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21. *Propagation and Apparent Attenuation of Elastic Waves in a Heterogeneous Medium with Certain Periodic Structures.*

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Abstract

The attenuation of seismic waves propagated in a heterogeneous medium is investigated. Heterogeneity of the medium is represented by the velocity distribution alone and the structure under consideration is expressed in Fourier cosine series. In this proplem the differential equation is of Hill's type, and conditions of instability are discussed. Since the unstable solutions obtained are of the standing waves, some modification and reformation for discussing the stability of the progressive waves are made.

It is concluded that the wave amplitude with wave length λ_n is affected by the structural component with wave length $L_n=\lambda_n/2$ and is independent of the other components, and that the function of the apparent attenuation for the specified wave amplitude is of hyperbolic secant but not of exponential. On the other hand, the phase of this specific wave is shifted by every component of the structure. In addition, the phase shift of waves passing through the medium is negligibly small; the phase velocity is apparently fixed for all frequencies.

A similar problem will arise in that of surface wave propagation along a rough surface.

In the appendix, Hill's equation is solved in forms convenient in discussing the stability of its solution.

1. Introduction

The attenuation of seismic waves has been studied by many authors from various points of view, theoretically and experimentally. Although we have not yet arrived at a satisfactory theory of attenuation, it is probable that the attenuation depends on the frequency of waves propagated. The specific dissipation function $1/Q$ seems to be independent of the frequency in many materials[1] over wide-band frequencies, in other words, the attenuation coefficient is proportional to frequency. However, in a Voigt solid it is proportional to the square of the frequency, whereas in a Maxwell solid it is independent of the frequency. If the earth materials involve any model of combinations of the Voigt and the Maxwell solids, the attenuation coefficient proportional to the frequency cannot be explained. On the other hand, if non-linear stress-strain relations are assumed, the attenuation coefficient is expressed by a rather complicated function[2]. L. Knopoff and G. J. F. MacDonald[3] proposed a non-linear term in the wave equation resulting from the stress and strain rate dependence of the rate of permanent deformation.

H. Jeffreys[4] suggested the dissipation of waves passing through a granular medium, by analogy with his firmoviscous law; theory of a Voigt model. The scattering of elastic waves by a tightly packed assembly of spherical obstacles was calculated by N. Yamakawa[5], who noticed that the effect of scattering should be taken into consideration for the attenuation.

The attenuation of waves passing through the dispersive medium was discussed by G. L. Lamb[6] and W. I. Futterman[7], by analogy with Kramer-Krönig dispersion relation in the electromagnetic field.

Seismic waves are propagated in various kinds of heterogeneous media in the earth. On the earth surface there are a lot of topographic irregularities; mountains, rivers, oceans, and so on. In a recent explosion seismic study[8], it was obtained that even the velocity of the upper mantle of the earth varies from region to region. If we consider the propagation of surface waves, the problem of wave propagation in such

1) e. g., L. KNOPOFF and G. J. F. MACDONALD, "Attenuation of Small Amplitude Stress Waves in Solids," *Rev. Modern Phys.*, **30** (1958), 1178-1192.

J. N. BRUNE, "Attenuation of Dispersed Wave Trains," *Bull. Seism. Soc. Amer.*, **52** (1962), 109-112.

2) e. g., C. LOMNITZ, "Linear Dissipation in Solids," *J. Appl. Phys.*, **28** (1957), 201-205.

3) L. KNOPOFF and G. J. F. MACDONALD, *loc. cit.*, 1).

4) H. JEFFREYS, *The Earth*, 4th ed., (Cambridge, 1959), pp. 107-109.

5) N. YAMAKAWA, "Scattering and Attenuation of Elastic Waves," *Geophys. Mag.*, **31** (1962), 63-95 and 97-103.

6) G. L. LAMB, Jr., "The Attenuation of Waves in a Dispersive Medium," *J. Geophys. Res.*, **67** (1962), 5273-5277.

7) W. I. FUTTERMAN, "Dispersive Body Waves," *J. Geophys. Res.*, **67** (1962), 5279-5291.

8) L. C. PAKISER, "Structure of the Crust and Upper Mantle in the Western United States," *J. Geophys. Res.*, **68** (1963), 5747-5756.

topographic irregularities will mathematically be equivalent to one represented by heterogeneity of the medium. R. Yoshiyama[9] investigated the problem from this point of view, and suggested that the wave with a specified wave length is attenuated according to the heterogeneity of the medium.

