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37. *Construction of Refraction Diagrams of Tsunami* [I].

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Abstract

The construction of wave refraction diagrams, which is a very tedious process by hand, was programmed for digital computers by Griswold. In this report, the program for wave refraction is also built up, allowing for a particular application to a tsunami. Water depths in the area to be studied, wave direction, and separation angle and ray separation of two neighbouring rays are fed into the computer. The computer converts the depth values to wave speed in the approximation of long wave, computes the path of the wave orthogonals, the refraction coefficients, and the wave heights, and calculates the time necessary for travelling the distance along the orthogonals.

1. Introduction

A trial of numerical calculation of wave refraction has already been made by Gale M. Griswold.[1]

The construction of wave refraction diagrams requires manual drafting techniques, which are time consuming and very subjective. The number of rays that can be constructed is limited by the ability of the draftsman. For these reasons, Griswold tried to make the programming for numerical calculation of wave refraction.

In this paper, the author has made also a program of the wave refraction, particularly as an attempt to construct *a refraction diagram of a tsunami*. The program made-up by the author is based on the codes of the IBM 7090 computer.

2. Construction of a Program

2, 1. *Definitions and Notations.*

γ : l/l_0, where l and l_0 are ray separations at an arbitrary

1) Gale M. GRISWOLD, *Jour. Geophy. Res.*, **68** (1963), 1715.
This paper will be referred to as paper I.

point and at a starting point of waves;
g : acceleration of gravity;
s : wave ray;
c : wave velocity;
τ : curve traced by l values along s;
θ : angle τ curve makes with s-axis;
K_τ : curvature of τ curve;
K_s : curvature of ray;
x, y : grid coordinates;
H : depth of water;
α : angle x makes with s curve.

These definitions and notations are used in the following discussions, unless otherwise stated.

2, 2. Method of Wave Ray. In constructing the refraction diagrams, we have two available methods, *i.e.*, of the wave ray and the wave front. The former is superior in accuracy to the latter and the wave front along the rays can be easily obtained as an extension of method of wave ray, as will be shown later on. Accordingly, we proceed to program the method of the ray. The details of ray method have already been described by Griswold in paper I, so that only outlines of the method are given in this paper allowing for a particular application to tsunamis.

Since our attention is focussed on a problem of drafting a refraction diagram of *tsunamis*, a wave speed may be considered, to the first order of approximation, as $\sqrt{gH}$. Then a wave front will move a distance Δs along the ray for a time increment Δt according to

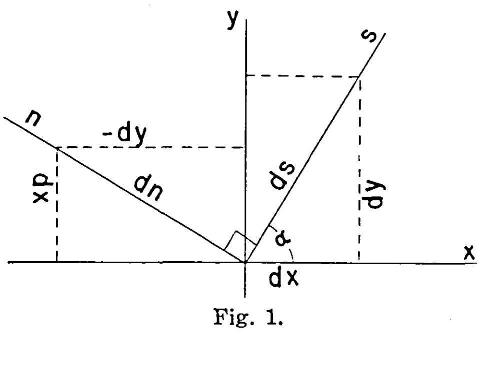

Fig. 1.

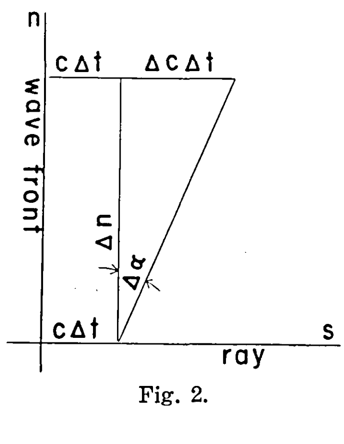

Fig. 2.

$$\Delta s = c \cdot \Delta t \quad \text{or} \quad ds = c \cdot dt ,$$

where

$$c = \sqrt{gH} .$$

Suppose that the wave velocity increases an amount Δc versus an increment Δn along the wave front orthogonal to the ray, it follows from Fig. 2 that

$$\Delta\alpha \doteqdot -\frac{\Delta c \cdot \Delta t}{\Delta n}$$

for an infinitesimal value of $\Delta\alpha$.

