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34. *Electromagnetic Induction in a Nearly Spherical Conductor.*

By Tsuneji RIKITAKE,

Earthquake Research Institute.

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Summary

A theory of electromagnetic induction in a perfect conductor of which the shape is slightly different from a sphere is developed. It is demonstrated by the theory that the induced field is strengthened over the portion of the earth where the top surface of the conducting part of the mantle is uplifted. It is also pointed out that the depth of the conductor determined on the assumption of spherical symmetry does not agree with that of a sphere having a radius equal to the mean radius of the non-spherical conductor.

1. Introduction

Chapman[1),2)], Price[3)] and others advanced theories of electromagnetic induction in a conductor with spherical symmetry. Applying these theories to the relation between inducing and induced parts of transient geomagnetic variations, it has been surmised that the earth's mantle becomes highly conducting below a depth of a few hundred kilometers[1)~4)]. An actual procedure of this kind of study is to carry out a spherical harmonic analysis of a geomagnetic variation and then to compare the coefficients of an external field with those of an internal one. It has been thought that irregularities in the observed data could be eliminated by the procedure, so that the electrical conductivity thus determined would represent that for the averaged state of the earth's interior.

Recent findings of geomagnetic variation anomaly indicate, however, that the irregularities in geomagnetic data are no longer accidental. Conversely, they reflect very well local or regional conductivity anomalies in the mantle. For example, Schmucker[5)] showed that a sharp contrast

1) S. CHAPMAN, *Phil. Trans. Roy. Soc. London A*, **218** (1919), 1.
2) S. CHAPMAN and J. BARTELS, *Geomagnetism*, Oxford Univ. Press. (1940), Ch. 22.
3) B. N. LAHIRI and A. T. PRICE, *Phil. Trans. Roy. Soc. London A*, **237** (1939), 509.
4) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **28** (1950), 45, 219, 263.
5) U. SCHMUCKER, *J. Geomag. Geoelectr.*, **15** (1964), 193.

in ΔZ (change in the vertical component) as found along an east-west profile in south-west U.S.A. at times of geomagnetic bay and similar change could be accounted for by assuming a stepwise depression of the top surface of the conducting part of the mantle from a depth of 160 km to 320 km.

As mantle conductivity anomalies have been found at a number of places over the earth[6], it seems natural to think that the surface of the conducting part of the mantle is undulatory. In that case a question is raised about the meaning of the depth of the conducting mantle which is determined on the assumption of spherical symmetry. Such a depth would certainly differ from the depth of a sphere having a mean radius.

It is the aim of this paper to see to what extent a geomagnetic variation is affected by a conducting core which is not exactly a sphere. As was argued by Price[7], a general theory of electromagnetic induction in a conductor, which is not spherically symmetric, is so difficult to develop that no theories have so far been advanced. Since no detailed information about regional or local characteristics of possible upheavals and depressions of the surface of the conducting part of the mantle has yet been put forward, it is doubtful whether it would be worthwhile at present trying to elaborate a general theory. Only a theory of electromagnetic induction in a perfect conductor of which the shape is slightly different from a sphere will be attempted in this paper.

2. Theory and numerical solution for a simple case

Let us take a perfect conductor of which the shape is given by

$$r=q_0a[1+\varepsilon u(\theta)] \, , \qquad (1)$$

in spherical polar coordinates. q_0, a and ε are constants, a being identified later as the earth's radius. Putting

$$\rho=r/a \, , \qquad (2)$$

the equation of the surface of the conductor can be written as

$$\rho=q_0[1+\varepsilon u(\theta)] \, . \qquad (3)$$

6) A summary of recent findings on mantle conductivity anomaly has been published in *J. Geomag. Geoelectr.*, **15**, No. 4.

7) A. T. PRICE, *J. Geomag. Geoelectr.*, **15** (1964), 241.

If we assume a uniform inducing field parallel to the $\theta=0$ axis, its magnetic potential is given by

$$W_e = ae_1\rho P_1(\cos\theta)\ , \tag{4}$$

from which the r and θ components are given as follows;

$$H_{re} = -e_1 P_1\ , \qquad H_{\theta e} = -e_1 dP_1/d\theta\ . \tag{5}$$

The potential and components of the induced field can be written as follows;

$$W_i = a\sum_n i_n \rho^{-n-1} P_n(\cos\theta)\ , \tag{6}$$

$$H_{ri} = \sum_n (n+1) i_n \rho^{-n-2} P_n\ , \quad H_{\theta i} = -\sum_n i_n \rho^{-n-2} dP_n/d\theta\ , \tag{7}$$

where $P_n(\cos\theta)$ is a Legendre function of degree n.

