon : \text{phase})$$

without any loss of efficiency of this method as compared with Simpson's formula.

Then the formula (6) becomes

$$\begin{aligned}\int_a^b &\psi(x)\sin(kx+\epsilon)dx\\ &=h[\alpha\{\psi(a)\cos(ka+\epsilon)-\psi(b)\cos(kb+\epsilon)\}\\ &\quad+\beta T_{2\gamma}+\gamma T_{2r-1}]\ ,\end{aligned} \qquad (15)$$

where

$$T_{2r}=\sum_{r=0}^{n}\psi(x_{2r})\sin(kx_{2r}+\epsilon)-\frac{1}{2}\{\psi(a)\sin(ka+\epsilon)+\psi(b)\sin(kb+\epsilon)\}\ ,$$

$$T_{2r-1}=\sum_{r=1}^{n}\psi(x_{2r-1})\sin(kx_{2r-1}+\epsilon)\ .$$

Using the formula (15), the integration (14), in effect, may be calculated. In the following section, the formula expression (14) will be modified further to a form which is more efficient for calculation.

4. Calculation of Coefficients a_j and b_j

In the preceding section, the expression ω^* $(=\sqrt{k^*\cdot\tanh k^*})$ is approximated by $a_j+b_jk^*$ in the interval $m_{j+1}\geqq k^*>m_j$. The line of approach of calculation of a_j and b_j is an experimental procedure by means of an electronic computer.

Referring to Fig. 1, the points A and B at $k^*=k_0^*$ and k_n^* are fixed in such a way that

Table 3. Coefficients a_j, b_j of the function $a_j+b_jk^*$ approximated in the intervals $m_j<k^*\leqq m_{j+1}$ (since the approximate functions involve the relative errors 0.2×10^{-4}, the computed values are retained to only six digits), and the divided numbers ($2n$) of each of the small intervals.

$m_j \sim m_{j+1}$	a_j	b_j	$2n$
$0.740750\times10^{-2}\sim0.274085\times10^{-1}$	0.117759×10^{-5}	0.999851×10^{0}	2
$0.274085\times10^{-1}\sim0.461595\times10^{-1}$	0.154989×10^{-4}	0.999329×10^{0}	2
$0.461595\times10^{-1}\sim0.642855\times10^{-1}$	0.545147×10^{-4}	0.998484×10^{0}	2
$0.642855\times10^{-1}\sim0.824115\times10^{-1}$	0.129085×10^{-3}	0.997324×10^{0}	2
$0.824115\times10^{-1}\sim0.100537\times10^{0}$	0.251295×10^{-3}	0.995841×10^{0}	2
$0.100537\times10^{0}\sim0.118663\times10^{0}$	0.432538×10^{-3}	0.994038×10^{0}	2
$0.118663\times10^{0}\sim0.136789\times10^{0}$	0.683979×10^{-3}	0.991919×10^{0}	2
$0.136789\times10^{0}\sim0.154915\times10^{0}$	0.101648×10^{-2}	0.989488×10^{0}	2
$0.154915\times10^{0}\sim0.173041\times10^{0}$	0.144059×10^{-2}	0.986751×10^{0}	2
$0.173041\times10^{0}\sim0.191167\times10^{0}$	0.196647×10^{-2}	0.983711×10^{0}	2
$0.191167\times10^{0}\sim0.209293\times10^{0}$	0.260391×10^{-2}	0.980377×10^{0}	2
$0.209293\times10^{0}\sim0.227419\times10^{0}$	0.336220×10^{-2}	0.976754×10^{0}	2
$0.227419\times10^{0}\sim0.245545\times10^{0}$	0.425022×10^{-2}	0.972849×10^{0}	2
$0.245545\times10^{0}\sim0.263671\sim10^{0}$	0.527632×10^{-2}	0.968670×10^{0}	2
$0.263671\times10^{0}\sim0.282110\times10^{0}$	0.645881×10^{-2}	0.964186×10^{0}	2
$0.282110\times10^{0}\sim0.300548\times10^{0}$	0.780956×10^{-2}	0.959398×10^{0}	2
$0.300548\times10^{0}\sim0.319299\times10^{0}$	0.933919×10^{-2}	0.954308×10^{0}	2
$0.319299\times10^{0}\sim0.338050\times10^{0}$	0.110592×10^{-1}	0.948921×10^{0}	2
$0.338050\times10^{0}\sim0.356801\times10^{0}$	0.129632×10^{-1}	0.943289×10^{0}	2
$0.356801\times10^{0}\sim0.375552\times10^{0}$	0.150567×10^{-1}	0.937422×10^{0}	2
$0.375552\times10^{0}\sim0.394303\times10^{0}$	0.173443×10^{-1}	0.931330×10^{0}	2
$0.394303\times10^{0}\sim0.413367\times10^{0}$	0.198511×10^{-1}	0.924973×10^{0}	2
$0.413367\times10^{0}\sim0.432743\times10^{0}$	0.226092×10^{-1}	0.918300×10^{0}	2
$0.432743\times10^{0}\sim0.452119\times10^{0}$	0.256071×10^{-1}	0.911373×10^{0}	2
$0.452119\times10^{0}\sim0.471495\times10^{0}$	0.288250×10^{-1}	0.904255×10^{0}	2
$0.471495\times10^{0}\sim0.490871\times10^{0}$	0.322641×10^{-1}	0.896961×10^{0}	2
$0.490871\times10^{0}\sim0.510559\times10^{0}$	0.359550×10^{-1}	0.889442×10^{0}	2
$0.510559\times10^{0}\sim0.530560\times10^{0}$	0.399356×10^{-1}	0.881646×10^{0}	2
$0.530560\times10^{0}\sim0.550561\times10^{0}$	0.441823×10^{-1}	0.873641×10^{0}	2
$0.550561\times10^{0}\sim0.570562\times10^{0}$	0.486626×10^{-1}	0.865504×10^{0}	2
$0.570562\times10^{0}\sim0.591188\times10^{0}$	0.534482×10^{-1}	0.857116×10^{0}	2
$0.591188\times10^{0}\sim0.611814\times10^{0}$	0.585492×10^{-1}	0.848488×10^{0}	2
$0.611814\times10^{0}\sim0.632440\times10^{0}$	0.638884×10^{-1}	0.839761×10^{0}	2
$0.632440\times10^{0}\sim0.653691\times10^{0}$	0.695460×10^{-1}	0.830815×10^{0}	2

(to be continued)

(continued)

$m_j \sim m_{j+1}$	a_j	b_j	$2n$
$0.653691\times10^{0} \sim 0.674942\times10^{0}$	0.755295×10^{-1}	0.821662×10^{0}	2
$0.674942\times10^{0} \sim 0.696193\times10^{0}$	0.817483×10^{-1}	0.812448×10^{0}	2
$0.696193\times10^{0} \sim 0.718069\times10^{0}$	0.882903×10^{-1}	0.803051×10^{0}	2
$0.718069\times10^{0} \sim 0.739945\times10^{0}$	0.951603×10^{-1}	0.793484×10^{0}	2
$0.739945\times10^{0} \sim 0.762446\times10^{0}$	0.102356×10^{0}	0.783759×10^{0}	2
$0.762446\times10^{0} \sim 0{\cdot}784947\times10^{0}$	0.109880×10^{0}	0.773890×10^{0}	2
$0.784947\times10^{0} \sim 0.807761\times10^{0}$	0.117676×10^{0}	0.763959×10^{0}	2
$0.807761\times10^{0} \sim 0.830887\times10^{0}$	0.125791×10^{0}	0.753912×10^{0}	2
$0.830887\times10^{0} \sim 0.854013\times10^{0}$	0.134167×10^{0}	0.743832×10^{0}	2
$0.854013\times10^{0} \sim 0.877764\times10^{0}$	0.142851×10^{0}	0.733663×10^{0}	2
$0.877764\times10^{0} \sim 0.902140\times10^{0}$	0.151958×10^{0}	0.723288×10^{0}	2
$0.902140\times10^{0} \sim 0.926516\times10^{0}$	0.161369×10^{0}	0.712856×10^{0}	2
$0.926516\times10^{0} \sim 0.951517\times10^{0}$	0.171072×10^{0}	0.702383×10^{0}	2
$0.951517\times10^{0} \sim 0.976518\times10^{0}$	0.181062×10^{0}	0.691885×10^{0}	2
$0.976518\times10^{0} \sim 0.100214\times10^{1}$	0.191324×10^{0}	0.681376×10^{0}	2
$0.100214\times10^{1} \sim 0.102839\times10^{1}$	0.201979×10^{0}	0.670744×10^{0}	2
$0.102839\times10^{1} \sim 0.105464\times10^{1}$	0.212889×10^{0}	0.660134×10^{0}	2
$0.105464\times10^{1} \sim 0.108152\times10^{1}$	0.224039×10^{0}	0.649563×10^{0}	2
$0.108152\times10^{1} \sim 0.110902\times10^{1}$	0.235548×10^{0}	0.638921×10^{0}	2
$0.110902\times10^{1} \sim 0.113714\times10^{1}$	0.247405×10^{0}	0.628230×10^{0}	2
$0.113714\times10^{1} \sim 0.116590\times10^{1}$	0.259595×10^{0}	0.617510×10^{0}	2
$0.116590\times10^{1} \sim 0.119527\times10^{1}$	0.272104×10^{0}	0.606781×10^{0}	2
$0.119527\times10^{1} \sim 0.122527\times10^{1}$	0.284914×10^{0}	0.596064×10^{0}	2
$0.122527\times10^{1} \sim 0.125527\times10^{1}$	0.297873×10^{0}	0.585487×10^{0}	2
$0.125527\times10^{1} \sim 0.128652\times10^{1}$	0.311092×10^{0}	0.574956×10^{0}	2
$0.128652\times10^{1} \sim 0.131840\times10^{1}$	0.324691×10^{0}	0.564387×10^{0}	2
$0.131840\times10^{1} \sim 0.135090\times10^{1}$	0.338511×10^{0}	0.553904×10^{0}	2
$0.135090\times10^{1} \sim 0.138403\times10^{1}$	0.352530×10^{0}	0.543527×10^{0}	2
$0.138403\times10^{1} \sim 0.141778\times10^{1}$	0.366724×10^{0}	0.533271×10^{0}	2
$0.141778\times10^{1} \sim 0.145278\times10^{1}$	0.381198×10^{0}	0.523062×10^{0}	2
$0.145278\times10^{1} \sim 0.148841\times10^{1}$	0.395931×10^{0}	0.512921×10^{0}	2
$0.148841\times10^{1} \sim 0.152466\times10^{1}$	0.410761×10^{0}	0.502957×10^{0}	2
$0.152466\times10^{1} \sim 0.156216\times10^{1}$	0.425789×10^{0}	0.493100×10^{0}	2
$0.156216\times10^{1} \sim 0.160091\times10^{1}$	0.441113×10^{0}	0.483291×10^{0}	2
$0.160091\times10^{1} \sim 0.164091\times10^{1}$	0.456695×10^{0}	0.473558×10^{0}	2
$0.164091\times10^{1} \sim 0.168216\times10^{1}$	0.472499×10^{0}	0.463926×10^{0}	2
$0.168216\times10^{1} \sim 0.172466\times10^{1}$	0.488489×10^{0}	0.454421×10^{0}	2
$0.172466\times10^{1} \sim 0.176841\times10^{1}$	0.504625×10^{0}	0.445065×10^{0}	2