Tending Δn to zero, the above expression becomes

$$d\alpha = -\left(\frac{dc}{dn}\right) \cdot dt ,$$

where dc/dn is the gradient of a wave speed along a curved wave front.

The substitution of the relation $ds = c \cdot dt$ into the expression mentioned above yields the equation for the curvature of a ray (K_s)

$$K_s = \frac{d\alpha}{ds} = -\frac{1}{c} \cdot \frac{dc}{dn} , \tag{1}$$

where d/ds denotes differentiation with respect to s along a curved ray.

Differentiation with respect to n can be described as

$$\frac{d}{dn} = \frac{dx}{dn} \cdot \frac{\partial}{\partial x} + \frac{dy}{dn} \cdot \frac{\partial}{\partial y} . \tag{2}$$

Using the angle of α, (2) becomes (see paper I)

$$\frac{d}{dn} = -\sin\alpha \cdot \frac{\partial}{\partial x} + \cos\alpha \cdot \frac{\partial}{\partial y} . \tag{3}$$

From (1) and (3), the curvature of the ray becomes as follows

$$K_s = \frac{d\alpha}{ds} = (\sin\alpha) \cdot \frac{1}{c} \cdot \frac{\partial c}{\partial x} - (\cos\alpha) \cdot \frac{1}{c} \cdot \frac{\partial c}{\partial y} , \tag{4}$$

where

$$c = \sqrt{gH} .$$

As far as the *refraction coefficient* is concerned, this may be calculated as each point along the ray by use of the theory developed by Munk and Arthur[2] for determining the wave intensity along a refracted ray. According to their theory, the equation to determine the wave intensity at any point along the ray is as follows:—

$$\frac{d^2\gamma}{ds^2} + p\cdot\frac{d\gamma}{ds} + q\cdot\gamma = 0\,, \tag{5}$$

where

$$\left.\begin{aligned} p &= -(\cos\alpha)\cdot\frac{1}{c}\cdot\frac{\partial c}{\partial x} - (\sin\alpha)\cdot\frac{1}{c}\cdot\frac{\partial c}{\partial y}\,,\\ q &= -(\sin^2\alpha)\cdot\frac{1}{c}\cdot\frac{\partial^2 c}{\partial x^2} - 2(\sin\alpha\cdot\cos\alpha)\cdot\frac{1}{c}\cdot\frac{\partial^2 c}{\partial x\partial y}\\ &\quad + (\cos^2\alpha)\cdot\frac{1}{c}\cdot\frac{\partial^2 c}{\partial y^2}\,. \end{aligned}\right\} \tag{6}$$

The equation (5) has been solved by two methods, *i.e.*, Kelvin's method of approximating the integral curve by a suitable circular arc, and the finite difference method which transforms the equation (5) into the finite difference equation. According to paper I, the former is superior, in accuracy, to the latter, while the former is inferior, in speed of calculation, to the latter. In this paper, putting an emphasis upon the accuracy of calculation, Kelvin's method is employed.

The curvature of the $l-s$ curve τ is (Fig. 3)

$$K_\tau = \frac{d\theta}{d\tau} = \frac{d\theta}{ds}\Big/\frac{d\tau}{ds}\,. \tag{7}$$

The angle θ the tau curve makes with the s-axis may be defined as one whose tangent is dl/ds. Thus the equation (7) becomes

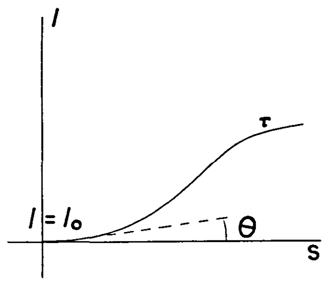

Fig. 3.

$$K_\tau = \frac{d/ds(\tan^{-1} dl/ds)}{\{1+(dl/ds)^2\}^{1/2}} = \frac{\{1+(dl/ds)^2\}^{-1}\, d^2l/ds^2}{\{1+(dl/ds)^2\}^{1/2}}\,. \tag{8}$$

From (5), the equation (8) becomes

$$K_\tau = -(p\tan\theta + ql)\cos^3\theta\,,$$

2) W. H. MUNK and R. R. ARTHUR, *Natl. Bur. Std., U. S. Circ. 521, Gravity Waves*, November 1952.

which may be written as

$$K_\tau = -(p \cdot \sin\theta + ql \cdot \cos\theta)\cos^2\theta\,. \tag{9}$$

Since p and q in (6) are functions of the position and direction of the ray, they may be calculated for any point along a ray.

