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8. *A Motion of Water Excited by an Earthquake* [I].
——*The Case of a Rectangular Basin* (*one-dimensional*)——

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Abstract

It is well known that the water surface of a pond, reservior and lake is disturbed on the occasion of an earthquake. In the present purview, the problem of the water motion accompanied by an earthquake is treated theoretically. The used model is a one-dimensional rectangular basin and the generator of water disturbance is given only at the bottom in a form of periodic, propagated waves. Then the following conclusions are obtained:—

(1) When the period of the ground motion $\ll$ the period of the n-th mode of the eigen oscillations, i.e. $\omega_0 \gg \gamma_n$ (ω_0: the angular frequency of the ground motion; γ_n: the angular frequency of the n-th mode of the eigen oscillations), the generated water waves move with the period of the earthquake at the time of the duration of the ground motion and, after the termination of the earthquake, with a period of the eigen oscillation.

(2) When $\omega_0 \ll \gamma_n$, the water waves produced by the ground motion have, in the midst of the earthquake, two kinds of periods which are relevant to that of the earthquake and those of the eigen oscillations of the lake water, and, after the earthquake, the former disappears and the latter only remain.

(3) When $\omega_0 \simeq \gamma_n$, the wave height of the n-th mode of the waves is in proportion to a time duration of the earthquake.

(4) When $kl \to 0$ (k: the wave number of the earthquake wave; l: the length of the lake), the whole surface of the water oscillates uniformly with a period of an earthquake and no higher modes of the waves are generated.

(5) When $kl \to n\pi$, the modes of the waves with a difference of even numbers from the n-th one are not produced on the surface of water. At the beginning of the earthquake, the modes of the waves on the lower side of the resonant mode (the n-th one) are more excited by the ground motion than those on the upper side (n: positive integers). When the lake is very shallow, a similar phenomenon takes place, that is to say, the *nearby* modes of the

waves on the lower side of the resonant one are more excited by the earthquake than those on the upper side.

(6) When the ground motion lasts unlimitedly, even though the amplitude of the earthquake is so small, the produced water waves increase in height infinitely to the extent that the linear theory used in this paper cannot be applied. Then the period of the water waves is nearly equal to that of the earthquake.

In the subsequent articles, other cases will be considered.

1. Introduction

It is a generally accepted fact that on the occasion of a great earthquake the motion of the water in a pond, reservoir and lake is caused by movement of the ground accompanied by the earthquake. Among these motions of the water, an eigen oscillation of the lake is a very curious and ambiguous phenomenon. The most difficult point is that when a period of an oscillation of the lake differs considerably from that of the earthquake. Although it is known that a remarkable upheaval or subsidence of the earth-crust produces a tsunami in the open sea, the displacement of the ground on the occasion of an earthquake seems not to be large, in amount, enough to generate the water motion of the lake. In an attempt to explain such phenomena, the author takes up the present problem in this and the subsequent papers. Firstly, in this paper, the case of a rectangular basin (one-dimensional) is treated.

2. Theory

Let the fluid be ideal (no viscosity), incompressible and the motion irrotational. When the co-ordinate axes are taken in the conventional manner, with the (x, y) plane at the undisturbed surface of water (since we are treating the case of a one-dimensional phenomenon, the y-axis is perpendicular to the paper), the z-axis vertically upwards, and the wave of small wave height is assumed, the governing equation and boundary conditions are (refer to Fig. 1):

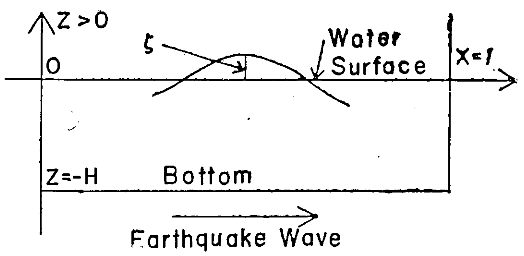

Fig. 1.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{1}$$

(ϕ is the velocity potential);

$$\zeta=\frac{1}{-g}\left(\frac{\partial\phi}{\partial t}+2\mu\phi\right)_{z=0} \tag{2}$$

(ζ is the wave height, g the acceleration of gravity, t a variable of time and μ the virtual viscosity);

$$\frac{\partial\zeta}{\partial t}=\left(\frac{\partial\phi}{\partial z}\right)_{z=0}; \tag{3}$$

$$\frac{\partial\phi}{\partial z}=\eta \tag{4}$$

(η is the velocity of the displacement of the bottom).

From (2) and (3), the relation

$$\frac{\partial^2\phi}{\partial t^2}+2\mu\frac{\partial\phi}{\partial t}+g\frac{\partial\phi}{\partial z}=0 \qquad (z=0) \tag{5}$$

is obtained.