(to be continued)

(continued)

m_j ~ m_{j+1}	a_j	b_j	$2n$
0.176841×10^1 ~0.181341×10^1	0.520872×10^0	0.435878×10^0	2
0.181341×10^1 ~0.185966×10^1	0.537191×10^0	0.426878×10^0	2
0.185966×10^1 ~0.190717×10^1	0.553549×10^0	0.418082×10^0	2
0.190717×10^1 ~0.195717×10^1	0.570116×10^0	0.409396×10^0	2
0.195717×10^1 ~0.200842×10^1	0.586851×10^0	0.400845×10^0	2
0.200842×10^1 ~0.206217×10^1	0.603697×10^0	0.392458×10^0	2
0.206217×10^1 ~0.211717×10^1	0.620616×10^0	0.384253×10^0	2
0.211717×10^1 ~0.217467×10^1	0.637556×10^0	0.376252×10^0	2
0.217467×10^1 ~0.223467×10^1	0.654660×10^0	0.368387×10^0	2
0.223467×10^1 ~0.229717×10^1	0.671871×10^0	0.360685×10^0	2
0.229717×10^1 ~0.236217×10^1	0.689137×10^0	0.353168×10^0	2
0.236217×10^1 ~0.242967×10^1	0.706412×10^0	0.345855×10^0	2
0.242967×10^1 ~0.249968×10^1	0.723653×10^0	0.338759×10^0	2
0.249968×10^1 ~0.257218×10^1	0.740824×10^0	0.331890×10^0	2
0.257218×10^1 ~0.264843×10^1	0.758032×10^0	0.325200×10^0	2
0.264843×10^1 ~0.272843×10^1	0.775376×10^0	0.318651×10^0	2
0.272843×10^1 ~0.281093×10^1	0.792676×10^0	0.312311×10^0	2
0.281093×10^1 ~0.289843×10^1	0.810020×10^0	0.306141×10^0	2
0.289843×10^1 ~0.298843×10^1	0.827387×10^0	0.300149×10^0	2
0.298843×10^1 ~0.308343×10^1	0.844734×10^0	0.294344×10^0	2
0.308343×10^1 ~0.318343×10^1	0.862273×10^0	0.288656×10^0	2
0.318343×10^1 ~0.328593×10^1	0.879757×10^0	0.283164×10^0	2
0.328593×10^1 ~0.339344×10^1	0.897170×10^0	0.277864×10^0	2
0.339344×10^1 ~0.350595×10^1	0.914717×10^0	0.272694×10^0	2
0.350595×10^1 ~0.362471×10^1	0.932475×10^0	0.267628×10^0	2
0.362471×10^1 ~0.374659×10^1	0.950298×10^0	0.262711×10^0	2
0.374659×10^1 ~0.387473×10^1	0.968176×10^0	0.257940×10^0	2
0.387473×10^1 ~0.400911×10^1	0.986335×10^0	0.253253×10^0	2
0.400911×10^1 ~0.414975×10^1	0.100476×10^1	0.248656×10^0	2
0.414975×10^1 ~0.429663×10^1	0.102346×10^1	0.244150×10^0	2
0.429663×10^1 ~0.444664×10^1	0.104223×10^1	0.239782×10^0	4
0.444664×10^1 ~0.460290×10^1	0.106108×10^1	0.235543×10^0	4
0.460290×10^1 ~0.476541×10^1	0.108021×10^1	0.231386×10^0	4
0.476541×10^1 ~0.493417×10^1	0.109962×10^1	0.227313×10^0	4
0.493417×10^1 ~0.510918×10^1	0.111931×10^1	0.223322×10^0	4
0.510918×10^1 ~0.529044×10^1	0.113928×10^1	0.219413×10^0	4
0.529044×10^1 ~0.547795×10^1	0.115953×10^1	0.215587×10^0	4
0.547795×10^1 ~0.567484×10^1	0.118020×10^1	0.211812×10^0	4

(to be continued)

(continued)

m_j ~ m_{j+1}	a_j	b_j	$2n$
0.567484×10^1 ~0.588110×10^1	0.120147×10^1	0.208065×10^0	4
0.588110×10^1 ~0.609361×10^1	0.122313×10^1	0.204381×10^0	4
0.609361×10^1 ~0.631237×10^1	0.124503×10^1	0.200787×10^0	4
0.631237×10^1 ~0.653738×10^1	0.126715×10^1	0.197283×10^0	4
0.653738×10^1 ~0.677489×10^1	0.128978×10^1	0.193822×10^0	4
0.677489×10^1 ~0.701865×10^1	0.131291×10^1	0.190407×10^0	4
0.701865×10^1 ~0.726866×10^1	0.133622×10^1	0.187086×10^0	6
0.726866×10^1 ~0.753117×10^1	0.135999×10^1	0.183817×10^0	6
0.753117×10^1 ~0.779993×10^1	0.138419×10^1	0.180603×10^0	6
0.779993×10^1 ~0.808119×10^1	0.140880×10^1	0.177448×10^0	6
0.808119×10^1 ~0.837495×10^1	0.143408×10^1	0.174320×10^0	6
0.837495×10^1 ~0.867496×10^1	0.145973×10^1	0.171257×10^0	6
0.867496×10^1 ~0.898747×10^1	0.148572×10^1	0.168261×10^0	6
0.898747×10^1 ~0.931248×10^1	0.151229×10^1	0.165305×10^0	6
0.931248×10^1 ~0.964999×10^1	0.153942×10^1	0.162391×10^0	6
0.964999×10^1 ~0.100000×10^2	0.156708×10^1	0.159525×10^0	6

$$\Delta\omega_i^*/\omega_i^* = \in_r \qquad (i=0, n) \tag{16}$$

($\in_r$ is a small value) and then the calculation of a_j and b_j can readily be made. Thereafter, the interval (k_0^*, k_n^*) is divided into $(n+1)$ divisions (n: large number) to compute the relative errors

$$E_i = |\Delta\omega_i^*|/\omega_i^* \tag{17}$$

at each point $k^*=k_i^*$ $(i=1, 2, 3, \cdots, n-1)$.

If

$$\in_r - \delta\in_r \leqq E_i \leqq \in_r \tag{18}$$

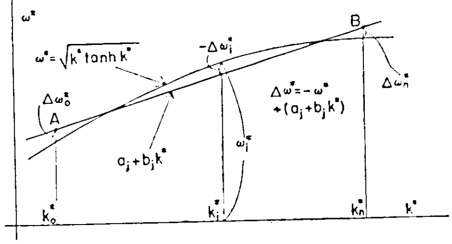

Fig. 1.

($\delta\in_r$ is a small positive value as compared with $\in_r$), $a_j+b_jk^*$ is employed as the approximated linear function of

$$\omega^*(=\sqrt{k^*\tanh k^*})$$

in the interval (k_0^*, k_n^*), which is set down as (m_j, m_{j+1}). But when the criterion (18) is not satisfied, re-computation of a_j and b_j must be made by choosing a new point A under the condition (16) (then, the point B is fixed). The way to search for a new point A is such that the point A, for $E_j > \epsilon_r$, must be moved a little to the right and, for $\epsilon_r - \delta\epsilon_r > E_j$, to the left in Fig. 1 to calculate (17) and to check the criterion (18) for the newly-selected point A. Such a process is iterated until the criterion (18) is fulfilled.

In such a way, the approximated function $a_j + b_j k^*$ is obtained subsequently for successive intervals (m_j, m_{j+1}) of k^*.

Suppose that

$$\epsilon_r = 0.2 \times 10^{-4}, \qquad \delta\epsilon_r = 0.2 \times 10^{-5},$$

the values calculated in the fore-going procedure are shown in Table 3.

5. Computation of Wave Height and Discussion

Although the second integration of (9) might be formally calculated using the expression (14) by means of Filon's method, it is better that (14) be modified a little to save time of computation, which is done in the following way.

By use of (15), the right-hand side of (14) becomes

$$\int_{k_1^*}^{k_2^*} = \sum_j h_j (U_j + V_j), \tag{19}$$

$$\begin{aligned} U_j = {} & \alpha_j^{(+)} \{\psi_A^{(2)}(m_j) \cos (c_j^{(+)} m_j + d_j^{(+)}) - \psi_A^{(2)}(m_{j+1}) \cos (c_j^{(+)} m_{j+1} + d_j^{(+)})\} \\ & + \alpha_j^{(-)} \{\psi_A^{(2)}(m_j) \cos (c_j^{(-)} m_j - d_j^{(-)}) - \psi_A^{(2)}(m_{j+1}) \cos (c_j^{(-)} m_{j+1} - d_j^{(-)})\} \\ & + \alpha_j^{(+)} \{\psi_B^{(2)}(m_j) \cos (c_j^{(+)} m_j + d_j^{(-)}) - \psi_B^{(2)}(m_{j+1}) \cos (c_j^{(+)} m_{j+1} + d_j^{(-)})\} \\ & + \alpha_j^{(-)} \{\psi_B^{(2)}(m_j) \cos (c_j^{(-)} m_j - d_j^{(+)}) - \psi_B^{(2)}(m_{j+1}) \cos (c_j^{(-)} m_{j+1} - d_j^{(+)})\}, \end{aligned} \tag{20}$$

$$\begin{aligned} V_j = {} & \beta_j^{(+)} T_{2r}(c_j^{(+)}, d_j^{(+)}) + \gamma_j^{(+)} T_{2r-1}(c_j^{(+)}, d_j^{(+)}) \\ & + \beta_j^{(-)} T_{2r}(c_j^{(-)}, d_j^{(-)}) + \gamma_j^{(-)} T_{2r-1}(c_j^{(-)}, d_j^{(-)}) \\ & + \beta_j^{(+)} T_{2r}(c_j^{(+)}, d_j^{(-)}) + \gamma_j^{(+)} T_{2r-1}(c_j^{(+)}, d_j^{(-)}) \\ & + \beta_j^{(-)} T_{2r}(c_j^{(-)}, d_j^{(+)}) + \gamma_j^{(-)} T_{2r-1}(c_j^{(-)}, d_j^{(+)}), \end{aligned} \tag{21}$$