A condition that the component of the magnetic field normal to the surface vanishes should be satisfied at the conductor. Hence

$$(H_{re}+H_{ri})\cos\alpha - (H_{\theta e}+H_{\theta i})\sin\alpha = 0\ , \tag{8}$$

holds at any point on the surface defined by (1) or (3). α being the angle between the normal and the line connecting the point to the origin, we have relations

$$\left.\begin{aligned} \cos\alpha &= 1/\{1+(f'/r)^2\}^{1/2}\ , \\ \sin\alpha &= (f'/r)/\{1+(f'/r)^2\}^{1/2}\ , \end{aligned}\right\} \tag{9}$$

where f stands for the righthand-side of (1) and f' denotes $df/d\theta$.

With the aid of (9), (8) leads to an equation

$$H_{re} + H_{ri} - (H_{\theta e} + H_{\theta i})\varepsilon u'(\theta)/[1+\varepsilon u(\theta)] = 0\ . \tag{10}$$

Putting (5) and (7) into (10), (10) reduces to

$$\sum_n (n+1) i_n \rho^{-n-2} P_n + q_0 \varepsilon \sum_n i_n \rho^{-n-3} \frac{dP_n}{d\theta} u'(\theta) = e_1\left(P_1 - \frac{\varepsilon \dfrac{dP_1}{d\theta} u'(\theta)}{1+\varepsilon u(\theta)}\right)\ , \tag{11}$$

in which ρ is a function of θ defined by (3). (11) is the equation solving which i_n's are determined.

Multiplying (11) by $P_N \sin\theta$ and integrating it with respect to θ from 0 to π, we obtain

$$\sum_n [(n+1)\alpha_{nN} + \varepsilon\beta_{nN}] q_0^{-n-2} i_n = (\xi_{1N} + \varepsilon\eta_{1N}) e_1\ , \tag{12}$$

where

$$\left.\begin{aligned}\alpha_{nN}&=\int_0^\pi\frac{P_nP_N\sin\theta d\theta}{[1+\varepsilon u(\theta)]^{n+2}}\ ,\\ \beta_{nN}&=\int_0^\pi\frac{\frac{dP_n}{d\theta}P_Nu'(\theta)\sin\theta d\theta}{[1+\varepsilon u(\theta)]^{n+3}}\ ,\\ \xi_{1N}&=\int_0^\pi P_1P_N\sin\theta d\theta\ ,\\ \eta_{1N}&=-\int_0^\pi\frac{\frac{dP_1}{d\theta}P_Nu'(\theta)\sin\theta d\theta}{1+\varepsilon u(\theta)}\ .\end{aligned}\right\}\qquad(13)$$

If the integrals in (13) can be calculated, (12) for various N values provides a set of simultaneous equations solving which we can determine i_n's.

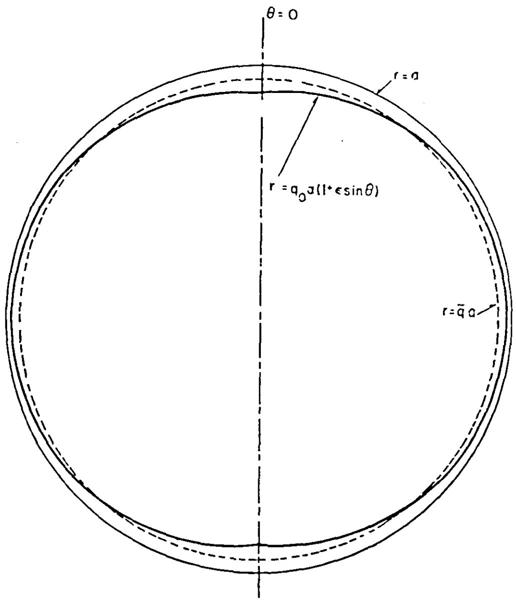

Fig. 1. Non-spherical conductor as defined by $r=q_0a(1+\varepsilon\sin\theta)$ and a sphere $(r=\bar{q}a)$ having a radius equal to the mean radius. $r=a$ denotes the earth's surface.