Let D be the displacement of the bottom. There exists a relation between η and D, viz.

$$\eta=\frac{\partial D}{\partial t}\ (z=-H). \tag{6}$$

Substituting the Fourier transforms

$$\left.\begin{aligned}\phi&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi' e^{-i\omega t}d\omega\\ \eta&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\eta' e^{-i\omega t}d\omega\\ D&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}D' e^{-i\omega t}d\omega\end{aligned}\right\} \tag{7}$$

into (1), (4), (5) and (6), we have:

$$\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial z^2}=0, \tag{1'}$$

$$-(\omega^2+i\cdot 2\mu\omega)\phi'+g\frac{\partial\phi'}{\partial z}=0 \qquad (z=0), \tag{5'}$$

$$\frac{\partial\phi'}{\partial z}=-i\omega D' \qquad (z=-H). \tag{6'}$$

Boundary Condition: The boundary conditions for the case treated in this paper are

$$\frac{\partial \phi}{\partial x}=0 \qquad (x=0,\ l) \tag{8}$$

and

$$\left.\begin{aligned} D&=D_c e^{-i\omega_0 t+ikx} \qquad (0<t<t_0) \\ &=0 \qquad \text{(otherwise)} \end{aligned}\right\}, \tag{9}$$

where the only real part has physical meaning (D_c : real constant). Putting the first expression of (7) into (8), (8) becomes

$$\frac{\partial \phi'}{\partial x}=0 \qquad (x=0,\ l). \tag{8'}$$

From the last expression of (7), the inverse Fourier transform of D is

$$D'=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} D e^{+i\omega\tau} d\tau.$$

Using (9), the above expression becomes as follows

$$D'=\frac{1}{\sqrt{2\pi}}\int_{0}^{t_0} D_c e^{-i\omega_0 t+ikx} e^{+i\omega\tau} d\tau. \tag{10}$$

Mode Solution: Using the boundary condition (8′), the solution of (1′) is given by

$$\phi'=\sum_{n=0}^{\infty} \phi'_n \cos\frac{n\pi}{l}x, \tag{11}$$

$$\frac{1}{\phi'_n}\cdot\frac{d^2\phi'_n}{dz^2}=a_n^2=\left(\frac{n\pi}{l}\right)^2. \tag{12}$$

Then the boundary conditions (5′) and (6′) becomes

$$-(\omega^2+i\cdot 2\mu\omega)\phi'_n+g\frac{d\phi'_n}{dz}=0 \qquad (z=0), \tag{13}$$

$$\frac{d\phi'_n}{dz}=-i\omega D'_n \qquad (z=-H), \tag{14}$$

where

$$\left.\begin{aligned} D&=\sum_{n=0}^{\infty} D_n \cos\frac{n\pi}{l}x, \\ D_n&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} D'_n e^{-i\omega t} d\omega. \end{aligned}\right\} \tag{15}$$

When $n=0$, the solution of (12) is

$$\phi'_0=A_0z+B_0, \tag{16}$$

where A_0 and B_0 are arbitrary constants.

When $n\geqq 1$, the solution of (12) becomes

$$\phi'_n=A_ne^{+a_nz}+B_ne^{-a_nz}, \tag{17}$$

where A_n and B_n are arbitrary contants.

Using the boundary conditions (13) and (14), the arbitrary constants can be determined. Then the expressions (16) and (17) become as follows

$$\phi'_0=\left(z+\frac{g}{\omega^2+i\cdot 2\mu\omega}\right)(-i\omega D'_0), \tag{16'}$$

$$\phi'_n=\frac{(\omega^2+i\cdot 2\mu\omega)\sinh a_nz+ga_n\cosh a_nz}{\omega^2+i\cdot 2\mu\omega-ga_n\tanh a_nH}\cdot\frac{(-i\omega D'_n)}{a_n\cosh a_nH}$$
$$(n=1,2,3,\cdots). \tag{17'}$$

From the first expression of (7) and (11), we have :—

$$\phi=\sum_{n=0}^{\infty}\phi_n\cos\frac{n\pi}{l}x, \tag{18}$$

$$\phi_n=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi'_ne^{-i\omega t}d\omega \qquad (n=0,1,2,\cdots). \tag{19}$$

Substituting (18) and (19) into (2), the expression of the wave height is as given below :

$$\zeta=\sum_{n=0}^{\infty}\zeta_n\cos\frac{n\pi}{l}x, \tag{20}$$

$$\zeta_n=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{-g}(-i\omega+2\mu)(\phi'_n)_{z=0}e^{-i\omega t}d\omega. \tag{21}$$

Setting down z in (16') and (17') equal to zero, (16') and (17') are reduced to

$$(\phi'_0)_{z=0}=\frac{-i\cdot g}{\omega+i\cdot 2\mu}D'_0, \tag{22}$$

$$(\phi'_n)_{z=0}=\frac{-i\cdot g\omega}{\omega^2+i\cdot 2\mu\omega-ga_n\tanh a_nH}\cdot\frac{D'_n}{\cosh a_nH} \tag{23}$$
$$(n=1,2,3,\cdots).$$

From the last expression of (7) and (15), D'_n is given by

$$D'_n=\frac{1}{Q_n}\int_0^l D'\cos\frac{n\pi}{l}xdx \qquad (n=0, 1, 2, \cdots)\,, \tag{24}$$

where

$$Q_n=\int_0^l \cos^2\frac{n\pi}{l}xdx.$$

Putting (22) into (21), the zeroth mode of the wave height is

$$\zeta_0=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} D'_0 e^{-i\omega t}d\omega \qquad (\mu=0).$$