$$\left.\begin{aligned} & \alpha_j^{(\pm)} = 1/\theta_j^{(\pm)} + \cos\theta_j^{(\pm)} \sin\theta_j^{(\pm)}/(\theta_j^{(\pm)})^2 - 2\sin^2\theta_j^{(\pm)}/(\theta_j^{(\pm)})^3, \\ & \beta_j^{(\pm)} = 2[(1 + \cos^2\theta_j^{(\pm)})/(\theta_j^{(\pm)})^2 - 2\sin\theta_j^{(\pm)}\cos\theta_j^{(\pm)}/(\theta_j^{(\pm)})^3], \\ & \gamma_j^{(\pm)} = 4\{\sin\theta_j^{(\pm)}/(\theta_j^{(\pm)})^3 - \cos\theta_j^{(\pm)}/(\theta_j^{(\pm)})^2\}, \\ & \theta_j^{(\pm)} = c_j^{(\pm)} h_j, \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned}
T_{2r}(c_j^{(\pm)}, d_j^{(\pm)}) &= \sum_{r=0}^{n} \psi_A^{(2)}(k_{2r}^*) \sin(c_j^{(\pm)} k_{2r}^* + d_j^{(\pm)}) \\
&\quad -\frac{1}{2}\{\psi_A^{(2)}(m_j) \sin(c_j^{(\pm)} m_j + d_j^{(\pm)}) \\
&\quad + \psi_A^{(2)}(m_{j+1}) \sin(c_j^{(\pm)} m_{j+1} + d_j^{(\pm)})\}, \\
T_{2r}(c_j^{(\pm)}, d_j^{(\mp)}) &= \sum_{r=0}^{n} \psi_B^{(2)}(k_{2r}^*) \sin(c_j^{(\pm)} k_{2r}^* + d_j^{(\mp)}) \\
&\quad -\frac{1}{2}\{\psi_B^{(2)}(m_j) \sin(c_j^{(\pm)} m_j + d_j^{(\mp)}) \\
&\quad + \psi_B^{(2)}(m_{j+1}) \sin(c_j^{(\pm)} m_{j+1} + d_j^{(\mp)})\}, \\
T_{2r-1}(c_j^{(\pm)}, d_j^{(\pm)}) &= \sum_{r=1}^{n} \psi_A^{(2)}(k_{2r-1}^*) \sin(c_j^{(\pm)} k_{2r-1}^* + d_j^{(\pm)}), \\
T_{2r-1}(c_j^{(\pm)}, d_j^{(\mp)}) &= \sum_{r=1}^{n} \psi_B^{(2)}(k_{2r-1}^*) \sin(c_j^{(\pm)} k_{2r-1}^* + d_j^{(\mp)}),
\end{aligned}\right\} \tag{23}$$

(in the expressions (22) and (23), the double signs ($\pm$) must be taken in the same order).

Using (23), the expression (21) becomes

$$\begin{aligned}
V_j = \sum_{r=0}^{n} g_r[&\psi_A^{(2)}(k_{2r}^*)\{\beta_j^{(+)} \sin(c_j^{(+)} k_{2r}^* + d_j^{(+)}) + \beta_j^{(-)} \text{sn}(c_j^{(-)} k_{2r}^* + d_j^{(-)}) \\
&+ \psi_B^{(2)}(k_{2r}^*)\{\beta_j^{(+)} \sin(c_j^{(+)} k_{2r}^* + d_j^{(-)}) + \beta_j^{(-)} \sin(c_j^{(-)} k_{2r}^* + d_j^{(+)})\}] \\
&+ \sum_{r=1}^{n} [\psi_A^{(2)}(k_{2r-1}^*)\{\gamma_j^{(+)} \sin(c_j^{(+)} k_{2r-1}^* + d_j^{(+)}) + \gamma_j^{(-)} \sin(c_j^{(-)} k_{2r-1}^* + d_j^{(-)})\} \\
&+ \psi_B^{(2)}(k_{2r-1}^*)\{\gamma_j^{(+)} \sin(c_j^{(+)} k_{2r-1}^* + d_j^{(-)}) + \gamma_j^{(-)} \sin(c_j^{(-)} k_{2r-1}^* + d_j^{(+)})\}],
\end{aligned}$$

where $g_r = 1/2$ for $r = 0, n$
$= 1$, otherwise.

Substituting (14′) into the above expression and after some reductions, we have

$$\begin{aligned}
V_j = &\sum_{r=0}^{n} g_r \frac{a^*}{\sqrt{2\pi k_{2r}^* r^*}} \frac{J_1(k_{2r}^* a^*)}{\cosh k_{2r}^*} \\
&\cdot[A_n(k_{2r}^* r^*)\{\beta_j^{(+)} \sin(c_j^{(+)} k_{2r}^* + d_j^{(+)}) + \beta_j^{(-)} \sin(c_j^{(-)} k_{2r}^* + d_j^{(-)})\} \\
&- B_n(k_{2r}^* r^*)\{\beta_j^{(+)} \sin(c_j^{(+)} k_{2r}^* + d_j^{(-)}) + \beta_j^{(-)} \sin(c_j^{(-)} k_{2r}^* + d_j^{(+)})\}] \\
&+ \sum_{r=1}^{n} \frac{a^*}{\sqrt{2\pi k_{2r-1}^* r^*}} \frac{J_1(k_{2r-1}^* a^*)}{\cosh k_{2r-1}^*} \\
&\cdot[A_n(k_{2r-1}^* r^*)\{\gamma_j^{(+)} \sin(c_j^{(+)} k_{2r-1}^* + d_j^{(+)}) + \gamma_j^{(-)} \sin(c_j^{(-)} k_{2r-1}^* + d_j^{(-)})\} \\
&- B_n(k_{2r-1}^* r^*)\{\gamma_j^{(+)} \sin(c_j^{(+)} k_{2r-1}^* + d_j^{(-)}) + \gamma_j^{(-)} \sin(c_j^{(-)} k_{2r-1}^* + d_j^{(+)})\}].
\end{aligned} \tag{21′}$$

The use of (21′) enables us to calculate V_j more efficiently than (21) does.

If one uses (21), the factor $\psi_A^{(2)}(k_{2r}^*)$, $\psi_B^{(2)}(k_{2r}^*)$, $\psi_A^{(2)}(k_{2r-1}^*)$ or $\psi_B^{(2)}(k_{2r-1}^*)$ must be computed iteratively for each calculation of sine term and the computation of ψ-term needs not a little time, as expressed in (14′). On the other hand, when (21′) is used, the waste of time in computation of ψ-terms for each calculation of sine terms may be avoided. According to the numerical experiment by an electronic computer, the time necessary for calculation of (21′) is about one-sixth of that for (21). Now when the specification of the values a^*, k_1^* and k_2^* is made, the second integration of (9) is calculable for successive values of r^*, t^* by using (19), (20), (21′), (22) and the coefficients a_j, b_j calculated in section 4.

Provided that

$$\left.\begin{aligned} a^* &= 10 \\ k_2^* &= 10 \end{aligned}\right\} ,$$

the error due to cut-off integration (R in (9)) is less than 0.0001, which has already been estimated in the previous paper.[7]

When one uses $z=5$ as the critical value between the ascending and asymptotic expressions of the Bessel function $J_\nu(z)(\nu=0, 1)$, accuracy of the five numerical digits is expected (then $A_n(z)$ and $B_n(z)$ in the asymptotic expression of $J_\nu(z)$ are retained up to five terms). Accordingly, in the integration (9), when $k_1^*=5/r^*$(viz. $k_1^* r^*=5$), an accuracy of about five numerical digits with respect to the Bessel functions is expected.

Then the integration (9) is rewritten as follows

$$\int_0^\infty = \int_0^{[5/r^*]} + \int_{[5/r^*]}^{10} \tag{9'}$$

where $[5/r^*]$ is the nearest m_j value of $5/r^*$.

Here, if one uses the values of the coefficients a_j, b_j in Table 3, divides each interval $m_j < k^* \leqq m_{j+1}$ of the second integration and the interval $(0, [5/r^*])$ into even numbers for numerical calculation, the integration (9′), in effect, can be made. And the determination of the divided numbers is carried out quite experimentally. The divided numbers of the second integration are shown in Table 3 and the interval $(0, [5/r^*])$ of the first integration is divided such that the divided interval $\varDelta k^*$ becomes about 0.01.

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7) T. MOMOI, *loc. cit.*, 1)

Table 4. The variation of the tsunami height versus time.

r^*	$\zeta_R(t^*=12)$	$\zeta_R(t^*=13)$	$\zeta_R(t^*=14)$	$\zeta_R(t^*=15)$
25.0	0.14588×10^{-1}	0.30791×10^{-1}	0.59557×10^{-1}	0.10362×10^{0}
27.5	0.18798×10^{-2}	0.46289×10^{-2}	0.10532×10^{-1}	0.22268×10^{-1}
30.0	0.22784×10^{-3}	0.64259×10^{-3}	0.13292×10^{-2}	0.33360×10^{-2}
32.5	0.24327×10^{-4}	0.10472×10^{-3}	0.13938×10^{-3}	0.37451×10^{-3}
35.0	-0.13803×10^{-4}	-0.29888×10^{-4}	0.56476×10^{-4}	0.78493×10^{-4}
37.5	-0.74928×10^{-5}	-0.45291×10^{-4}	0.40584×10^{-4}	0.55958×10^{-4}
40.0	0.12842×10^{-4}	0.48511×10^{-5}	0.49164×10^{-5}	0.14964×10^{-4}
42.5	0.21505×10^{-4}	0.41297×10^{-4}	-0.12534×10^{-4}	-0.14657×10^{-4}
45.0	0.14722×10^{-4}	0.30494×10^{-4}	0.43181×10^{-6}	-0.58924×10^{-5}
47.5	0.63811×10^{-5}	0.28836×10^{-5}	0.19801×10^{-4}	0.18241×10^{-4}
50.0	0.69863×10^{-5}	-0.39426×10^{-5}	0.23614×10^{-4}	0.26695×10^{-4}

r^*	$\zeta_R(t^*=16)$	$\zeta_R(t^*=17)$	$\zeta_R(\zeta^*=18)$	$\zeta_R(t^*=19)$
25.0	0.16237×10^{0}	0.22933×10^{0}	0.28961×10^{0}	0.32315×10^{0}
27.5	0.42886×10^{-1}	0.76014×10^{-1}	0.12312×10^{0}	0.18116×10^{0}
30.0	0.76927×10^{-2}	0.16249×10^{-1}	0.31463×10^{-1}	0.56227×10^{-1}
32.5	0.10703×10^{-2}	0.26457×10^{-2}	0.57270×10^{-2}	0.11809×10^{-1}
35.0	0.14967×10^{-3}	0.41449×10^{-3}	0.81008×10^{-3}	0.18333×10^{-2}
37.5	0.45737×10^{-4}	0.10012×10^{-3}	0.86799×10^{-4}	0.19934×10^{-3}
40.0	0.28999×10^{-4}	0.36687×10^{-4}	0.34027×10^{-5}	0.11905×10^{-4}
42.5	0.18826×10^{-4}	0.80804×10^{-5}	0.18769×10^{-5}	0.15673×10^{-4}
45.0	0.11325×10^{-4}	-0.56505×10^{-5}	0.87579×10^{-5}	0.29820×10^{-4}
47.5	0.78720×10^{-5}	-0.72813×10^{-5}	0.14335×10^{-4}	0.33977×10^{-4}
50.0	0.82394×10^{-5}	-0.97261×10^{-6}	0.17314×10^{-4}	0.30098×10^{-4}