S. Homma[10] investigated the propagation of elastic plane waves along the surface with topographic irregularities. He regarded the irregularity as the additional mass loaded on the horizontal plane, and obtained the solution of the disturbance, the condition of resonance and possibility of oscillatory seismograms. According to that treatment, R. Sato[11] developed the problem to the P or SV incident wave of the unit function type in the displacement potential. F. Gilbert and L. Knopoff[12] investigated it following an idea similar to Homma's and gave an integral representation by superposition of the functions arising in the theory of Lamb's problem for isolated surface line sources.

R. Yoshiyama[13] recently treated, in detail, the propagation in a periodic structure with a sinusoidal fluctuation, and concluded that the apparent attenuation resulting from heterogeneity of the medium is not of the exponential function but of hyperbolic secant one. In this paper, the author investigates propagation of elastic waves in a heterogeneous medium in which heterogeneity is expressed in terms of Fourier series.

2. General solutions in a heterogeneous medium with periodic structures

Solutions were obtained by R. Yoshiyama[14] for the wave propagation in a heterogeneous medium where the variation of the wave velocity is periodic. In that calculation, the independent variable, distance, is

9) R. YOSHIYAMA, "Waves through a Heterogeneous Medium," *Zisin*, **13** (1941), 363-366, (in Japanese).

10) S. HOMMA, "On the Effect of Topography on the Surface Oscillation," *Quart. J. Seism.*, **11**, (1941), 349-364; **12** (1942), 17-23 and 24-36, (in Japanese).

S. HOMMA, "On the Effect of Surface Heterogeneity on the Surface Oscillation," *Quart. J. Seism.*, **12** (1942), 37-51, (in Japanese).

11) R. SATO, "On Rayleigh Waves generated at Rough Surfaces (1)," *Zisin*, [ii], **8** (1955), 121-137, (in Japanese).

12) F. GILBERT and L. KNOPOFF, "Seismic Scattering from Topographic Irregularities," *J. Geophys. Res.*, **65** (1960), 3437-3444.

13) R. YOSHIYAMA, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction," *Bull. Earthq. Res. Inst.*, **38** (1960), 467-478.

14) R. YOSHIYAMA, *loc. cit.*, 13).

changed into a variable equivalent to a travel time, and a new function is introduced in a form of the product of the displacement and the square root of the impedance. By doing so, conditions of the stability of the wave are easily discussed in the stability chart of Mathieu's functions.

It will be covenient to give here a brief sketch of R. Yoshiyama's method. For brevity, it is assumed that the velocity of the medium varies in x-dircetion only, and that a wave is propagated in the same direction. The wave equation is given by the form

$$\rho\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right), \tag{1}$$

where ρ and E are the density and the elastic parameter, respectively, which in general are certain functions of x. The velocity of propagation is

$$c(x)=\sqrt{\frac{E}{\rho}}\,. \tag{2}$$

If the displacement u is substituted by a new function

$$\varphi=u\sqrt{\rho c}\,, \tag{3}$$

and variable τ, which is equivalent to the travel time, is defined as

$$\tau=\int\frac{dx}{c(x)}\,, \tag{4}$$

the wave equation is written in the form

$$\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial^2\varphi}{\partial\tau^2}-\alpha^2\varphi\,, \tag{5}$$

where

$$\alpha^2=-\sqrt{\frac{c}{\rho}}\frac{d}{dx}\left(\rho c^2\frac{d}{dx}\frac{1}{\sqrt{\rho c}}\right). \tag{6}$$

Here, α^2 is a function of space coordinate only but not of time.

Now, to study the behaviour of plane waves with a time factor $\exp(i\omega t)$, it is assumed that the density ρ is constant throughout the whole medium, and that the velocity fluctuates slowly in the x-direction with small magnitude. The velocity in such a medium may be expressed by the form

$$c(x)=c_0(1+\sum_{r=1}^{r_0}\varepsilon_r\cos 2r\gamma x)\ , \tag{7}$$

where π/γ is the longest structural wave length involved in the medium under consideration. The magnitude of velocity fluctuation ε is assumed as small. Neglecting all terms of the order ε^2 and higher, the travel time is expressed by the form

$$\tau=\frac{x}{c_0}\left(1-\sum_r\varepsilon_r\frac{\sin 2r\gamma x}{2r\gamma x}\right)\ , \tag{8}$$