By use of (24) and the bottom condition (10), the above expression becomes

$$\zeta_0=D_\sigma\cdot\frac{i}{kl}\cdot\{-e^{-i\omega_0 t+ikl}+e^{-i\omega_0 t}\}\int_0^{t_0}\delta(\tau-t)d\tau. \tag{25}$$

Taking a real part of (25), the zeroth mode of the wave height is

$$\zeta_0=D_\sigma\cdot\frac{1}{kl}\cdot\{\sin\omega_0 t-\sin(\omega_0 t-kl)\}\int_0^{t_0}\delta(\tau-t)d\tau. \tag{25'}$$

Likewise, substituting (23) into (21), the higher modes of the wave height $(n=1, 2, 3, \cdots)$ are

$$\zeta_n=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\cosh a_n H}\cdot\int_{-\infty}^{\infty} D'_n e^{-i\omega t}d\omega$$
$$+\frac{1}{\sqrt{2\pi}}\cdot\frac{\gamma_n^2}{\cosh a_n H}\cdot\int_{-\infty}^{\infty}\frac{D'_n e^{-i\omega t}}{\omega^2+i\cdot 2\mu\omega-\gamma_n^2}d\omega, \tag{26}$$

where

$$\gamma_n=\sqrt{g a_n \tanh a_n H}. \tag{27}$$

Using (24) and the boundary condition (10), the first term of the right-hand side of (26) becomes :—

$$\text{the first term of (26)} \quad =D_\sigma\cdot\frac{T_n}{\cosh a_n H}\cdot e^{-i\omega_0 t}\cdot\int_0^{t_0}\delta(\tau-t)d\tau, \tag{28}$$

where

$$T_n=\frac{2kl}{(kl)^2-(n\pi)^2}\cdot[(-1)^n\sin kl+i\cdot\{1-(-1)^n\cos kl\}], \tag{29}$$

and the second term is reduced to the following :—

$$\text{the second term of (26)} \quad = D_\sigma \cdot \frac{T_n}{\cosh a_n H} \cdot \frac{\gamma_n^2}{2\pi} \cdot \int_0^{t_0} e^{-i\omega_0 \tau} V_n d\tau, \tag{30}$$

where

$$V_n = \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-\tau)}}{\omega^2 + i \cdot 2\mu\omega - \gamma_n^2} d\omega. \tag{31}$$

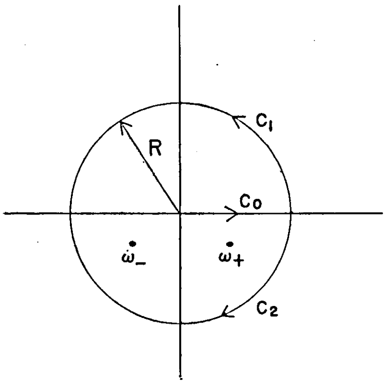

Fig. 2.

In order to calculate the integration (31), the complex plane is used. Referring to Fig. 2, there exist the next relations

$$\int_{\sigma_0} + \int_{\sigma_1} = 0 \tag{32}$$

and

$$\int_{\sigma_0} + \int_{\sigma_2} = 2\pi i \cdot \{\text{Res}\,(\omega_+) + \text{Res}\,(\omega_-)\}, \tag{33}$$

where ω_+ and ω_- are the roots of the denominator of the integrand of (31), i.e.

$$\omega_\pm = -i\mu \pm \sqrt{-\mu^2 + \gamma_n^2}. \tag{34}$$

When R tends to infinity,[1]

$$\int_{\sigma_1} \to 0 \qquad \text{for } t < \tau \tag{35}$$

and

$$\int_{\sigma_2} \to 0 \qquad \text{for } t > \tau. \tag{36}$$

From (31), (32) and (35),

$$V_n = 0 \qquad \text{for } t < \tau. \tag{37}$$

Since

$$\left.\begin{aligned} \text{Res}\,(\omega_+) &= \frac{e^{-i\omega_-(t-\tau)}}{\omega_- - \omega_+} \\ \text{and} \qquad \text{Res}\,(\omega_-) &= \frac{e^{-i\omega_+(t-\tau)}}{\omega_+ - \omega_-} \end{aligned}\right\}, \tag{38}$$

1) R. TAKAHASHI, *Bull. Earthq. Res. Inst.*, **20** (1942), 375.

we have the expression of V_n by use of (31), (33), (34), (36) and (38) i.e.

$$V_n = 2\pi \cdot \frac{\sin \gamma_n (t-\tau)}{\gamma_n} \qquad \text{for } t>\tau, \tag{39}$$

where $\mu=0$.

Substituting (37) and (39) into (30),

$$\text{the second term of (26)} = D_\sigma \cdot \frac{T_n \cdot \gamma_n}{\cosh a_n H} \cdot \left\{ \begin{matrix} \int_0^{t_0} \\ \int_0^{t} \end{matrix} \right\} \cdot e^{-i\omega_0 \tau} \sin (t-\tau)\gamma_n d\tau, \tag{40}$$

where the integrations on the upper and lower sides correspond to the cases for $t>t_0$ and $t_0>t>0$ respectively.