r^*	$\zeta_R(t^*=20)$	$\zeta_R(t^*=21)$	$\zeta_R(t^*=22)$	$\zeta_R(t^*=23)$
25.0	0.31516×10^{0}	0.26884×10^{0}	0.20463×10^{0}	0.15354×10^{0}
27.5	0.24103×10^{0}	0.28966×10^{0}	0.30937×10^{0}	0.29071×10^{0}
30.0	0.92810×10^{-1}	0.14180×10^{0}	0.19730×10^{0}	0.24946×10^{0}
32.5	0.23110×10^{-1}	0.42010×10^{-1}	0.70482×10^{-1}	0.10977×10^{0}
35.0	0.43669×10^{-2}	0.87726×10^{-2}	0.17010×10^{-1}	0.31291×10^{-1}
37.5	0.77296×10^{-3}	0.13236×10^{-2}	0.29986×10^{-2}	0.66494×10^{-2}
40.0	0.16633×10^{-3}	0.96037×10^{-4}	0.38119×10^{-3}	0.11910×10^{-2}
42.5	0.14621×10^{-4}	0.73590×10^{-6}	0.72287×10^{-4}	0.20112×10^{-3}
45.0	-0.42181×10^{-4}	0.48563×10^{-4}	0.85448×10^{-4}	0.15474×10^{-4}
47.5	-0.41465×10^{-4}	0.68805×10^{-4}	0.84034×10^{-4}	-0.21311×10^{-4}
50.0	-0.84466×10^{-5}	0.50096×10^{-4}	0.47401×10^{-4}	-0.11212×10^{-4}

(to be continued)

(continued)

r^*	$\zeta_R(t^*=24)$	$\zeta_R(t^*=25)$	$\zeta_R(t^*=26)$	$\zeta_R(t^*=27)$
25.0	0.13762×10^{0}	0.15204×10^{0}	0.16429×10^{0}	0.14465×10^{0}
27.5	0.24033×10^{0}	0.17993×10^{0}	0.13620×10^{0}	0.12388×10^{0}
30.0	0.28561×10^{0}	0.29298×10^{0}	0.26675×10^{0}	0.21419×10^{0}
32.5	0.15844×10^{0}	0.21014×10^{0}	0.25469×10^{0}	0.27941×10^{0}
35.0	0.53573×10^{1}	0.85217×10^{-1}	0.12621×10^{0}	0.17370×10^{0}
37.5	0.12992×10^{-1}	0.23539×10^{-1}	0.40399×10^{-1}	0.65881×10^{-1}
40.0	0.25127×10^{-2}	0.49091×10^{-2}	0.94549×10^{-2}	0.17757×10^{-1}
42.5	0.38408×10^{-3}	0.81275×10^{-3}	0.17907×10^{-2}	0.37403×10^{-2}
45.0	-0.81444×10^{-5}	0.12069×10^{-3}	0.36574×10^{-3}	0.66410×10^{-3}
47.5	-0.56615×10^{-4}	0.41059×10^{-4}	0.13756×10^{-3}	0.10705×10^{-3}
50.0	-0.19931×10^{-4}	0.31007×10^{-4}	0.64680×10^{-4}	0.18876×10^{-4}

r^*	$\zeta_R(t^*=28)$	$\zeta_R(t^*=29)$	$\zeta_R(t^*=30)$	$\zeta_R(t^*=31)$
25.0	0.92731×10^{-1}	0.39676×10^{-1}	0.12669×10^{-1}	0.44623×10^{-2}
27.5	0.13637×10^{0}	0.14713×10^{0}	0.12890×10^{0}	0.80467×10^{-1}
30.0	0.15727×10^{0}	0.11981×10^{0}	0.11189×10^{0}	0.12466×10^{0}
32.5	0.27549×10^{0}	0.24204×10^{0}	0.18922×10^{0}	0.13735×10^{0}
35.0	0.22053×10^{0}	0.25620×10^{0}	0.27030×10^{0}	0.25665×10^{0}
37.5	0.10031×10^{0}	0.14197×10^{0}	0.18698×10^{0}	0.22755×10^{0}
40.0	0.31086×10^{-1}	0.50940×10^{-1}	0.78972×10^{-1}	0.11483×10^{0}
42.5	0.72535×10^{-2}	0.13380×10^{-1}	0.23620×10^{-1}	0.39420×10^{-1}
45.0	0.13046×10^{-2}	0.27668×10^{-2}	0.54651×10^{-2}	0.10240×10^{-1}
47.5	0.14776×10^{-3}	0.48135×10^{-3}	0.10441×10^{-2}	0.21674×10^{-2}
50.0	0.34797×10^{-5}	0.92801×10^{-4}	0.18587×10^{-3}	0.39660×10^{-3}

r^*	$\zeta_R(t^*=32)$	$\zeta_R(t^*=33)$	$\zeta_R(t^*=34)$	$\zeta_R(t^*=35)$
25.0	-0.18956×10^{-1}	-0.72836×10^{-1}	-0.13495×10^{0}	-0.17549×10^{0}
27.5	0.25495×10^{-1}	-0.73371×10^{-2}	-0.18060×10^{-1}	-0.35584×10^{-1}
30.0	0.13213×10^{0}	0.11328×10^{0}	0.66475×10^{-1}	0.12137×10^{-1}
32.5	0.10570×10^{0}	0.10184×10^{0}	0.11417×10^{0}	0.11923×10^{0}
35.0	0.21791×10^{0}	0.16600×10^{0}	0.11923×10^{0}	0.93866×10^{-1}
37.5	0.25455×10^{0}	0.25874×10^{0}	0.23679×10^{0}	0.19431×10^{0}
40.0	0.15640×10^{0}	0.19793×10^{0}	0.23174×10^{0}	0.24962×10^{0}
42.5	0.62146×10^{-1}	0.92326×10^{-1}	0.12906×10^{0}	0.16906×10^{0}
45.0	0.18021×10^{-1}	0.30403×10^{-1}	0.48790×10^{-1}	0.73908×10^{-1}
47.5	0.40831×10^{-2}	0.77096×10^{-2}	0.13916×10^{-1}	0.23578×10^{-1}
50.0	0.76143×10^{-3}	0.16099×10^{-2}	0.32118×10^{-2}	0.59186×10^{-2}

(to be continued)

(continued)

r^*	$\zeta_R(t^*=36)$	$\zeta_R(t^*=37)$	$\zeta_R(t^*=38)$	$\zeta_R(t^*=39)$
25.0	-0.18859×10^{0}	-0.19977×10^{0}	-0.21257×10^{0}	-0.20324×10^{0}
27.5	-0.79969×10^{-1}	-0.14089×10^{0}	-0.18452×10^{0}	-0.19577×10^{0}
30.0	-0.24270×10^{-1}	-0.38540×10^{-1}	-0.51937×10^{-1}	-0.88160×10^{-1}
32.5	0.97948×10^{-1}	0.51121×10^{-1}	-0.19569×10^{-2}	-0.39190×10^{-1}
35.0	0.93103×10^{-1}	0.10467×10^{0}	0.10652×10^{0}	0.82825×10^{-1}
37.5	0.14424×10^{0}	0.10334×10^{0}	0.83534×10^{-1}	0.85390×10^{-1}
40.0	0.24484×10^{0}	0.21652×10^{0}	0.17163×10^{0}	0.12412×10^{0}
42.5	0.20632×10^{0}	0.23310×10^{0}	0.24207×10^{0}	0.22905×10^{0}
45.0	0.10574×10^{0}	0.14241×10^{0}	0.17999×10^{0}	0.21212×10^{0}
47.5	0.38182×10^{-1}	0.58816×10^{-1}	0.86078×10^{-1}	0.11895×10^{0}
50.0	0.10607×10^{-1}	0.18234×10^{-1}	0.29981×10^{-1}	0.46864×10^{-1}

r^*	$\zeta_R(t^*=40)$	$\zeta_R(t^*=41)$	$\zeta_R(t^*=42)$	$\zeta_R(t^*=43)$
25.0	-0.15547×10^{0}	-0.90434×10^{-1}	-0.39792×10^{-1}	-0.11736×10^{-1}
27.5	-0.19041×10^{0}	-0.18691×10^{0}	-0.17569×10^{0}	-0.13594×10^{0}
30.0	-0.14360×10^{0}	-0.18896×10^{0}	-0.19981×10^{0}	-0.18344×10^{0}
32.5	-0.55328×10^{-1}	-0.67943×10^{-1}	-0.97711×10^{-1}	-0.14589×10^{0}
35.0	0.35852×10^{-1}	-0.16428×10^{-1}	-0.53399×10^{-1}	-0.70381×10^{-1}
37.5	0.95783×10^{-1}	0.94384×10^{-1}	0.67890×10^{-1}	0.19935×10^{-1}
40.0	0.89006×10^{-1}	0.75021×10^{-1}	0.78988×10^{-1}	0.87137×10^{-1}
42.5	0.19550×10^{0}	0.15019×10^{0}	0.10625×10^{0}	0.76780×10^{-1}
45.0	0.23140×10^{0}	0.23207×10^{0}	0.21197×10^{0}	0.17479×10^{0}
47.5	0.15471×10^{0}	0.18879×10^{0}	0.21511×10^{0}	0.22708×10^{0}
50.0	0.69665×10^{-1}	0.98282×10^{-1}	0.13137×10^{0}	0.16550×10^{0}

r^*	$\zeta_R(t^*=44)$	$\zeta_R(t^*=45)$	$\zeta_R(t^*=46)$	$\zeta_R(t^*=47)$
25.0	0.23825×10^{-2}	-0.23028×10^{-2}	-0.33549×10^{-1}	-0.71937×10^{-1}
27.5	-0.71936×10^{-1}	-0.16841×10^{-1}	0.44505×10^{-2}	-0.98114×10^{-3}
30.0	-0.16326×10^{0}	-0.14536×10^{0}	-0.11311×10^{0}	-0.58481×10^{-1}
32.5	-0.18827×10^{0}	-0.19876×10^{0}	-0.17628×10^{0}	-0.14301×10^{0}
35.0	-0.81710×10^{-1}	-0.10652×10^{0}	-0.14763×10^{0}	-0.18500×10^{0}
37.5	-0.31094×10^{-1}	-0.66706×10^{-1}	-0.83103×10^{-1}	-0.93325×10^{-1}
40.0	0.81870×10^{-1}	0.52759×10^{-1}	0.45192×10^{-2}	-0.45326×10^{-1}
42.5	0.67513×10^{-1}	0.73052×10^{-1}	0.78918×10^{-1}	0.69409×10^{-1}
45.0	0.12997×10^{0}	0.90242×10^{-1}	0.66340×10^{-1}	0.61198×10^{-1}
47.5	0.22003×10^{0}	0.19389×10^{0}	0.15423×10^{0}	0.11121×10^{0}
50.0	0.19552×10^{0}	0.21547×10^{0}	0.21998×10^{0}	0.20610×10^{0}