In order to solve the differential equation, two boundary conditions for θ and γ are required. These conditions take different forms so as to suit the different conditions at the starting points of the rays.

Suppose that the rays start, parallel to each other, in deep water, the values of γ and θ are 1 and 0 respectively. This requires that the evaluation of γ commences in deep water alone and progresses towards shallow water. Such requirements are, for the case of a tsunami, possible only for the *far-field* tsunamis, of which the rays become parallel approximately to each other near the coasts. In the case of the *near-field* tsunamis, these conditions must take different forms from those of the *far-field* tsunamis.

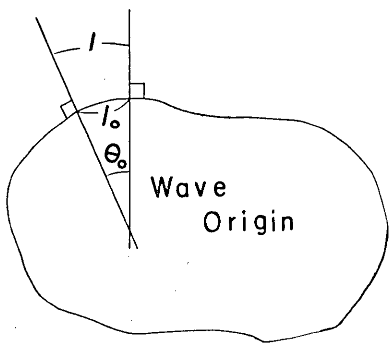

Fig. 4.

When a wave origin is located near the coast, the form of the origin must be taken into consideration in the construction of a refraction diagram.

Suppose that the rays of the *near-field* tsunamis leave, perpendicular to a margin of the wave origin, from the wave source (Fig. 4), the values of γ and θ are given as $1(=l_0/l_0)$ and θ_0 respectively, where θ_0 is an angle of the separation of the neighbouring rays.

Then from Fig. 3, there exists relations

$$dl = ds \cdot \tan\theta_0\,, \tag{10}$$

$$\Delta\tau = (\Delta s^2 + \Delta l^2)^{1/2}\,, \tag{11}$$

along tau curve.

Substituting (10) into (11), the relation, at the starting point,

$$\Delta\tau = (1+\theta_0^2)^{1/2} \cdot \Delta s \tag{12}$$

is obtained, where $\tan\theta_0 \simeq \theta_0$ is used (this approximation is, in general, valid for the two neighbouring rays).

Now, using the above conditions at the starting points, a refraction diagram can be drafted on the map of depth covered by mesh points.

2, 3. Arrival Time and Wave Height. In calculation, a ray is extrapolated by iteration from the n-th to the $(n+1)$-th points by step Δs_0. Δs_0 is a representative length, of which the size may be chosen so as to obtain the most suitable expression of the variation of depth. Then a time necessary to travel the distance Δs_0 is given by $\Delta s_0/c_{av}$ (c_{av} is the average velocity between the n-th and $(n+1)$-th points). If one sums up the values of $\Delta s_0/c_{av}$ along a curved ray from a starting point to the coast, an arrival time of tsunamis is easily calculated.

Provided that energy is transferred, without loss, within a canal composed hypothetically by walls of two neighbouring rays, the conservation of energy yields the following relation[3] :—

$$\frac{\zeta_1}{\zeta_2}=K\cdot\gamma^{1/2}, \tag{13}$$

where

$$K=(l_1/l_2)^{1/2},$$

$$\gamma=\left(\frac{c_1}{c_2}\right)\cdot\left(\frac{1+2k_1H_1\,\mathrm{cosech}\,2k_1H_1}{1+2k_2H_2\,\mathrm{cosech}\,2k_2H_2}\right),$$

k: wave number of a wave,

ζ: wave height,

and the quantities subscripted by 1 and 2 stand for those relevant to the depths H_1 and H_2 respectively.

Since our attention is confined to a problem of a tsunami, the expression (13) becomes, by use of $kH\ll1$,

$$\frac{\zeta_2}{\zeta_1}=\left(\frac{c_1}{c_2}\right)^{1/2}\cdot\left(\frac{l_1}{l_2}\right)^{1/2}, \tag{14}$$

which is the well-known Green's formula.

In the near future, a more generalized form will be employed in the calculation of wave height.