Now putting (28) and (40) into (26), and after a few reductions, the expression of the higher mode becomes:

$$\zeta_n = D_\sigma \cdot \frac{T_n}{\cosh a_n H} \cdot \frac{\chi_n}{2} \qquad \text{for } t>t_0, \tag{41}$$

where

$$\chi_n = \frac{\gamma_n}{\omega_0+\gamma_n} \cdot \{ e^{-i(\omega_0+\gamma_n)t_0 + i\gamma_n t} - e^{+i\gamma_n t} \} - \frac{\gamma_n}{\omega_0-\gamma_n} \cdot \{ e^{-i(\omega_0-\gamma_n)t_0 - i\gamma_n t} - e^{-i\gamma_n t} \}.$$

Substituting (29) into (41) and taking the real part alone, the higher modes of waves become

$$\zeta_n = \frac{D_\sigma}{\cosh a_n H} \cdot \frac{kl}{(kl)^2-(n\pi)^2} \cdot \{ U_n(0) + (-1)^{n+1} U_n(l) \} \qquad (t>t_0),$$

where

$$\begin{aligned} U_n(\xi) = & \frac{2\gamma_n}{\omega_0+\gamma_n} \cdot \sin \frac{t_0}{2}(\omega_0+\gamma_n) \\ & \times \cos \left\{ \gamma_n t + k\xi - \frac{t_0}{2}(\omega_0+\gamma_n) \right\} \\ & - \frac{2\gamma_n}{\omega_0-\gamma_n} \cdot \sin \frac{t_0}{2}(\omega_0-\gamma_n) \\ & \times \cos \left\{ -\gamma_n t + k\xi - \frac{t_0}{2}(\omega_0-\gamma_n) \right\}. \end{aligned}$$

Setting down

$$\left.\begin{aligned} \alpha_{(\pm)} &= \frac{t_0}{2}(\omega_0 \pm \gamma_n) \\ \beta_n &= \frac{1}{2}(kl - n\pi) \end{aligned}\right\}, \tag{42}$$

the above expression becomes, after some reduction,

$$\begin{aligned} \xi_n = \frac{1}{2} \cdot \frac{D_\sigma}{\cosh a_n H} \cdot \frac{n\pi + 2\beta_n}{n\pi + \beta_n} \cdot \frac{\sin \beta_n}{\beta_n} \cdot \gamma_n t_0 \\ \times \left\{ \frac{\sin \alpha_{(+)}}{\alpha_{(+)}} \cdot \sin(\gamma_n t - \alpha_{(+)} + \beta_n) \right. \\ \left. + \frac{\sin \alpha_{(-)}}{\alpha_{(-)}} \cdot \sin(\gamma_n t + \alpha_{(-)} - \beta_n) \right\} \\ \text{for } t > t_0. \end{aligned} \tag{43}$$

In a manner similar to the fore-going reduction, the higher mode of the wave heights for the case $o < t < t_0$ is as follows:—

$$\begin{aligned} \zeta_n = \frac{D_\sigma}{\cosh a_n H} \cdot \frac{n\pi + 2\beta_n}{n\pi + \beta_n} \cdot \frac{\sin \beta_n}{\beta_n} \cdot \cos(\omega_0 t - \beta_n) \\ + \frac{1}{2} \cdot \frac{D_\sigma}{\cosh a_n H} \cdot \frac{n\pi + 2\beta_n}{n\pi + \beta_n} \cdot \frac{\sin \beta_n}{\beta_n} \cdot \gamma_n t_0 \\ \times \left\{ -\frac{\sin \tau_{(+)}}{\tau_{(+)}} \cdot \sin(\tau_{(-)} - \beta_n) + \frac{\sin \tau_{(-)}}{\tau_{(-)}} \cdot \sin(\tau_{(+)} - \beta_n) \right\}, \end{aligned} \tag{44}$$

where

$$\tau_{(\pm)} = \frac{t}{2}(\omega_0 \pm \gamma_n) \tag{44'}$$

and the first term is derived from (28) and (29), and the second group of terms is obtained by (30)–(39), the lower side of (40) and (29).

Dimensionless Form of Mode Solution: For convenience of later discussion, the above obtained solutions are expressed, in dimensionless form, as follows:

$$\zeta^* = \zeta_0^* + \sum_{n=1}^{\infty} \zeta_n^* \cos \frac{n\pi}{l^*} x^*, \tag{45}$$

where

$$
\left.\begin{aligned}
\zeta_0^* &= D_0^* \cdot \frac{1}{k^* l^*} \cdot \{\sin \omega_0^* t^* - \sin(\omega_0^* t^* - k^* l^*)\}, \\
\zeta_n^* &= \frac{D_0^*}{\cosh \alpha_n^*} \cdot \frac{n\pi + 2\beta_n^*}{n\pi + \beta_n^*} \cdot \frac{\sin \beta_n^*}{\beta_n^*} \cdot \Big[\cos(\omega_0^* t^* - \beta_n^*) \\
&\quad + \frac{1}{2} \cdot \gamma_n^* t^* \cdot \Big\{ -\frac{\sin \tau_{(+)}^*}{\tau_{(+)}^*} \cdot \sin(\tau_{(-)}^* - \beta_n^*) \\
&\quad + \frac{\sin \tau_{(-)}^*}{\tau_{(-)}^*} \cdot \sin(\tau_{(+)}^* - \beta_n^*) \Big\} \Big], \\
&\qquad \text{for } 0 < t < t_0
\end{aligned}\right\}, \tag{46}
$$