(to be continued)

(continued)

r^*	$\zeta_R(t^*=48)$	$\zeta_R(t^*=49)$	$\zeta_R(t^*=50)$	$\zeta_R(t^*=51)$
25.0	-0.84701×10^{-1}	-0.65586×10^{-1}	-0.34616×10^{-1}	-0.79844×10^{-2}
27.5	-0.15633×10^{-1}	-0.37325×10^{-1}	-0.65836×10^{-1}	-0.81378×10^{-1}
30.0	-0.43525×10^{-2}	0.16932×10^{-1}	0.92556×10^{-3}	-0.27675×10^{-1}
32.5	-0.11575×10^{0}	-0.87780×10^{-1}	-0.46039×10^{-1}	0.26658×10^{-3}
35.0	-0.19322×10^{0}	-0.16674×10^{0}	-0.12482×10^{0}	-0.89509×10^{-1}
37.5	-0.11415×10^{0}	-0.14852×10^{0}	-0.17961×10^{0}	-0.18431×10^{0}
40.0	-0.79424×10^{-1}	-0.94345×10^{-1}	-0.10254×10^{0}	-0.11969×10^{0}
42.5	0.37239×10^{-1}	-0.11100×10^{-1}	-0.58906×10^{-1}	-0.90766×10^{-1}
45.0	0.67394×10^{-1}	0.70429×10^{-1}	0.56694×10^{-1}	0.21707×10^{-1}
47.5	0.76274×10^{-1}	0.57593×10^{-1}	0.55642×10^{-1}	0.61716×10^{-1}
50.0	0.17536×10^{0}	0.13449×10^{0}	0.93948×10^{-1}	0.64017×10^{-1}

r^*	$\zeta_R(t^*=52)$	$\zeta_R(t^*=53)$	$\zeta_R(t^*=54)$	$\zeta_R(t^*=55)$
25.0	0.46524×10^{-2}	-0.70335×10^{-2}	-0.35717×10^{-1}	-0.53398×10^{-1}
27.5	-0.63465×10^{-1}	-0.24737×10^{-1}	0.18539×10^{-2}	0.20050×10^{-2}
30.0	-0.48491×10^{-1}	-0.61971×10^{-1}	-0.69222×10^{-1}	-0.57849×10^{-1}
32.5	0.22012×10^{-1}	0.41940×10^{-2}	-0.33810×10^{-1}	-0.60195×10^{-1}
35.0	-0.63036×10^{-1}	-0.33080×10^{-1}	0.14065×10^{-2}	0.20144×10^{-1}
37.5	-0.15497×10^{0}	-0.10764×10^{0}	-0.66479×10^{-1}	-0.40674×10^{-1}
40.0	-0.14830×10^{0}	-0.17302×10^{0}	-0.17296×10^{0}	-0.14083×10^{0}
42.5	-0.10398×10^{0}	-0.11002×10^{0}	-0.12325×10^{0}	-0.14621×10^{0}
45.0	-0.26520×10^{-1}	-0.71933×10^{-1}	-0.10090×10^{0}	-0.11177×10^{0}
47.5	0.61650×10^{-1}	0.43732×10^{-1}	0.60849×10^{-2}	-0.41674×10^{-1}
50.0	0.50358×10^{-1}	0.50871×10^{-1}	0.55845×10^{-1}	0.52175×10^{-1}

r^*	$\zeta_R(t^*=56)$	$\zeta_R(t^*=57)$	$\zeta_R(t^*=58)$	$\zeta_R(t^*=59)$
25.0	-0.45062×10^{-1}	-0.21902×10^{-1}	-0.58148×10^{-3}	0.30523×10^{-2}
27.5	-0.12561×10^{-1}	-0.32808×10^{-1}	-0.49014×10^{-1}	-0.45263×10^{-1}
30.0	-0.24002×10^{-1}	0.75228×10^{-2}	0.85683×10^{-2}	-0.16398×10^{-1}
32.5	-0.64349×10^{-1}	-0.57022×10^{-1}	-0.44423×10^{-1}	-0.21935×10^{-1}
35.0	0.44383×10^{-2}	-0.35538×10^{-1}	-0.67064×10^{-1}	-0.68989×10^{-1}
37.5	-0.20830×10^{-1}	0.93643×10^{-3}	0.14135×10^{-1}	0.11390×10^{-2}
40.0	-0.90825×10^{-1}	-0.46828×10^{-1}	-0.21799×10^{-1}	-0.99287×10^{-2}
42.5	-0.16496×10^{0}	-0.16029×10^{0}	-0.12545×10^{0}	-0.74021×10^{-1}
45.0	-0.11542×10^{0}	-0.12497×10^{0}	-0.14258×10^{0}	-0.15550×10^{0}
47.5	-0.84086×10^{-1}	-0.10955×10^{0}	-0.11773×10^{0}	-0.11892×10^{0}
50.0	0.30386×10^{-1}	-0.92872×10^{-2}	-0.56192×10^{-1}	-0.95351×10^{-1}

(to be continued)

(continued)

r^*	$\zeta_R(t^*=60)$	$\zeta_R(t^*=61)$	$\zeta_R(t^*=62)$	$\zeta_R(t^*=63)$
25.0	-0.16002×10^{-1}	-0.37947×10^{-1}	-0.37250×10^{-1}	-0.17646×10^{-1}
27.5	-0.19383×10^{-1}	0.35622×10^{-2}	0.21023×10^{-2}	-0.16282×10^{-1}
30.0	-0.39243×10^{-1}	-0.44780×10^{-1}	-0.36869×10^{-1}	-0.19562×10^{-1}
32.5	0.42230×10^{-2}	0.10968×10^{-1}	-0.12206×10^{-1}	-0.42367×10^{-1}
35.0	-0.49908×10^{-1}	-0.29509×10^{-1}	-0.13905×10^{-1}	0.10040×10^{-4}
37.5	-0.35888×10^{-1}	-0.69084×10^{-1}	-0.71820×10^{-1}	-0.46353×10^{-1}
40.0	-0.15616×10^{-3}	0.57588×10^{-2}	-0.52293×10^{-2}	-0.36209×10^{-1}
42.5	-0.29596×10^{-1}	-0.67626×10^{-2}	-0.17045×10^{-2}	-0.18538×10^{-2}
45.0	-0.14636×10^{0}	-0.10942×10^{0}	-0.57859×10^{-1}	-0.14676×10^{-1}
47.5	-0.12484×10^{0}	-0.13730×10^{0}	-0.14480×10^{0}	-0.13172×10^{0}
50.0	-0.11681×10^{0}	-0.12170×10^{0}	-0.12030×10^{0}	-0.12300×10^{0}

r^*	$\zeta_R(t^*=64)$	$\zeta_R(t^*=65)$	$\zeta_R(t^*=66)$	$\zeta_R(t^*=67)$
25.0	0.27387×10^{-3}	0.92050×10^{-3}	-0.15972×10^{-1}	-0.30948×10^{-1}
27.5	-0.32895×10^{-1}	-0.35837×10^{-1}	-0.21239×10^{-1}	0.11214×10^{-2}
30.0	0.75589×10^{-3}	0.59101×10^{-2}	-0.12960×10^{-1}	-0.34932×10^{-1}
32.5	-0.49295×10^{-1}	-0.31463×10^{-1}	-0.10676×10^{-1}	-0.42426×10^{-5}
35.0	0.54374×10^{-2}	-0.96946×10^{-2}	-0.37887×10^{-1}	-0.51012×10^{-1}
37.5	-0.17607×10^{-1}	-0.33014×10^{-2}	-0.72014×10^{-3}	-0.28689×10^{-2}
40.0	-0.67218×10^{-1}	-0.71446×10^{-1}	-0.44373×10^{-1}	-0.96396×10^{-2}
42.5	-0.30246×10^{-2}	-0.13203×10^{-1}	-0.37483×10^{-1}	-0.63236×10^{-1}
45.0	0.49681×10^{-2}	0.37850×10^{-2}	-0.46333×10^{-2}	-0.11790×10^{-1}
47.5	-0.93282×10^{-1}	-0.42521×10^{-1}	-0.19298×10^{-2}	0.13537×10^{-1}
50.0	-0.13075×10^{0}	-0.13308×10^{0}	-0.11647×10^{0}	-0.77393×10^{-1}

Following the above-mentioned procedure, the wave heights ζ_R versus time parameter t^* for the positions r^* in the range from 25 to 50 are computed. The values of the computed wave heights are tabulated in Table 4, of which the figures are shown in Fig. 2. To check the accuracy of the integration, the doubled number of the divisions is used in integration for some particular examples. According to the results (refer to Table 5), the integrations seem to have, in accuracy, about four digits below the point.

Looking through all the figures in Fig. 2, one of the most outstanding features is an appearance of small valleys on the largest first crests which have already been found, on a small scale, in the figures of the preceding works[8),9)]. And such saddle-shaped valleys are often found in the records

8) T. MOMOI, *loc. cit.*, 1)
9) T. MOMOI, *loc. cit.*, 2)

Table 5. The tsunami heights calculated with a doubled number of divisions of integration.

t^*	$\zeta_R(\gamma^*=25)$	$\zeta_R(\gamma^*=50)$	t^*	$\zeta_R(\gamma^*=25)$	$\zeta_R(\gamma^*=50)$
12	0.14584×10^{-1}	0.70198×10^{-5}	40	-0.15548×10^{0}	0.69639×10^{-1}
13	0.30791×10^{-1}	-0.38185×10^{-5}	41	-0.90409×10^{-1}	0.98255×10^{-1}
14	0.59553×10^{-1}	0.22854×10^{-4}	42	-0.39827×10^{-1}	0.13134×10^{0}
15	0.10362×10^{0}	0.25964×10^{-4}	43	-0.12045×10^{-1}	0.16557×10^{0}
16	0.16237×10^{0}	0.84568×10^{-5}	44	0.23706×10^{-2}	0.19549×10^{0}
17	0.22933×10^{0}	-0.93728×10^{-6}	45	-0.21790×10^{-2}	0.21544×10^{0}
18	0.28961×10^{0}	0.16538×10^{-4}	46	-0.33584×10^{-1}	0.21994×10^{0}
19	0.32315×10^{0}	0.30967×10^{-4}	47	-0.72038×10^{-1}	0.20607×10^{0}
20	0.31516×10^{0}	-0.86730×10^{-5}	48	-0.84711×10^{-1}	0.17532×10^{0}
21	0.26884×10^{0}	0.59738×10^{-4}	49	-0.65601×10^{-1}	0.13446×10^{0}
22	0.20463×10^{0}	0.48637×10^{-4}	50	-0.34647×10^{-1}	0.93913×10^{-1}
23	0.15354×10^{0}	-0.14983×10^{-4}	51	-0.80183×10^{-2}	0.63980×10^{-1}
24	0.13761×10^{0}	-0.20986×10^{-4}	52	0.46416×10^{-2}	0.50320×10^{-1}
25	0.15203×10^{0}	0.39856×10^{-4}	53	-0.70519×10^{-2}	0.50832×10^{-1}
26	0.16428×10^{0}	0.68397×10^{-4}	54	-0.35754×10^{-1}	0.55806×10^{-1}
27	0.14464×10^{0}	0.16483×10^{-4}	55	-0.53427×10^{-1}	0.52134×10^{-1}
28	0.92722×10^{-1}	0.37629×10^{-5}	56	-0.45082×10^{-1}	0.30344×10^{-1}
29	0.39666×10^{-1}	0.92084×10^{-4}	57	-0.22661×10^{-1}	-0.93309×10^{-2}
30	0.12659×10^{-1}	0.18587×10^{-3}	58	-0.62000×10^{-3}	-0.56236×10^{-1}
31	0.44516×10^{-2}	0.37665×10^{-3}	59	0.30307×10^{-2}	-0.95397×10^{-1}
32	-0.18967×10^{-1}	0.74123×10^{-3}	60	-0.16046×10^{-1}	-0.11686×10^{0}
33	-0.72848×10^{-1}	0.15890×10^{-2}	61	-0.38802×10^{-1}	-0.12175×10^{0}
34	-0.13497×10^{0}	0.31900×10^{-2}	62	-0.37286×10^{-1}	-0.12035×10^{0}
35	-0.17550×10^{0}	0.58967×10^{-2}	63	-0.17685×10^{-1}	-0.12305×10^{0}
36	-0.18860×10^{0}	0.10584×10^{-1}	64	0.22764×10^{-3}	-0.13080×10^{0}
37	-0.19979×10^{0}	0.18210×10^{-1}	65	0.88591×10^{-3}	-0.13313×10^{0}
38	-0.21259×10^{0}	0.29957×10^{-1}	66	-0.16015×10^{-1}	-0.11652×10^{0}
39	-0.20326×10^{0}	0.46838×10^{-1}	67	-0.30993×10^{-1}	-0.77450×10^{-1}