As an example we take $\varepsilon=0.1$ and $u(\theta)=\sin\theta$. In such a case the meridian section of the conductor takes a shape as shown in Fig. 1 for $q_0=0.8837$. A sphere having the mean radius ($\bar{q}=0.94$, a typical value for the conducting part of the mantle as obtained from studies based on spherical symmetry) of the conductor is also shown in Fig. 1 together with that for $q=1$ (the earth's surface). It is obvious in this case that α_{nN}'s and β_{nN}'s for even n's and N's are always zero. Although (12) provides a set of simultaneous equations of infinite number, only i_1, i_3, i_5, i_7, i_9 and i_{11} are here taken into account neglecting higher degree terms.

The coefficients of i_n's and the righthand members of the equations are calculated as indicated in Table 1. Most of the integral involved

Table 1. The coefficients and the righthand members of simultaneous equations (12).

i_1	i_3	i_5	i_7	i_9	i_{11}	Righthand member*
1	0.1867	0.0379	0.0190	0.0138	−0.0148	0.4418
0.0458	1	0.1136	0.1020	0.0254	−0.0044	−0.0052
0.0094	0.0853	1	0.1802	0.0806	0.0759	−0.0009
0.0049	0.0224	0.3044	1	0.2321	0.0793	−0.0003
0.0004	0.0089	0.1412	0.1768	1	0.4124	−0.0002
0.0012	0.0013	0.0079	0.0458	0.2463	1	−0.0001

* Unit: e_1

are numerically calculated by means of Simpson's method.

The equations are easily solved because the diagonal coefficients are fairly large. In Table 2 are given i_n's.

Table 2. i_n's in units of e_1.

n	i_n
1	0.4466
3	−0.0252
5	−0.0027
7	−0.0012
9	0.0007
11	−0.0006

It is seen that i_1 is much larger than the other i_n's. This is reasonable because the conductor differs from a sphere only slightly. However, the fact that i_n's are finite, though small, for higher values of n is important. Judging from the fact that i_n's become smaller as n becomes large, the procedure in which the terms for large n's were ignored may be justified though no exact theory of convergence has been advanced.

3. Discussion

First of all, let us examine how well the condition that the normal component vanishes is satisfied at the conductor. The normal components of the inducing and induced fields along a meridian section of the surface of the conductor can be expressed as functions of θ by

$$\left.\begin{aligned} H_{ne} &= H_{re}\cdot\cos\alpha - H_{\theta e}\sin\alpha\,, \\ H_{ni} &= H_{ri}\cdot\cos\alpha - H_{\theta i}\sin\alpha\,. \end{aligned}\right\} \tag{14}$$

H_{ne} and H_{ni} can easily be calculated from (5) and (7) with the values of i_n. In Fig. 2 are shown H_{ne} and H_{ni} in units of e_1. It is apparent that H_{ne} is approximately cancelled out by H_{ni}. The values of $H_{ne}+H_{ni}$ at $\theta=0°, 10°, 20°, \cdots, 180°$ are also plotted in Fig. 2. They are so small that we can practically regard them as zero when