which will be approximated by x/c_0, as $(\sin 2r\gamma x)/(2r\gamma x)$ is of the order of magnitude of ε even at small x and is far smaller than unity with the distance travelled. Then, putting z for $\gamma c_0\tau$, equation (5) is written as the form

$$\frac{d^2\varphi}{dz^2}+(\nu^2+2\sum_{r=1}^{r_0}\varepsilon_r r^2\cos 2rz)\varphi=0\ . \tag{9}$$

$$z=\gamma c_0\tau\approx\gamma x\ ,$$

since, by using relation (8),

$$\left|\cos 2r\gamma x-\cos 2rz\right|<\sum_s\varepsilon_s\frac{r}{s}|\sin 2s\gamma x|<|\varepsilon|\ , \tag{10}$$

where ν denotes $\omega/\gamma c_0$ which alone is dependent of the frequency in this equation. This type of differential equation (9) is classified as Hill's equation. So long as r_0 is not large, each coefficient in the summation $\varepsilon_r r^2$ will remain within the order ε, and the solution can be approximated in the following form within the order ε (see Appendix in this paper); when ν is equal to an integer n,

$$\varphi=Ae^{\mu z}y\left(z,\ \frac{\pi}{4}\right)+Be^{-\mu z}y\left(z,\ -\frac{\pi}{4}\right)\ ,$$

where

$$\mu=\frac{1}{2}n\varepsilon_n\ , \tag{11}$$

and

$$y\left(z,\ \pm\frac{\pi}{4}\right)=\sin\left(nz\mp\frac{\pi}{4}\right)+\frac{\varepsilon_n}{8}\sin\left(3nz\mp\frac{\pi}{4}\right)$$
$$+\frac{1}{2}\sum_{r\neq n}\varepsilon_r\left[\frac{r\sin\{(n+2r)z\mp\pi/4\}}{n+r}-\frac{r\sin\{(n-2r)z\mp\pi/4\}}{n-r}\right]\ .$$

On the other hand, when ν is no integer,

$$
\begin{aligned}
\varphi = & A\sin\left(\nu z-\frac{\pi}{4}\right)+B\sin\left(\nu z+\frac{\pi}{4}\right) \\
& +\frac{A}{2}\sum_r \varepsilon_r\left[\frac{r\sin\{(\nu+2r)z-\pi/4\}}{\nu+r}-\frac{r\sin\{(\nu-2r)z-\pi/4\}}{\nu-r}\right] \\
& +\frac{B}{2}\sum_r \varepsilon_r\left[\frac{r\sin\{(\nu+2r)z+\pi/4\}}{\nu+r}-\frac{r\sin\{(\nu-2r)z+\pi/4\}}{\nu-r}\right]. \qquad (12)
\end{aligned}
$$

It follows from these solutions that periods of the wave affected by each component of heterogeneity are specified: this condition is given by the form

$$\omega_n = c_0 n\gamma\ , \qquad (13)$$

or, stated with respect to the wave length of the specified wave λ_n,

$$\lambda_n = 2\pi/n\gamma\ .$$

Since $\pi/n\gamma$ corresponds to the n-th structural wave length L_n, λ_n is equal to $2L_n$;

$$\lambda_n = 2L_n\ . \qquad (14)$$

If n is specified, these relations agree with those deduced by R. Yoshiyama[15] who treated the medium with a periodic structure that fluctuates regularly.

3. Propagation of waves in a heterogeneous medium

The expressions of the displacement in the heterogeneous medium obtained in the preceding paragraph give that of a standing wave but not of a progressive wave. Therefore, to study the wave propagation, some modification and reformation seem to be necessary. Moreover to study the stability of a progressive wave, the case in which ν is equal to a certain integer is discussed in detail.

It is assumed in this paragraph, that a heterogeneous medium with width x_0 is inserted between two identically homogeneous media, and that its heterogeneity is expressed by Fourier series.