(mareograms) of the actual tsunamis. It is noteworthy that such saddle-shaped valleys, in the numerical experiments, take place in spite of the uniform elevation of a circular portion of the bottom. Therefore, it would be too hasty to consider from this fact that, when a saddle-shaped crest appears in the mareogram of the actual tsunami, the configuration of the bottom deformation would be of a saddle-like form. In the present study, the analysis is restricted only to the numerical one and theoretical explanation for the saddle-shaped valleys is submitted for the future article.

As shown in the latter part of Fig. 2, the waves of the later phase undulate mainly under the level of the still water, which has already been noted in the previous work.[9] The point of difference in the behavior of the previons study[9] is the revelation of the top of the second crest above the still water surface. As the waves leave the wave origin, the crests of the later phase are in a sense appearing gradually above the surface of the still water. Anyway, the long waves derived from "long wave approximation" are still dominant for the range of the present purview.

The inserted broken lines stand for the points of the waves progressing at the speed of the long wave ($\sqrt{gH}$).

As far as the fronts of the waves are concerned, they appear to be propagated with a long wave velocity for the waves in the range of $r^* = 25 \sim 50$. For one-dimentional case, Ichiye[10] has found, as a result of numerical analysis, that the velocity of the wave front is faster than that of the long wave. Such a phenomenon is considered due to a difference of the forms of the wave origins, that is to say, one is cylindrical (one-dimensional) and the other circular (two-dimensional). On the other hand, a train of the waves following the front advances at a speed of less than long wave velocity.

Comparing two amounts of an elevation and depression of the first crest (the largest crest in the figure), it is found that the former (an elevation) is a little larger than the latter (a depression). As already shown,[11] in the central part of the wave origin this phenomenon is reversed. Within the wave generating area, the larger depression of the water surface in the central part is considered to be caused by the waves moving in the direction of the center. But, in the range ($r^* = 25 \sim 50$) of the present study, the progressive waves are interpreted to be, in magnitude, dominant as compared with the retrogressive ones.

In this paper, the analysis is restricted to the numerical one and the numerically obtained results in the above will be dealt with by a theoretical method in the subsequent papers.

10) T. ICHIYE, *Jour. Ocean. Soc. Japan*, **14** (1958), 35.
11) T. MOMOI, *loc. cit.*, 1)

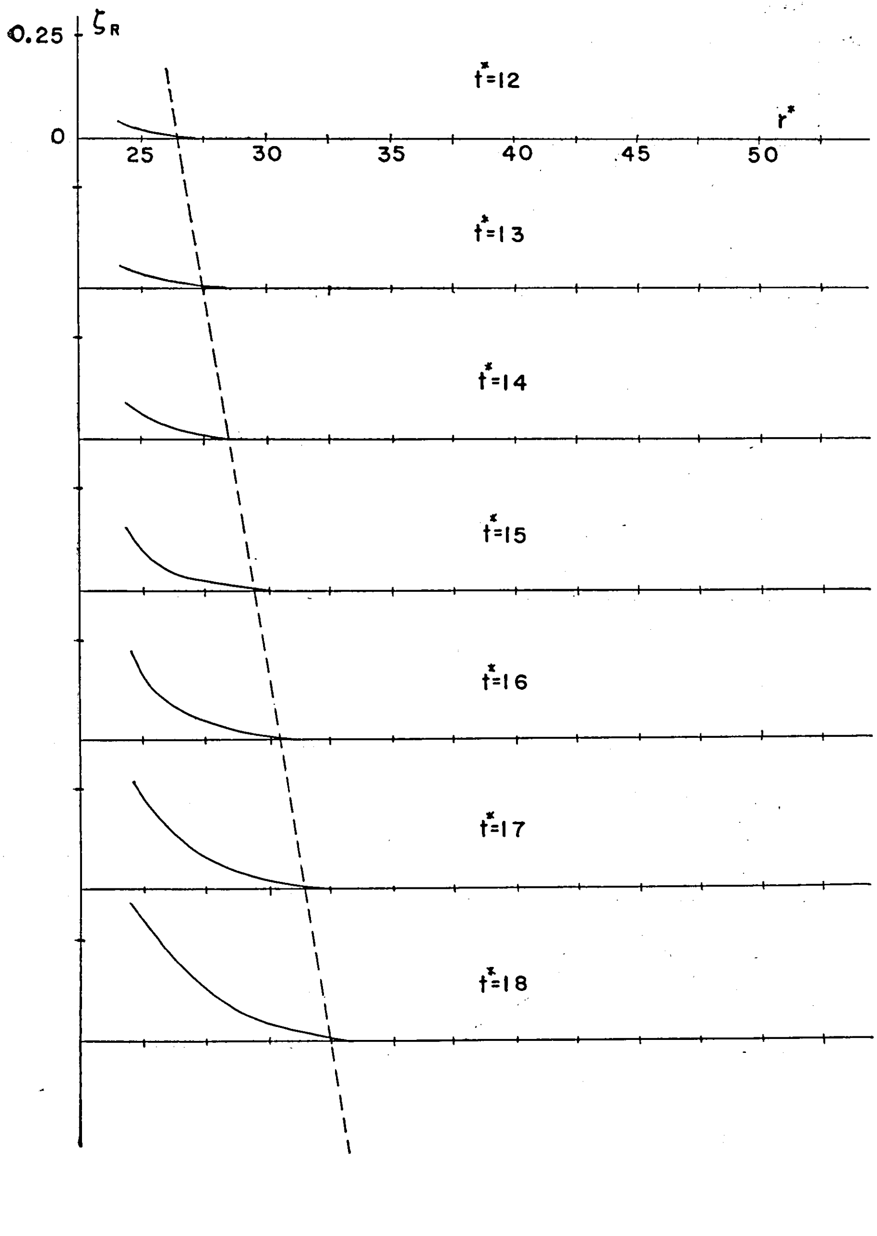

Fig. 2, a. Variation of the wave height.

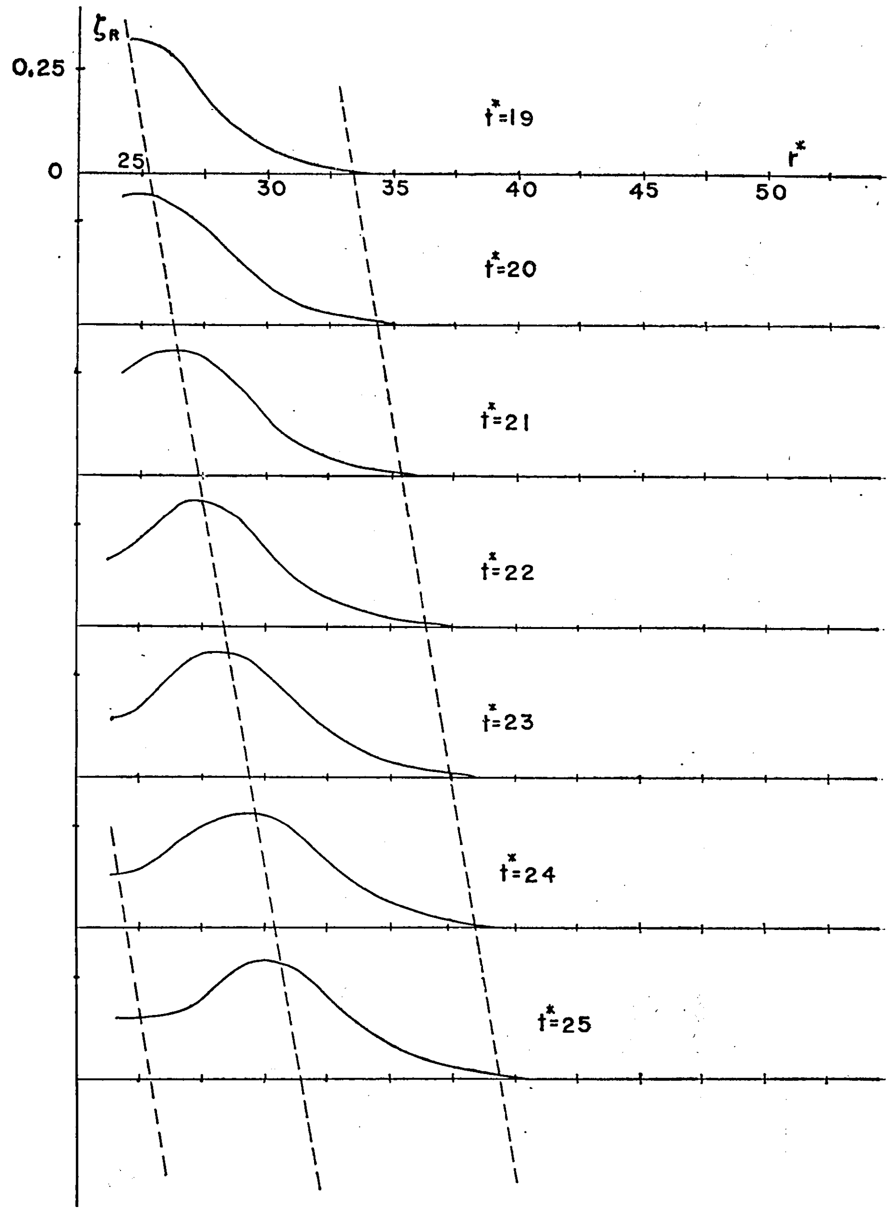

Fig. 2, b. Variation of the wave height.