At and beyond the points where the rays intersect with each other, a wave height of a tsunami might not be calculated supposedly by the relation (13) or (14). But these expressions are used, as the first approximation, in this paper.

3) M. HONMA and K. AKI, *Suirigaku* (Iwanami 1962), 503-504, (in Japanese).

2, 4. Programming. The path of a ray is determined through a field of depth by calculating the curvature of the ray and extrapolating a finite distance along the curve of a ray already obtained.

The curvature K_s of the ray, calculated at the initial point, is used to obtain the change of the angle the ray makes with the x-axis when progressing a finite distance along the ray, *i. e.*, from (4),

$$\Delta\alpha = K_s \cdot \Delta s_0 \,, \tag{15}$$

where Δs_0 has a size suitable to depict a variation of the depth, as mentioned in section 2, 3.

A new α_{n+1} in the $(n+1)$-th interval (the interval extrapolated by Δs_0 from the n-th to $(n+1)$-th points is called the (n+1)-th interval) is obtained by adding $\Delta\alpha$ to the initial α_n and a new position along the ray is determined by calculating the change occurring in x and y through the finite distance Δs_0 for a new α_{n+1}, *i. e.*,

$$\alpha_{n+1} = \alpha_n + \Delta\alpha \,, \tag{16}$$

$$\left.\begin{aligned} &\Delta x_0 = \Delta s_0 \cos \alpha_{n+1} \,, \\ &x_{n+1} = x_n + \Delta x_0 \,, \\ &\Delta y_0 = \Delta s_0 \sin \alpha_{n+1} \,, \\ &y_{n+1} = y_n + \Delta y_0 \,, \end{aligned}\right\} \tag{17}$$

where x and y subscripted by n and $n+1$ are related with the points n and $n+1$.

The method of calculation of K_s at the n-th point (which is designated by $(K_s)_n$) is that

$$(K_s)_n = (\sin \alpha_n) \cdot \frac{1}{\tilde{c}_n} \cdot \frac{(\tilde{c}_{n+1})_x - (\tilde{c}_{n-1})_x}{2 \cdot \Delta x} - (\cos \alpha_n) \cdot \frac{1}{\tilde{c}_n} \cdot \frac{(\tilde{c}_{n+1})_y - (\tilde{c}_{n-1})_y}{2 \cdot \Delta y} \,, \tag{18}$$

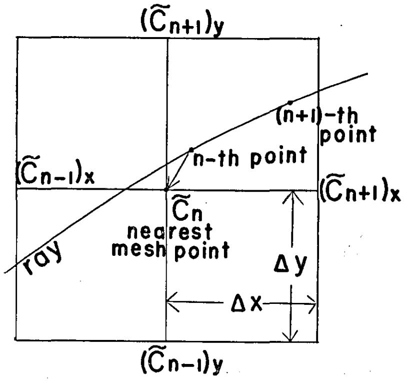

Fig. 5.

where $\tilde{c}_n$ is a velocity at the *nearest* mesh point of the n-th point on the ray (see Fig. 5), $(\tilde{c}_{n+1})_x$ and $(\tilde{c}_{n+1})_y$ the velocities at the adjacent *mesh* points on the positive sides of the n-th *mesh* point (instead of the n-th point on the ray), $(\tilde{c}_{n-1})_x$ and $(\tilde{c}_{n-1})_y$ the velocities at the adjacent *mesh* points on the negative sides, and Δx and Δy the mesh sizes of x and y coordinates.

Firstly, the curvature $(K_s)_n^{n+1}$ of the $(n+1)$-th interval ($(K_s)_n^{n+1}$ is the curvature at the $(n+1)$-th *interval* instead of *point*) is approximated by $(K_s)_n$ of the n-th point to obtain x_{n+1} and y_{n+1} by use of (15)–(17). Secondly, after searching for the nearest mesh point $(n+1)$ of (x_{n+1}, y_{n+1}), the curvature $(K_s)_{n+1}$ at the $(n+1)$-th point is obtained from (18). Thirdly, the curvature obtained in such a way, $(K_s)_{n+1}$, is averaged with $(K_s)_n$ to obtain a corrected $(K_s)_n^{n+1}$ at the $(n+1)$-th interval. Substituting $(K_s)_n^{n+1}$ obtained above into (15) and using (16)–(18), a corrected $(K_s)_n^{n+1}$ is calculated following the aforementioned procedure. Such an iteration is carried on until an appropriate criterion is attained, which is expressed as