and

$$
\left.\begin{aligned}
\zeta_0^* &= 0, \\
\zeta_n^* &= \frac{1}{2} \cdot \frac{D_0^*}{\cosh \alpha_n^*} \cdot \frac{n\pi + 2\beta_n^*}{n\pi + \beta_n^*} \cdot \frac{\sin \beta_n^*}{\beta_n^*} \cdot \gamma_n^* t_0^* \\
&\quad \times \Big\{ \frac{\sin \alpha_{(+)}^*}{\alpha_{(+)}^*} \cdot \sin(\gamma_n^* t^* - \alpha_{(+)}^* + \beta_n^*) \\
&\quad + \frac{\sin \alpha_{(-)}^*}{\alpha_{(-)}^*} \cdot \sin(\gamma_n^* t^* + \alpha_{(-)}^* - \beta_n^*) \Big\}, \\
&\qquad \text{for } t_0 < t
\end{aligned}\right\}, \tag{47}
$$

and where the notation used in (45)–(47) is as given below:

$$
\left.\begin{aligned}
&\zeta^* = \zeta/H, \quad \zeta_m^* = \zeta_m/H, \quad x^* = x/H, \\
&D_0^* = D_0/H, \quad \omega_0^* = \omega_0 \sqrt{\frac{H}{g}}, \quad t^* = t\sqrt{\frac{g}{H}}, \\
&k^* = kH, \quad l^* = l/H, \quad \alpha_n^* = \alpha_n H, \\
&\beta_n^* = \beta_n, \quad \gamma_n^* = \gamma_n \sqrt{\frac{H}{g}}, \quad t_0^* = t_0 \sqrt{\frac{g}{H}}, \\
&\tau_{(\pm)}^* = \tau_{(\pm)}, \quad \alpha_{(\pm)}^* = \alpha_{(\pm)} \\
&\begin{pmatrix} m = 0, 1, 2, \cdots \\ n = 1, 2, 3, \cdots \end{pmatrix}.
\end{aligned}\right\} \tag{48}
$$

As shown in the above, the zeroth mode of waves (ζ_0^*) is produced only in the interval of the duration of an earthquake wave. After the termination of a ground motion, the higher modes of waves excepting the zeroth one remain alone.

Unlimited Duration of Ground Motion: When the duration of an earthquake wave is unlimited, i.e., $t_0 \to \infty$, the expressions of (47) disappear and merely (46) is taken into consideration.

For sufficiently large t,

$$\frac{\sin \eta t}{\eta} \simeq \pi \delta(\eta).$$

In the above expression, provided that η is non-zero, the value of the left-hand member of the above equation is considered actually to zero. Therefore, the higher modes of waves of (46) becomes

$$\zeta_n^* = \frac{D_0^*}{\cosh a_n^*} \cdot \frac{n\pi + 2\beta_n^*}{n\pi + \beta_n^*} \cdot \frac{\sin \beta_n^*}{\beta_n^*} \cdot \Big\{ \cos(\omega_0^* t^* - \beta_n^*) + \frac{\pi}{2} \cdot \gamma_n^* \cdot \delta(\omega_0^* - \gamma_n^*) \cdot \sin(\gamma_n^* t^* - \beta_n^*) \Big\}. \tag{49}$$

As shown in the above equation, as t_0 and t $(t_0 > t)$ become large, the only mode for $\omega_0^* = \gamma_n^*$ remains and this mode is eventually infinite in magnitude, unless $\beta_n^* = m\pi$ $(m = 1, 2, 3, \cdots)$, of which the case will be discussed afterwards.

Now the expression (49) denotes that *a long duration of the seismic waves, even though the amplitude is so small, is sufficient to produce an oscillation of the lake water.*

Finite Duration of Ground Motion: The actual situation of the earthquake waves may differ from the model (9) and a gradual transition of the motion of the ground to a quiet state, instead of a sudden one as shown in (9), is seen in the actual earthquake, but the model (9), as the first approximation, is used to explain the behavior of the motion of the lake water on the occassion of an earthquake.

(i) When $\omega_0^* \gg \gamma_n^*$, the approximation

$$\tau_{(\pm)}^* \simeq \frac{1}{2} \omega_0^* t^*$$

is possible. Using this relation, the second expression of (46) becomes

$$\zeta_n^* = \frac{D_0^*}{\cosh a_n^*} \cdot \frac{n\pi + 2\beta_n^*}{n\pi + \beta_n^*} \cdot \frac{\sin \beta_n^*}{\beta_n^*} \cdot \Big\{ \cos(\omega_0^* t^* - \beta_n^*) + \frac{\gamma_n^*}{\omega_0^*} \cdot \sin \beta_n^* \cdot \sin \gamma_n^* t^* \Big\} \qquad \text{for } t_0 > t > 0.$$

In the above expression, the contribution of the second term is very small owing to $\omega_0^* \gg \gamma_n^*$. Hence, it turns out that *the water waves excited by an earthquake have the same period as those of an earthquake in the midst of the upheaval*. On the contrary, *for $t > t_0$, the generated water waves oscillate with a period of an eigen oscillation of the lake*, as shown in the expression (47).