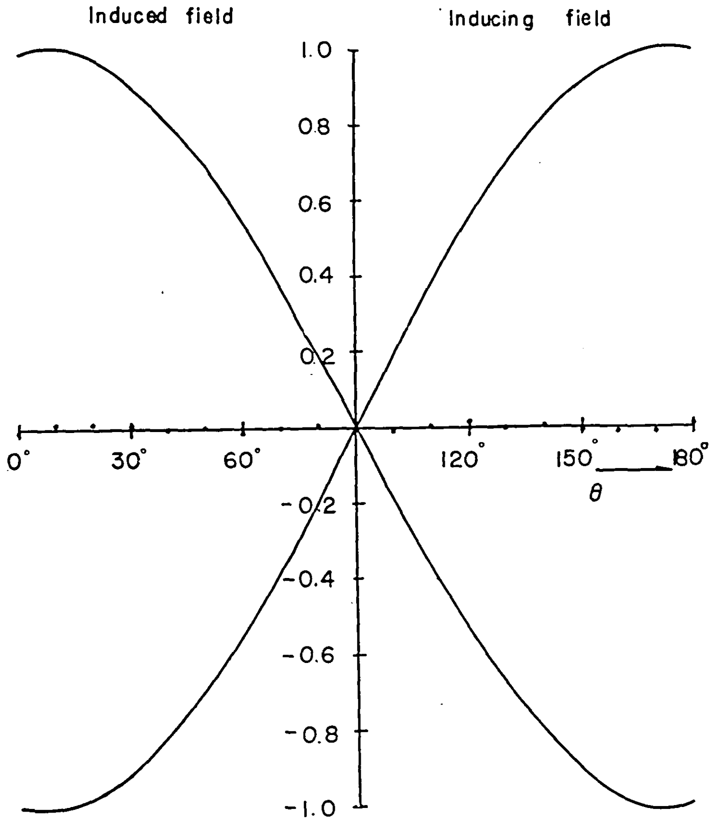

Fig. 2. The inducing and induced fields (normal to the surface of the conductor) at the surface of the conductor. The total field is shown by dots at every 10 degrees of θ.

the accuracy of estimation is taken into account. It can therefore be said that the fulfilment of the boundary condition is satisfactory.

The induced field components at $r=a$ or $\rho=1$ are calculated from (7) and are shown in Figs. 3 and 4 in which the components which would have been observed when the conductor is a sphere of radius $\bar{q}a$, the mean radius of the non-spherical conductor, are also shown. It is indicated in Figs. 3 and 4 that the field components become large over the portion where the depth of the conductor gets smaller and that they become small for large depths of the conductor.

Going back to Table 2, let us suppose that a spherical harmonic analysis of a geomagnetic variation resulted in the coefficients of the internal potential as given in the table, while the only coefficient of the

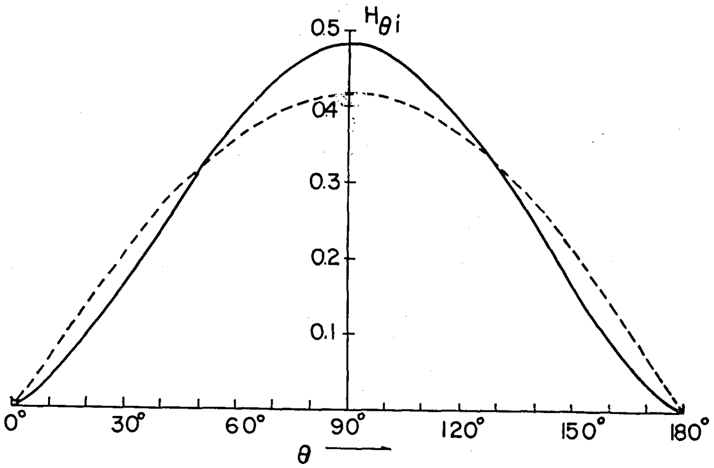

Fig. 3. The distribution of the induced horizontal magnetic field in units of e_1 over the earth's surface. The broken line shows the similar distribution when the conductor is a sphere which has a radius equal to the mean radius of the non-spherical conductor.

external potential is e_1 for $n=1$. If it is assumed that no other information is available, one is tempted to compare i_1 with e_1 ignoring i_3, i_5, $\cdots$ which are much smaller than i_1. If we do this and assume a spherical conductor within the earth, the ratio $i_1/e_1=0.4466$ leads to a conclusion that the radius of the conductor amounts to $0.963a$ as long as the perfect conductor approximation is valid. It is therefore demonstrated that the spherical conductor treatment leads to an apparent radius $0.963a$ in contrast with $0.94a$ of the mean radius of the actual model. The difference between the two values is significant, the apparent depth being 235 km which is a considerable underestimate because the mean depth is 382 km.

In the light of the above discussion, we see that what we determine from a spherical conductor treatment results in a size parameter which considerably differs from that of a non-spherical conductor. It is therefore doubtful whether the existing studies of the electrical state within the earth provide accurate estimates of the size of the conducting part of the mantle. It is rather suspected that the depth hitherto determined might largely be controlled by uplifted portions of the conducting part of the mantle.