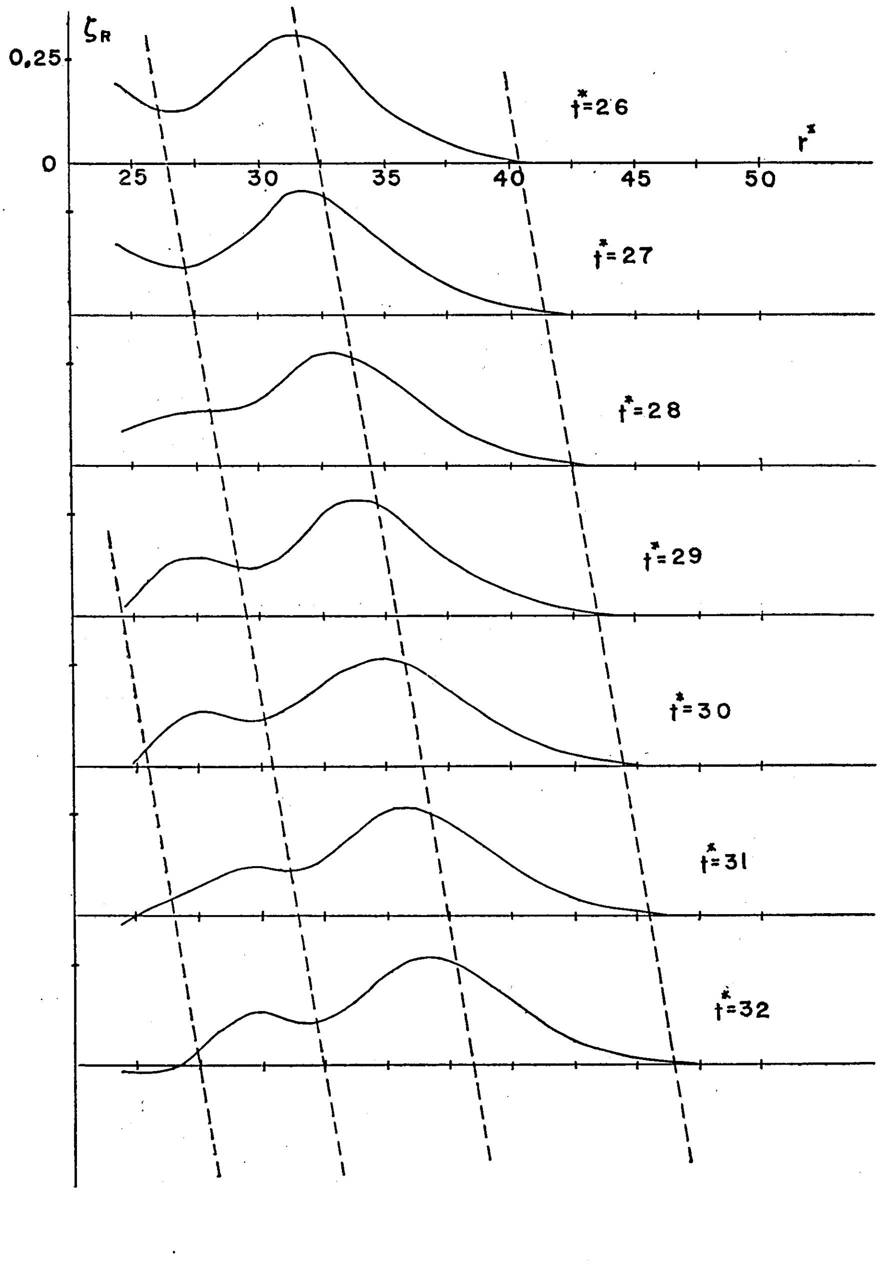

Fig. 2, c. Variation of the wave height.

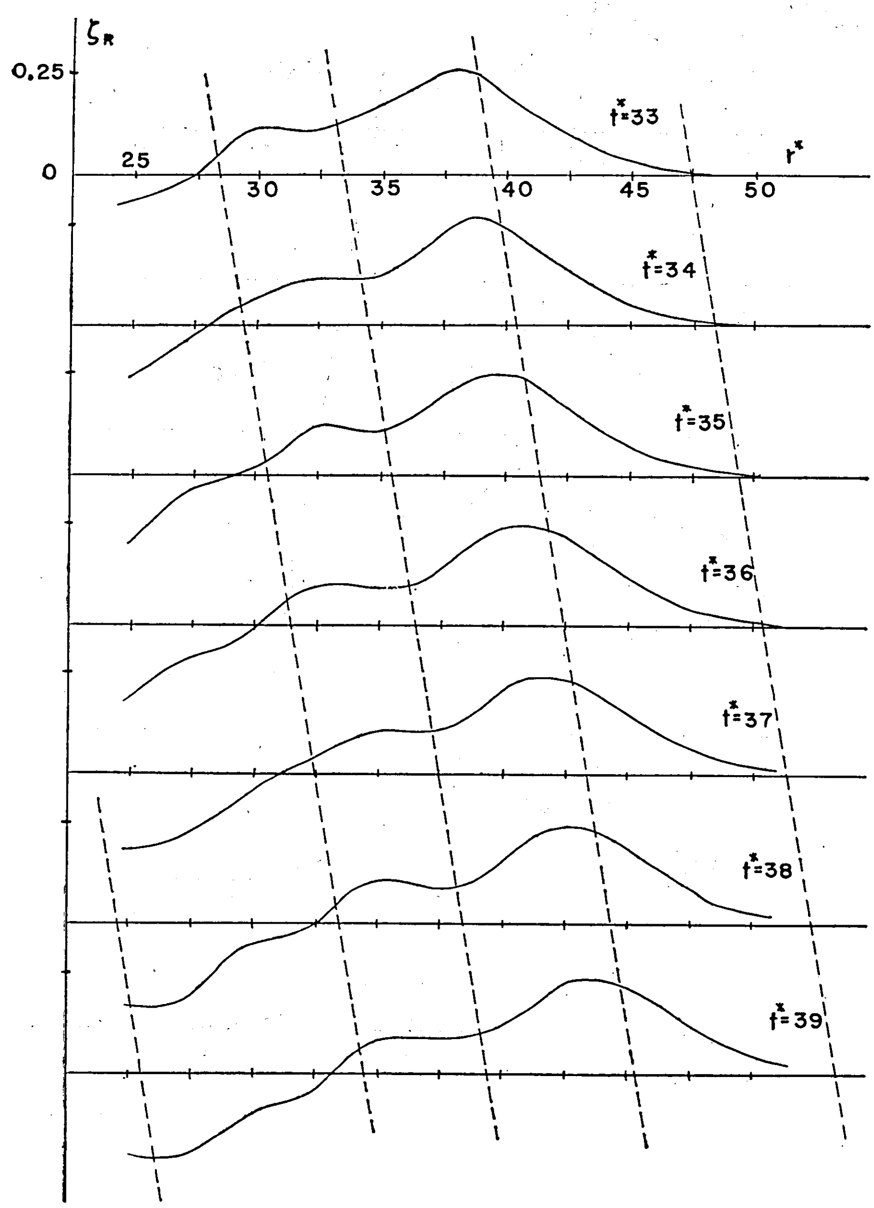

Fig. 2, d. Variation of the wave height.

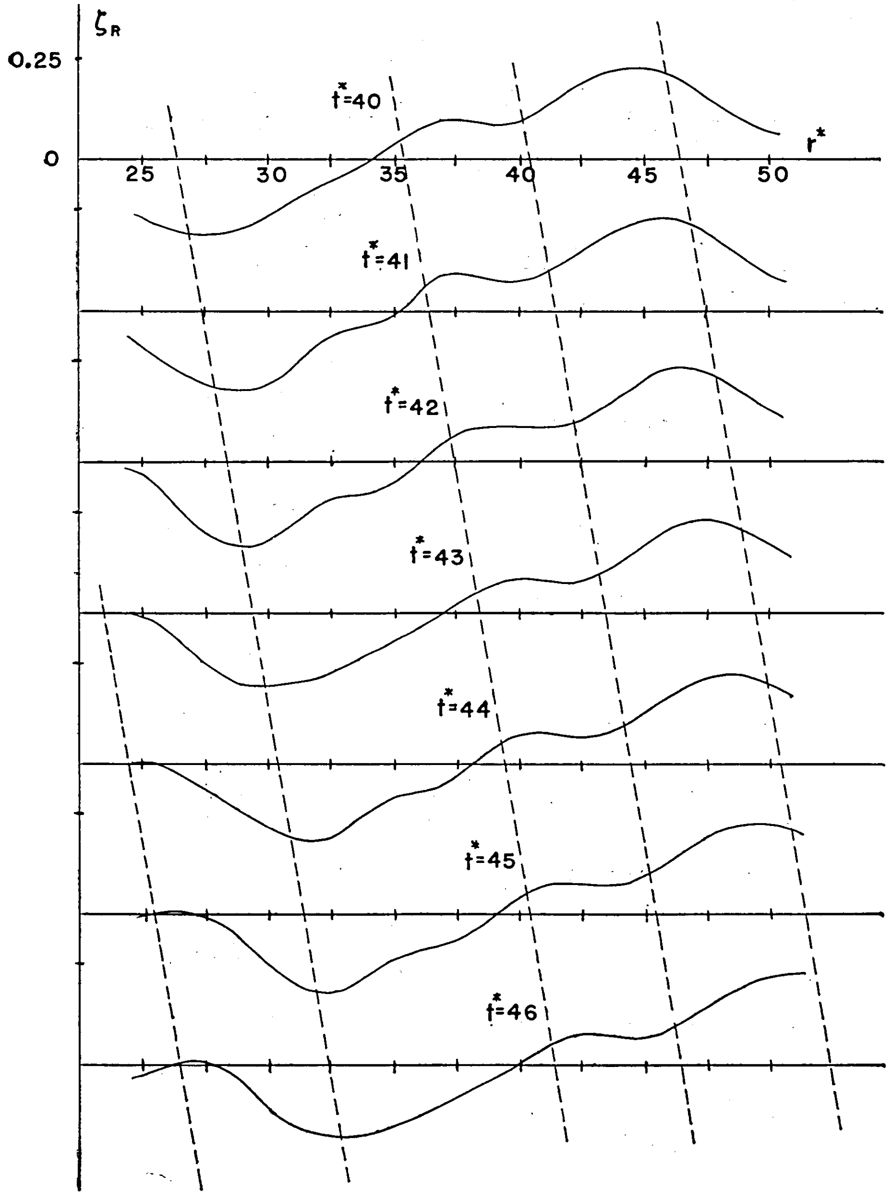

Fig. 2, e. Variation of the wave height.