(ii) When $\omega_0^* \ll \gamma_n^*$, the approximation

$$\tau_{(\pm)}^* \simeq \pm \frac{1}{2} \gamma_n^* t^*$$

is valid, which makes the following reduction possible.

From (46), the higher mode of the waves for $t_0 > t > 0$ is reduced, after a few reductions, to the form

$$\zeta_n^* = \frac{D_\sigma^*}{\cosh \alpha_n^*} \cdot \frac{n\pi + 2\beta_n^*}{n\pi + \beta_n^*} \cdot \frac{\sin \beta_n^*}{\beta_n^*} \cdot \{\cos(\omega_0^* t^* - \beta_n^*) + \sin \omega_0^* t^* \cdot \sin \beta_n^* - 2 \cos \beta_n^* \cdot \sin \tau_{(+)}^* \cdot \sin \tau_{(-)}^*\}.$$

From the above equation and (47), we find that, *when $\omega_0^* \ll \gamma_n^*$, the water waves produced by the ground motion have, in the midst of the earthquake, two kinds of periods which are relevant to the earthquake wave and the eigen oscillations of the lake water and, after the earthquake, the former disappears and the latter only remain.*

(iii) When $\omega_0^* \to \gamma_n^*$, the following approximations can be made:

$$\frac{\sin \tau_{(+)}^*}{\tau_{(+)}^*} = \frac{\sin \omega_0^* t^*}{\omega_0^* t^*}, \quad \frac{\sin \tau_{(-)}^*}{\tau_{(-)}^*} = 1 \quad (t^* : \text{finite}),$$

$$\frac{\sin \alpha_{(+)}^*}{\alpha_{(+)}^*} = \frac{\sin \omega_0^* t_0^*}{\omega_0^* t_0^*}, \quad \frac{\sin \alpha_{(-)}^*}{\alpha_{(-)}^*} = 1 \quad (t_0^* : \text{finite}).$$

By use of the above expressions, (46) and (47) become as follows: For $t_0 > t > 0$ (from (46)),

$$\zeta_n^* = \frac{D_\sigma^*}{\cosh \alpha_n^*} \cdot \frac{n\pi + 2\beta_n^*}{n\pi + \beta_n^*} \cdot \frac{\sin \beta_n^*}{\beta_n^*} \cdot \left\{\cos(\omega_0^* t^* - \beta_n^*) + \frac{1}{2} \cdot \sin \omega_0^* t^* \cdot \sin \beta_n^* + \frac{1}{2} \cdot \gamma_n^* t^* \cdot \sin(\omega_0^* t^* - \beta_n^*)\right\}$$

and for $t > t_0$ (from (47)),

$$\zeta_n^* = \frac{D_\sigma^*}{\cosh \alpha_n^*} \cdot \frac{n\pi + 2\beta_n^*}{n\pi + \beta_n^*} \cdot \frac{\sin \beta_n^*}{\beta_n^*} \cdot \frac{1}{2} \cdot \{\sin \omega_0^* t_0^* \cdot \sin(\omega_0^* t^* - \omega_0^* t_0^* + \beta_n^*) + \omega_0^* t_0^* \cdot \sin(\omega_0^* t^* - \beta_n^*)\}.$$

The above two expressions denote that, *as the period of the earthquake wave is equal to that of the eigen oscillation of the lake water, the wave height of the corresponding mode of the lake water is in proportion to a time duration of the earthquake.*

(iv) When $k^*l^* \to 0$, which denotes that the length of an earthquake wave is sufficiently large as compared with that of a lake, the higher modes of the waves are reduced to zero, which are derived readily from (42), (46) and (47).

On the other hand, the expressions of the zeroth mode are described as follows:

for $t_0 > t > 0$ (from (46)),

$$\zeta_0^* \to D_C^* \cos \omega_0^* t^*,$$

for $t > t_0$ (from (47)),

$$\zeta_0^* = 0.$$

From the above result, it is found that, *when $k^*l^* \to 0$, the whole surface of the water oscillates uniformly with the period of an earthquake and the motion of water is limited to the zeroth mode alone.*

In the actual earthquake, the water of a bottle, reservoir and lake is excited by the vibrations of the both wall and bottom. When $k^*l^* \to 0$, in general, the energy contribution from the wall to cause the disturbance of water is not so small compared to that from the bottom, so that the model of the bottom vibration used in the present paper may be considered to be inappropriate. But if the depth is so small as compared with the length of the bottom, the result obtained in the above is interpreted as still applicable.