It has been demonstrated in Section 2 that many spherical harmonic terms are required in discussing electromagnetic induction in a non-

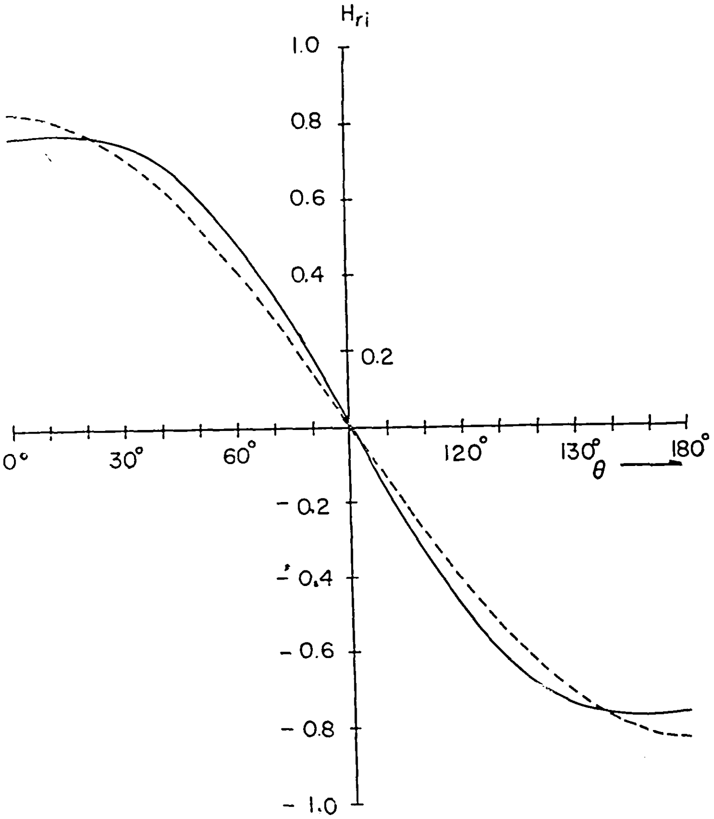

Fig. 4. The distribution of the induced vertical magnetic field in units of e_1. The broken line shows a similar one for the mean sphere.

spherical conductor even if the inducing field is uniform. For a conductor of much more complicated shape, which is likely to be the case of the earth, it would be of great difficulty to reach the actual shape of the conducting part of the mantle. However, the fact that large or small induced fields should be observed respectively over the portions of the earth where the conductor lies at small or large depths suggests a method of tackling the problem. Dividing the earth's surface into a number of areas, the depth of the conducting part of the mantle can be determined for each area on the basis of induction in a spherical body. Subsequently, we may form a non-spherical conductor model of which an approximate shape is obtained from the above regional study. Such a model could be improved by applying a theory similar to the one advanced in this paper in such a way as to account for the external-internal relation of geomagnetic variation.

The theory in Section 2 requires no restriction for ε, so that ε could be larger than 0.1 which was taken for the present numerical example. But it is feared for a large ε that many harmonic functions would be required in order to represent the induced field. It would not be practicable to apply the theory to a case in which the undulation is large.

4. Conclusion

A theory of electromagnetic induction in a non-spherical conductor makes it clear that the induced magnetic field is affected fairly largely even if the shape of the conductor differs from a sphere only slightly. The size of a spherical conductor that fits in the relation between the external and internal fields differs considerably from that of a sphere having a mean radius of the conductor. In a particular case studied in this paper, the depth of the conductor thus determined results in an underestimate of the mean depth though such a conclusion is likely to be largely controlled by the type of inducing field. As the surface of the conducting part of the earth's mantle seems likely to be undulatory, as has been suggested from recent findings of geomagnetic variation anomaly, the existing knowledge on the distribution of the electrical conductivity in the mantle should be reexamined although we need much more theoretical as well as observational work for that purpose.

The theory presented here is only suitable for a simple case in which perfect conductivity and small deviation from a sphere are assumed. Such a theory, however, could be extended to a conductor having a little more complicated shape and a finite conductivity.

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34. ひずんだ導体球中の電磁感応

地震研究所 力 武 常 次

球形よりわずかにずれている完全導体の電磁感応を調べた．一様磁場による感応の場合にも，高次の球関数で表わされる induced field があらわれる．

地球マントルの導電的部分が，球形よりずれているときに，これを球と仮定して，地磁気変化の解析より求めた半径は，平均の半径と一致しない．最近の地磁気変化異常の研究は，導電的部分表面の凹凸を示唆しているから，従来球対称の仮定のもとに求められてきた地球内部の電気的性質は，再吟味する必要がある．従来求められている導電的部分表面までの深さは，そのふくらんだ部分の影響をより強くうけて，見かけ上浅く求められている可能性がある．