$$|\{(K_s)_n^{n+1}\}_1 - \{(K_s)_n^{n+1}\}_2| < \varepsilon \cdot |\{(K_s)_n^{n+1}\}_1| , \tag{19}$$

where $(K_s)_n^{n+1}$ subscripted by 1 and 2 stands for the curvatures before and after iteration respectively; ε the relative error of $(K_s)_n^{n+1}$, which is taken as 0.00001 in this work. But if the mesh size (Δx or Δy) is too large for the mesh map to describe the variation of depth, a loop of the iteration may not exit. Therefore, when a certain number of iterations is repeated (which is taken as 10 in this paper), the $(n+1)$-th point is accepted as the n-th point of the ray to proceed with an estimation of the next point through which the ray passes.

As far as the size of Δs_0 is concerned, no mention is made of this so far. An entire portion of the work uses the value

$$\Delta s_0 = \frac{1}{2}\Delta s = \frac{1}{2}\sqrt{(\Delta x)^2 + (\Delta y)^2} .$$

From the point of view of accuracy, such a value of Δs_0 seems to be suitable for the construction of the ray on the map of the depth of mesh size Δx or Δy.

The curvature $(K_\tau)_n$ evaluated at the n-th point is used to determine the change for an increment $\Delta\tau = (\Delta s_0^2 + \Delta l^2)^{1/2}$ as

$$\Delta\theta = K_\tau \cdot \Delta\tau , \tag{20}$$

(from (7)).

As mentioned in section 2, 2, an increment $\Delta\tau$ at the starting point of the ray is expressed as (12). Except for the use of (12) at the starting point and the calculations of $\Delta\tau$ and $\Delta\theta$, the method of an iteration for K_τ works on the same lines as that of K_s. That is to say that: (i) at the first step of the iteration, by use of an increment of l in the n-th interval (Δl_n), the variation of $\Delta\tau$ in the $(n+1)$-th interval is approximated by

$$\Delta\tau=(\Delta s_0^2+\Delta l_n^2)^{1/2} .$$

(ii) the substitution of $\Delta\tau$ for (20) yields, as the first approximation, the variation of θ in the $(n+1)$-th interval, viz. $\Delta\theta_{n+1}$, where K_τ evaluated at the n-th point of the ray, $(K_\tau)_n$, is used: (iii) by use of the obtained $\Delta\theta_{n+1}$, $\Delta\tau$ and the relations

$$\left.\begin{aligned} \theta_{n+1}&=\theta_n+\Delta\theta_{n+1} , \\ \Delta l_{n+1}&=\Delta\tau\cdot\sin\theta_{n+1} , \\ l_{n+1}&=l_n+\Delta l_{n+1} , \\ \gamma_{n+1}&=l_{n+1}/l_0 , \end{aligned}\right\}$$

(θ_n and l_n, γ_n: the values of θ in the n-th interval and l, γ at the n-th point of the ray), the wave intensity along the ray can be calculated: (iv) this process of the iteration is made parallel with the calculation of the ray, K_s, so that $(K_\tau)_{n+1}$ at the $(n+1)$-th point is obtained through the relation (9) by use of the values of p and q at the $(n+1)$-the point (x_{n+1}, y_{n+1}) which is evaluated on the occasion of the computation of $(K_s)_{n+1}$: (v) the curvature $(K_\tau)_{n+1}$ at the $(n+1)$-th point is averaged with $(K_\tau)_n$ at the n-th point to obtain the averaged curvature in the $(n+1)$-th interval

$$(K_\tau)_n^{n+1}=\frac{1}{2}\cdot\{(K_\tau)_n+(K_\tau)_{n+1}\} :$$

(vi) before proceeding to the next step of the iteration, the criterion

$$|\{(K_\tau)_n^{n+1}\}_1-\{(K_\tau)_n^{n+1}\}_2|<\varepsilon\cdot|\{(K_\tau)_n^{n+1}\}_1|$$