(v) When $k^*l^* \to n\pi$ or $2l^* = n\lambda$ (λ is the wave length of an earthquake wave), the n-th mode of the waves becomes as follows:

for $t_0 > t > 0$ (from (46)),

$$\zeta_n^* = \frac{D_C^*}{\cosh \alpha_n^*} \cdot \left\{ \cos \omega_0^* t^* + 2 \cdot \frac{(\gamma_n^*)^2}{(\omega_0^*)^2 - (\gamma_n^*)^2} \sin \tau_{(+)}^* \sin \tau_{(-)}^* \right\};$$

for $t > t_0$ (from (47)),

$$\zeta_n^* = \frac{D_C^*}{\cosh \alpha_n^*} \cdot \left\{ \frac{\gamma_n^*}{\omega_0^* + \gamma_n^*} \sin \alpha_{(+)}^* \sin (\gamma_n^* t^* - \alpha_{(+)}^*) + \frac{\gamma_n^*}{\omega_0^* - \gamma_n^*} \sin \alpha_{(-)}^* \sin (\gamma_n^* t^* + \alpha_{(-)}^*) \right\}.$$

As shown in the above, if $\omega_0^* \neq \gamma_n^*$, the n-th mode of the waves has always a finite value. In derivation of the above two equations, the reductions of the factor with respect to β_n^*

$$\frac{n\pi+2\beta_n^*}{n\pi+\beta_n^*}\cdot\frac{\sin\beta_n^*}{\beta_n^*}\to 1 \qquad (\beta_n^*\to 0) \tag{50}$$

is used. Let the left-hand side of the above be equal to Q_n, viz.

$$Q_n=\frac{n\pi+2\beta_n^*}{n\pi+\beta_n^*}\cdot\frac{\sin\beta_n^*}{\beta_n^*}. \tag{51}$$

As k^*l^* tends to $n\pi$ $(m\neq n)$,

$$Q_m=\frac{2n}{m+n}\cdot\frac{\sin\frac{\pi}{2}(m-n)}{\frac{\pi}{2}(m-n)}. \tag{52}$$

When $k^*l^*\to n\pi$, Q_n *factor* (*the expression* (51) *is called in such a way in the following*) *tends to* 1 (*from* (50)) *and* Q_m *factor for* m $(m\neq n)$ *is proved, in absolute value, smaller than a unit* (Q_n *value for a resonant mode*) in the following way :—

Q_m factor is, at first, separated into partial fractions, i.e.

$$Q_m=\frac{2}{\pi}\sin\frac{\pi}{2}(n-m)\cdot\left\{\frac{1}{m+n}+\frac{1}{n-m}\right\}.$$

Taking the absolute value of the above equation,

$$\begin{aligned}|Q_m| &\leqq \frac{2}{\pi}\left|\sin\frac{\pi}{2}(n-m)\right|\cdot\left\{\frac{1}{m+n}+\frac{1}{|n-m|}\right\}\\ &\leqq \frac{2}{\pi}\left\{\frac{1}{m+n}+\frac{1}{|n-m|}\right\}.\end{aligned} \tag{53}$$

Since m and n are positive integers different from each other, the inequalities

$$|m-n|\geqq 1 \quad \text{and} \quad m+n\geqq 3$$

are valid. Substituting these relations into the inequality (53), the value of Q_n factor, when $k^*l^* \to n\pi$, is verified to take a maximum (a unit), i.e.,

$$\begin{aligned} |Q_m| &\leqq \frac{2}{\pi}\cdot\left(\frac{1}{3}+1\right) \\ &= \frac{8}{3\pi} \\ &< 1 \quad (= Q_n \text{ for a resonant mode}). \end{aligned}$$

Here, we consider on which side (the lower or upper side) of the resonant mode more energy is transferred by the motion of a ground with a wave length $\lambda = \frac{2l}{n}$ $(k^*l^* = n\pi)$.

When $k^*l^* \to n\pi$, let p be a positive integer such that $n > p$. Then the Q factors on both sides of the resonant mode are expressed from (52) as follows:

for the lower side,

$$Q_{n-p} = \frac{2}{\pi}\cdot\frac{\sin\frac{\pi}{2}p}{p}\cdot\frac{2n}{2n-p}\,; \tag{54}$$

for the upper side,

$$Q_{n+p} = \frac{2}{\pi}\cdot\frac{\sin\frac{\pi}{2}p}{p}\cdot\frac{2n}{2n+p}. \tag{55}$$

If p is an even number, the above two equations vanish, that is to say,

when $k^*l^* \to n\pi$, the modes of the waves with a difference of even numbers to the n-th one are not produced on the surface of water.

If p is an odd integer, we have from (54) and (55)

$$|Q_{n-p}| > |Q_{n+p}|. \tag{56}$$

Since the expressions of the higher mode of the waves (46) have the factor $1/\cosh \alpha_n^*$ with respect to n excepting the Q factor, the dependence of the amplitude of the waves on n is considered to be expressed by the factor[2)]

$$R_n = \frac{Q_n}{\cosh a_n^*}, \tag{57}$$

if t^* is not so large, and there exists an inequality

$$\cosh a_{n-p}^* < \cosh a_{n+p}^* \tag{58}$$

for $n>p>0$.