The displacement u and stress S in a medium may be stated in terms of the matrix equations, omitting the common time factor exp($i\omega t$), as follows:

15) R. YOSHIYAMA, *loc. cit.*, 13).

$$\begin{pmatrix} u_1 \\ S_1 \end{pmatrix} = \begin{pmatrix} \dfrac{\exp(-ik_1x)}{\sqrt{\rho c_1}}, & \dfrac{\exp(ik_1x)}{\sqrt{\rho c_1}} \\ -i\omega\sqrt{\rho c_1}\exp(-ik_1x), & i\omega\sqrt{\rho c_1}\exp(ik_1x) \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \equiv \boldsymbol{M}_1(x) \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \qquad \text{for } x \leq 0,$$

$$\begin{pmatrix} u_2 \\ S_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\exp(\mu z)y(z, \pi/4)}{\sqrt{\rho c_2}}, & \dfrac{\exp(-\mu z)y(z, -\pi/4)}{\sqrt{\rho c_2}} \\ \omega\sqrt{\rho c_2}\exp(\mu z)F\left(z, \dfrac{\pi}{4}\right), & \omega\sqrt{\rho c_2}\exp(-\mu z)F\left(z, -\dfrac{\pi}{4}\right) \end{pmatrix}$$

$$\times \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \equiv \boldsymbol{M}_2(x) \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \qquad \text{for } 0 \leq x \leq x_0, \tag{15}$$

and

$$\begin{pmatrix} u_3 \\ S_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\exp\{-ik_3(x-x_0)\}}{\sqrt{\rho c_3}} \\ -i\omega\sqrt{\rho c_3}\exp\{-ik_3(x-x_0)\} \end{pmatrix} A_3 \equiv \boldsymbol{v}_3(x) A_3, \qquad \text{for } x_0 \leq x,$$

where k is the wave number in the homogenous medium, ω/c, and ρ is assumed as constant throughout the whole medium; z and $y(z, \pm\pi/4)$ are given by expressions (9) and (11) or (12) of the preceding paragraph, respectively, and $F(z, \pm\pi/4)$ are expressed by the form

$$F\left(z, \pm\frac{\pi}{4}\right) = \frac{\gamma c_0}{\omega}\left\{\frac{dy(z, \pm\pi/4)}{dz} + \left(\pm\mu - \frac{1}{2c}\frac{dc}{dz}\right) y\left(z, \pm\frac{\pi}{4}\right)\right\}. \tag{16}$$

In these expressions (15) and (16), μ should be taken as zero for the case in which ν tends to no integer. In the first medium, A_1 and B_1 are connected with amplitudes of incident and reflected waves, respectively, and A_3 in the last medium gives the amplitude of the transmitted wave. A_2 and B_2 in the intermediate heterogeneous medium give the amplitudes of the attenuated standing waves in the incident and reverse directions, respectively.

Equating the adjacent matrices at each boundary, we have equations

$$\boldsymbol{M}_1(0)\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \boldsymbol{M}_2(0)\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \boldsymbol{M}_1^{-1}(0)\boldsymbol{M}_2(0)\begin{pmatrix} A_2 \\ B_2 \end{pmatrix},$$

and

$$\boldsymbol{M}_2(x_0)\begin{pmatrix} A_2 \\ B^2 \end{pmatrix} = \boldsymbol{v}_3(x_0)A_3, \quad \text{or} \quad \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \boldsymbol{M}_2^{-1}(x_0)\boldsymbol{v}_3(x_0)A_3,$$

where M^{-1} denotes the inverse of matrix M. Therefore, combining them, the relation between the amplitudes of the incident and reflected waves is expressed in terms of the amplitude of the transmitted wave, as follows:

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_1^{-1}(0) M_2(0) M_2^{-1}(x_0) v_3(x_0) A_3 \equiv N A_3 .$$

Substituting the solutions (11) or (12) and (15) into these matrices, the following expression is obtained;

$$N = \begin{pmatrix} \dfrac{i}{2D}\left\{e^{-\mu z_0} f_{32}\left(z_0, -\dfrac{\pi}{4}\right) g_{12}\left(0, \dfrac{\pi}{4}\right) - e^{\mu z_0} f_{32}\left(z_0, \dfrac{\pi}{4}\right) g_{12}\left(0, -\dfrac{\pi}{4}\right)\right\} \\ \dfrac{-i}{2D}\left\{e^{-\mu z_0} f_{32}\left(z_0, -\dfrac{\pi}{4}\right) f_{12}\left(0, \dfrac{\pi}{4}\right) - e^{\mu z_0} f_{32}\left(z_0, \dfrac{\pi}{4}\right) f_{12}\left(0, -\dfrac{\pi}{4}\right)\right\} \end{pmatrix}, \tag{17}$$