is used to examine whether the obtained $(K_\tau)_n^{n+1}$ falls in the range of the relative error which is determined appropriately beforehand ($(K_\tau)_n^{n+1}$ subscripted by 1 and 2 denote those before and after the iteration. At the first step of the iteration, $(K_\tau)_n^{n+1}$ is approximated by $(K_\tau)_n$.): (vii) when the criterion is satisfied, the process of the iteration is terminated to determine the *refraction factor* as $\gamma_{n+1}^{-1/2}$ using γ_{n+1} of the last iteration and proceed to obtain the refraction factor at the next point through which the ray passes. (viii) after the finish of the iteration, the relative wave height (ζ/ζ_0: ζ and ζ_0 are the wave heights at point and starting point of the ray) at the $(n+1)$-th point is calculated from (14) as

$$\zeta_{n+1}/\zeta_0=(c_0/c_{n+1})^{1/2}\cdot\gamma_{n+1}^{-1/2},$$

where c_0 is the wave velocity at the starting point: (ix) the time necessary for travelling the distance from the starting point to the coast at which the ray is to arrive is calculated following the procedure described in section 2, 3: (x) the processes mentioned above are repeated until the ray reaches the coast.

There may be a better method for programming, but the program is constructed in the fore-going ways in this work. Table 1 represents a type-out of the positions, directions, separations of the ray, the arrival times and wave heights of the waves by an electronic computer.

Table 1. Type-out of the positions, directions, arrival times, separations of the ray, and relative wave heights at any point of the ray by an electronic computer, where ARR. TIME, BETA and W. H. stated in the table stand for arrival time, γ and wave height respectively.

X-COMP. (GRID UNIT)	Y-COMP. (GRID UNIT)	DIRECTION (RADIAN)	ARR.TIME (MIN.)	BETA (NON-DIM.)	W.H. (NON-DIM.)
24.000	13.000	2.662	0.	1.000	1.000
23.290	13.296	2.661	1.166	1.001	0.999
22.581	13.593	2.659	2.333	1.004	0.998
21.873	13.892	2.655	3.493	1.009	0.991
21.166	14.194	2.651	4.636	1.018	0.976
20.460	14.496	2.649	5.745	1.020	0.957
19.755	14.799	2.649	6.838	1.006	0.967
19.046	15.097	2.658	7.959	0.975	1.004
18.329	15.382	2.679	9.167	0.929	1.086
17.601	15.650	2.711	10.444	0.881	1.115
16.862	15.895	2.748	11.802	0.838	1.217
16.113	16.123	2.779	13.331	0.800	1.318
15.360	16.341	2.793	14.997	0.769	1.383
14.609	16.562	2.789	16.786	0.747	1.465
13.860	16.790	2.777	18.813	0.730	1.612
13.114	17.022	2.771	21.026	0.729	1.614
12.359	17.235	2.803	23.462	0.740	1.772
11.588	17.409	2.866	26.172	0.753	1.757
10.804	17.539	2.938	29.347	0.758	2.083
10.006	17.593	3.057	33.838	0.743	2.501
9.321	17.262	3.686	43.033	0.473	7.410

RAY HAS REACHED THE COAST.

37. 津波の屈折図の作製 [I]

地震研究所 桃 井 高 夫

従来，津波の屈折図は人の手によつて作製されていた．これは非常に労力を要し，かつ可成，個人差のある図がかかれる可能性がある．

屈折図を電子計算機によつてかかせる試みは，すでに Griswold により（これは津波に限らず一般の波に対しておこなわれ，津波に対しては少々非能率的なプログラムと考えられる）おこなわれている．

この津波の屈折図に関するプログラム作製は国際協力の課題の一つになつていて，米国の Coast and Geodetic Survey でもプログラムを作ろうとしているようである．そこで筆者は，日本においてもそのプログラムの作製の必要を感じ，国際的見地より IBM 7090 の FORTRAN 言語を用いてプログラミングをおこなつた．プログラムの方法は，まず水深，波の出発時における方向，および隣接する二つの ray のなす角と間隔を電子計算機に入力として与え，出力として ray の座標，方向，隣接する二つの ray 間隔，波高および到達時刻を得る方法である．