Now, from (56)–(58), the following is obtained:

$$\frac{|Q_{n-p}|}{\cosh a_{n-p}^*} \gg \frac{|Q_{n+p}|}{\cosh a_{n+p}^*}. \tag{59}$$

Now, from (59), *the modes of the waves on the lower side of the resonnant mode are more excited by the ground motion than those on the upper side at the initial stage of th earthquake.*

If the depth of the lake is very shallow as compared with the length, i.e. $l^* \gg 1$, the variations of $\cosh a_n^*$ and $\gamma_n^* (=\sqrt{a_n^* \tanh a_n^*})$ versus n can be regarded nearly zero by virtue of the smallness of the variation of $a_n^* \left(=\frac{n\pi}{l^*}\right)$. Then the dependence of the amplitude of the n-th mode of the waves on n might be described by Q_n (refer to the footnote 2)).

Now, from (56) and the fact mentioned above, it is concluded that,

2) R_n, the amplitude factor relevant to n, seems to have to include one more factor γ_n^*, but the existences of $\tau_{(\pm)}^* \left(=\frac{t^*}{2}(\omega_0^* \pm \gamma_n^*)\right)$ in the denominators suppress the contribution of γ_n^* on R_n such that: when $\omega_0^* \gg \gamma_n^*$, the expression of the wave height is, from section (i),

$$\zeta_n^* = D_c^* R_n \cos(\omega_0^* t^* - \beta_n^*),$$

which does not depend on γ_n^*;
when $\omega_0^* \ll \gamma_n^*$, the wave height is described, from section (ii), as

$$\zeta_n^* = D_G^* \cdot R_n \cdot \{\cos(\omega_0^* t^* - \beta_n^*) + \sin \omega_0^* t^* \cdot \sin \beta_n^* - 2\cos \beta_n^* \cdot \sin \tau_{(+)}^* \cdot \sin \tau_{(-)}^*\}$$

in which the amplitude factor does not have γ_n^*;
finally, when $\omega_0^* \approx \gamma_n^*$, from section (iii), we have

$$\zeta_n^* = D_G^* \cdot R_n \cdot \left\{\cos(\omega_0^* t^* - \beta_n^*) + \frac{1}{2}\sin \omega_0^* t^* \sin \beta_n^* + \frac{1}{2}\gamma_n^* t^* \cdot \sin(\omega_0^* t^* - \beta_n^*)\right\}.$$

The last expression denotes that the amplitude factor is almost independent of γ_n^*, unless t^* is very large.

Thus, from the result discussed in the above, in the midst of the duration of the earthquake, the dependence of the amplitude of the waves on n may be considered to be described by the expression (57).

when the lake is very shallow, the nearby modes of the waves on the lower side of the resonant mode are more excited by the earthquake than those on the upper side.

The above conclusion is valid only for the nearby modes of the resonant one, because the property of a constant of the factors cosh α_n^* and γ_n^* cannot be held for large variation of n.

In the present paper, the ground motion is given only at the bottom and other cases will be treated in the future.

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8. 地震によつておこされた水の運動 [I]

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大地震の際に，池，貯水地，湖などの水面がじよう乱を受けることは良く知られた現象である．併しこれに対する理論的な解析は未だおこなわれていない．そこで筆者は本報告および後の報告でこの問題を理論的に究明しようと考え，まず本報告では，一次元の矩形の湖の水底に周期的な伝播波を有限時間与えるモデルを用いて解析を試みた．そして次のような結論を得た．

(1) (地震波の周期)≪(湖水の固有振動の第n番目のモードの周期) のとき，発生した水波は地震波継続中は地震波の周期で動き，地震波通過後は湖水の固有振動周期で動く．

(2) (地震波の周期)≫(湖水の固有振動の第n番目のモードの周期) のとき，発生した水波は2種類の周期をもつ，すなわち一つは地震波の周期をもち，他の一つは湖水の固有振動の周期に対するものである．地震波通過後は前者の周期は消え，後者の周期による振動のみが残る．

(3) (地震波の周期)≃(湖水の固有振動の第 n 番目のモードの周期) のとき，湖水振動の第 n 番目のモードの波高は地震波の継続時間に比例して増加する．

(4) (地震波の波長)≫(湖水の長さ) のとき，湖水は地震波の周期で振動し，しかも第零次モードのみが生じ，高次モードは発生しない．

(5) 2×(湖水の長さ)=n×(地震波の波長) (n=1, 2, 3, ･･･) のとき，すなわち地震波の波長が湖水の第 n 次モードの波長と一致するとき，n 次モードと偶数次だけ異なる湖水のモードは発生しない．そして，地震波発生の初期では共鳴モードより低次のモードはそれより高次のモードより大なるエネルギーを地震波より受ける．また，湖が浅く，長い場合に共鳴モードの近くのモードに限り，共鳴モードより低次側のモードの方が高次側のモードより，より大きなエネルギーを受ける．

(6) 地震が非常に長く続くとき，たとえ地震波の振幅が小さくとも，湖水の振動の振幅は非常に大きくなる．そして発生水波の周期は地震波の周期に近くなる．