where

$$D = |M_2(x_0)| \sqrt{c_{12} \cdot c_{32}} = \left\{ y\left(z_0, \frac{\pi}{4}\right) F\left(z_0 - \frac{\pi}{4}\right) - y\left(z_0, -\frac{\pi}{4}\right) F\left(z_0, \frac{\pi}{4}\right) \right\} \sqrt{c_{12} \cdot c_{32}} ,$$

$$\left.\begin{aligned} f_{j2}\left(\zeta, \pm\frac{\pi}{4}\right) &= F\left(\zeta, \pm\frac{\pi}{4}\right) + i c_{j2} y\left(\zeta, \pm\frac{\pi}{4}\right) \\ g_{j2}\left(\zeta, \pm\frac{\pi}{4}\right) &= F\left(\zeta, \pm\frac{\pi}{4}\right) - i c_{j2} y\left(\zeta, \pm\frac{\pi}{4}\right) \end{aligned}\right\} \quad \begin{pmatrix} \zeta = 0, & z_0 \\ j = 1, & 3 \end{pmatrix},$$

and

$$c_{12} = c_1 / c_2(0) , \quad c_{32} = c_3 / c_2(z_0) .$$

Since heterogeneity of the medium is expressed in Fourier series, γx_0 or approximately z_0 is equal to π, that is, $\gamma x_0 = z_0 = \pi$, and the velocity and its gradient are continuous at each boundary. Thence, the solutions $y(z_0, \pm\pi/4)$ and $F(z_0, \pm\pi/4)$ are simply expressed as follows, remembering $\exp(2inz_0)$ as being equal to unity;

$$y\left(z_0, \pm\frac{\pi}{4}\right) = \sin\left(nz_0 \mp \frac{\pi}{4}\right)\left\{1 + \frac{\varepsilon_n}{8} + \sum_{r \neq n} \frac{r^2 \varepsilon_r}{n^2 - r^2}\right\} ,$$

and

$$F\left(z_0, \pm\frac{\pi}{4}\right) = \cos\left(nz_0 \mp \frac{\pi}{4}\right)\left\{1 - \frac{\varepsilon_n}{8} - \sum_{r \neq n} \frac{r^2 \varepsilon_r}{n^2 - r^2}\right\} .$$

Therefore, the expression of the matrix N is given by the form

$$N=\begin{pmatrix} \exp(inz_0)\left\{\cosh\mu z_0-i\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{r^2\varepsilon_r}{n^2-r^2}\right)\sinh\mu z_0\right\} \\ -i\exp(inz_0)\sinh\mu z_0 \end{pmatrix}. \tag{18}$$

Now, the complex transmission and reflection coefficients T and R, are defined as the ratio of the amplitudes of waves passing through and reflected from the heterogeneous medium to that of incident waves, respectively, and are expressed by the formulae

$$T=\frac{\exp(-inz_0)}{\cosh\mu z_0-i\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{r^2\varepsilon_r}{n^2-r^2}\right)\sinh\mu z_0}$$
$$=\frac{\exp(-inz_0)}{\cosh\mu z_0}\exp\left\{i\tan^{-1}\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{r^2\varepsilon_r}{n^2-r^2}\right)\tanh\mu z_0\right\},$$

and

$$R=\frac{-i\sinh\mu z_0}{\cosh\mu z_0-i\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{r^2\varepsilon_r}{n^2-r^2}\right)\sinh\mu z_0}$$
$$=\tanh\mu z_0\cdot\exp\left\{-i\frac{\pi}{2}+i\tan^{-1}\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{r^2\varepsilon_r}{n^2-r^2}\right)\tanh\mu z_0\right\}.$$

As the argument of arctangent in phases is of the small order, these are approximated by the forms;

$$T=\frac{\exp(-inz_0)}{\cosh\mu z_0}\exp\left\{i\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{r^2\varepsilon_r}{n^2-r^2}\right)\tanh\mu z_0\right\},$$

and (19)

$$R=\tanh\mu z_0\exp\left\{-i\frac{\pi}{2}+i\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{r^2\varepsilon_r}{n^2-r^2}\right)\tanh\mu z_0\right\},$$

where

$$\mu=\frac{1}{2}n\varepsilon_n.$$

If all the ε_r's for r besides an integer n vanish, these formulae agree with those obtained by R. Yoshiyama[16].