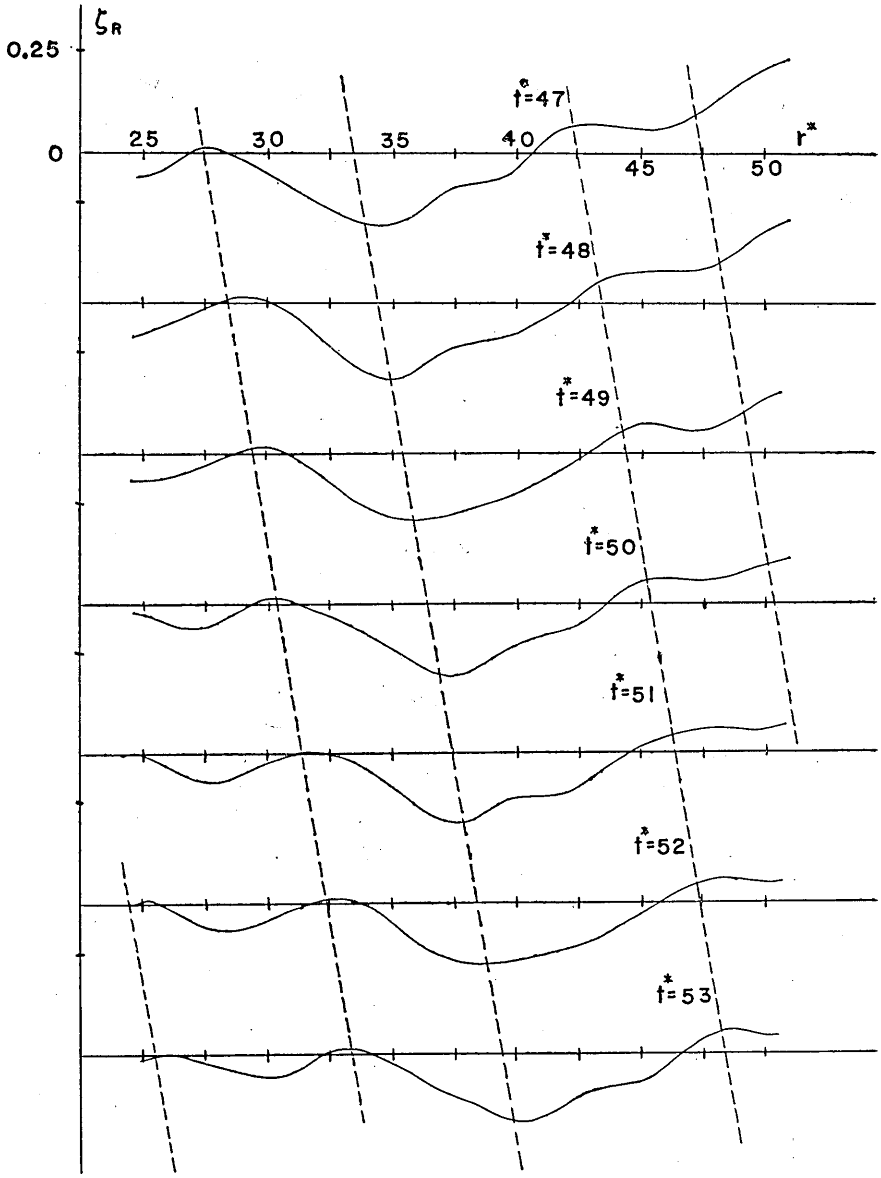

Fig. 2, f. Variation of the wave height.

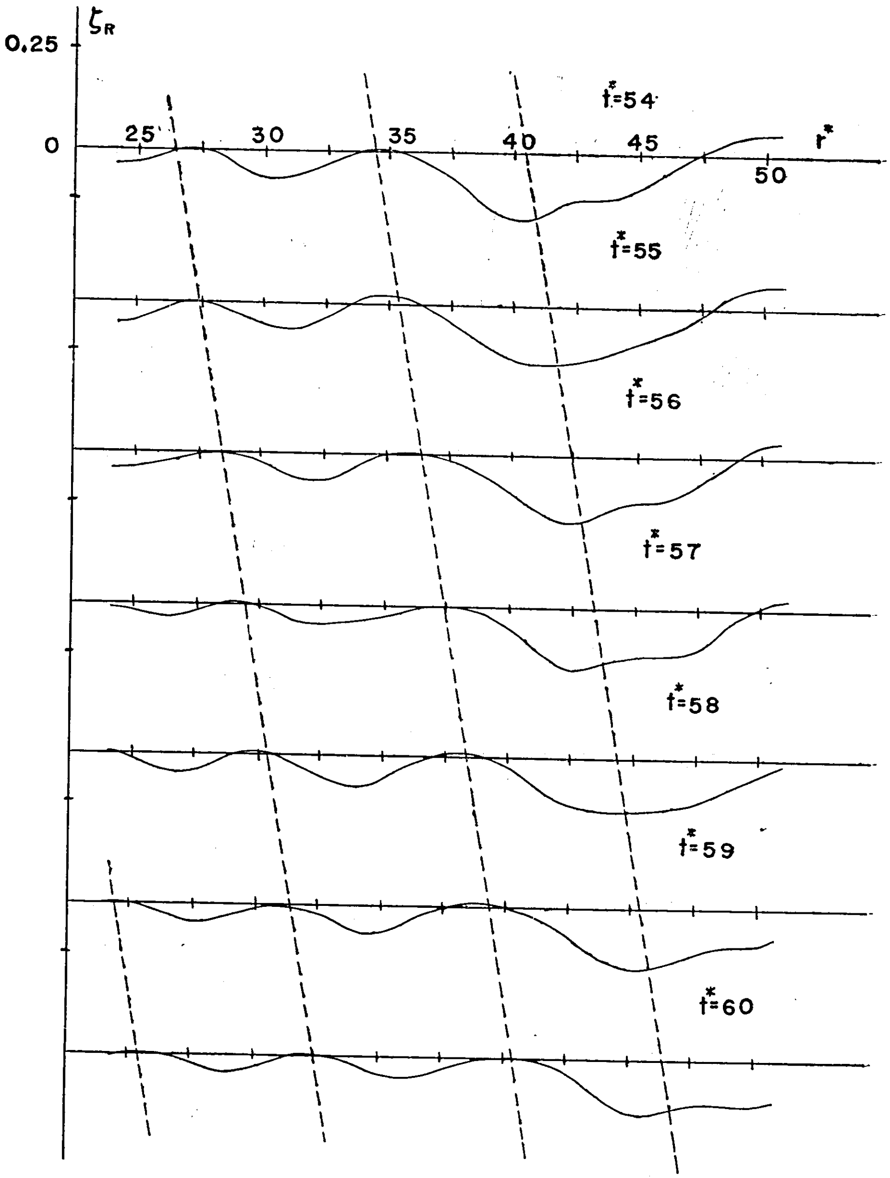

Fig. 2, g. Variation of the wave height.

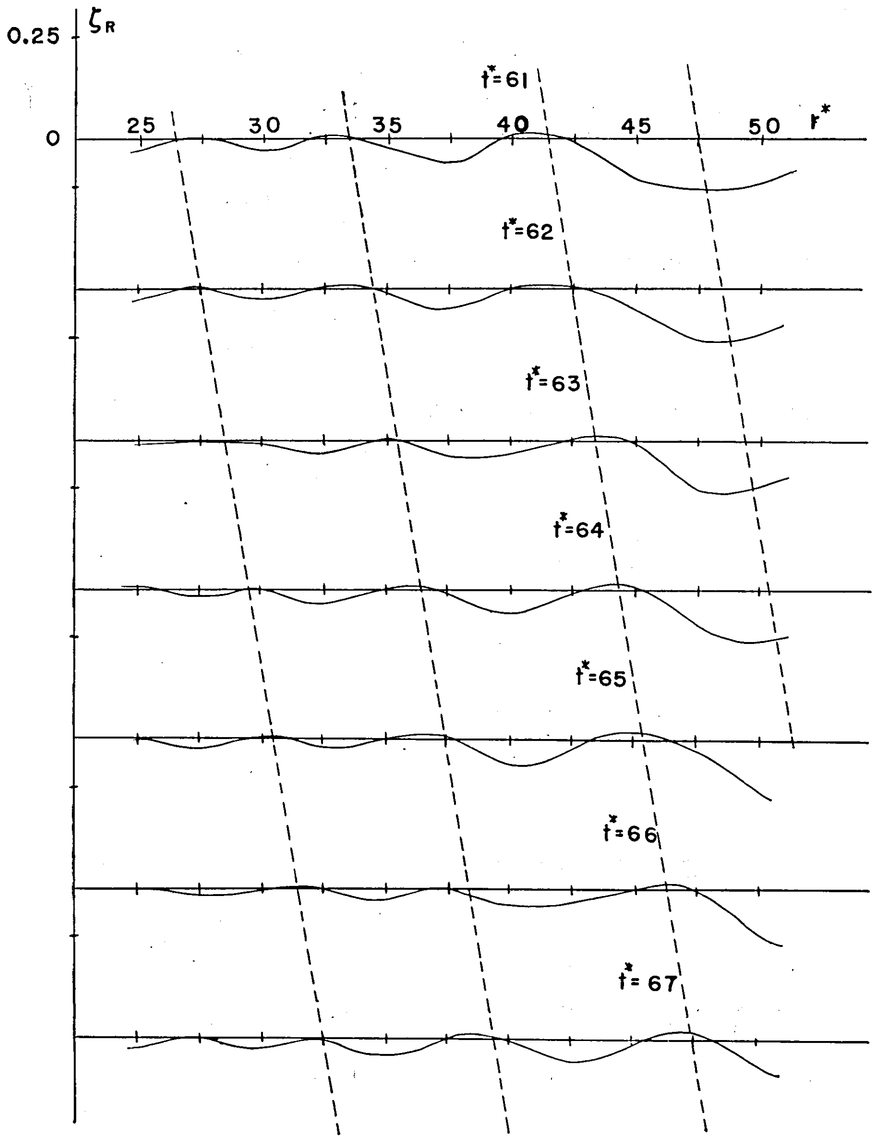

Fig. 2, h. Variation of the wave height.

5. 波源域における津波 [III]

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数値実験的に津波を取り扱うときにおこる最も困難な点は形式的な積分表示に含まれている振動因子 $\cos \omega^* t^*$, $J_0(k^* r^*)$ が t^* および r^* の増加につれて，数値積分の能率を著しく低下させることである．これを避けるために *Filon* の方法というのが存在する．筆者はこれを用いて，すでに報告ずみの第 1，第 2 報に続いて，波源域近傍の津波の解析を r^* が 25 から 50 の範囲について試みた．そして次の結論を得た．すなわち，

（1） 円形の水底の部分が一様に隆起したにもかかわらず，第一波の山の部分に鞍状の小さな谷が現われる．

（2） 津波の最先端部は長波の速度で進むように見える．第一波以下（第一波を含み）の波はすべて，長波の速度より遅い速度で進む．

（3） 第一波の上り（山の高さ）は下り（谷の深さ）より大きい．