The formulae (19) are interpreted as follows: the modulus of them

16) R. YOSHIYAMA, *loc. cit.*, 13).

depends only on the n-th component of the structure, whereas the phase angle of them depends on all its components. This dependence of the modulus is important in the interpretation of wave propagation. If heterogeneity of the medium is expressed in Fourier series, each component of the structure makes the amplitude of the specified wave alone attenuate, the wave length of which is associated with twice the structural wave length under consideration. As to the phase, every component of the structure which is of the order of the fluctuation should be taken into account as corrections. The result on the modulus of them agrees with those deduced by R. Yoshiyama, while one on the phase is slightly different from that deduction.

In particular, when the factor μ is negligibly small or zero, these coefficients are easily obtained as follows;

$$T=\exp(-i\nu z_0), \quad \text{and} \quad R=0.$$

Therefore, the waves are transmitted in the medium without any modification and no reflection of waves appears.

In every case, the conservation of energy flux holds within the incident, transmitted and reflected waves, since

$$|T|^2+|R|^2=\operatorname{sech}^2\mu z_0+\tanh^2\mu z_0=1.$$

4. Stability of progressive waves and some remarks upon them

It is noticeable that the transmission and reflection coefficients representing the effect of heterogeneity of the medium depend on the wave length, and that this dependence is a function of the fluctuation of each component whose structural wave length is equal to half a wave length of the wave. The maximum apparent attenuation to the n-th component may be written as $\operatorname{sech}(\mu z_0)$. The value μz_0 is equal to $(\varepsilon_n/2)\gamma_n x_0$, where γ_n is $n\gamma$, and may also be written as $(\varepsilon_n/2)(x_0/c_0)\omega_n$ from relation (13). As a consequence of Fourier analysis of the structure, the thickness of the heterogeneous medium x_0 is taken as equal to half a wave length of the wave with the frequency ω_1 which is regarded as the fundamental one. Therefore, the frequency ω_n corresponds to that of the n-th higher mode.

The case in which ν is nearly equal to n is considered. From equation (9) and relations (A–8), we see that

$$\nu = n\left(1 + \frac{1}{2}\varepsilon_n \cos 2\sigma\right),$$

and

$$\mu' = \frac{1}{2} n\varepsilon_n \sin 2\sigma,$$

where the prime is added in order to avoid confusion. Referring that ν is $\omega/\gamma c_0$ and multiplying z_0 in the lower expression, the following relations are derived, respectively,

$$\omega = \omega_n\left(1 + \frac{1}{2}\varepsilon_n \cos 2\sigma\right),$$

and (20)

$$\mu' z_0 = \frac{1}{2}\varepsilon_n \frac{x_0}{c_0}\omega_n \sin 2\sigma \equiv \mu z_0 \sin 2\sigma.$$

Waves with frequencies between $\omega_n(1 \pm \varepsilon_n/2)$ are made to attenuate, and in these frequency ranges the transmission and reflection coefficients (19) should be written in the forms

$$T = \frac{\exp\left\{-inz_0 + i(2\varepsilon - \cos 2\sigma)\dfrac{\tanh \mu' z_0}{\sin 2\sigma}\right\}}{\cosh \mu' z_0 \left\{1 - \cos 2\sigma(4\varepsilon - \cos 2\sigma)\dfrac{\tanh^2 \mu' z_0}{\sin^2 2\sigma}\right\}},$$

and

$$R = \frac{(1 - 2\varepsilon \cos 2\sigma)}{\cosh \mu' z_0} \frac{\sinh \mu z_0}{\sin 2\sigma} \frac{\exp\left\{-i\dfrac{\pi}{2} + i(2\varepsilon - \cos 2\sigma)\dfrac{\tanh \mu' z_0}{\sin 2\sigma}\right\}}{\left\{1 - \cos 2\sigma(4\varepsilon - \cos 2\sigma)\dfrac{\tanh^2 \mu' z_0}{\sin^2 2\sigma}\right\}},$$

where ε denotes $\varepsilon_n/8 + \sum \varepsilon_r r^2/(n^2 - r^2)$, for brevity. If the order of $(\mu' z_0)^2$ is neglected, these formulae are approximated by the forms

$$T = \frac{\exp(-inz_0)}{\cosh(\mu z_0 \sin 2\sigma)} \exp\{i\mu z_0(2\varepsilon - \cos 2\sigma)\},$$

and (21)

$$R = \frac{(1 - 2\varepsilon \cos 2\sigma)}{\cosh(\mu z_0 \sin 2\sigma)} \cdot \mu z_0 \exp\left\{-i\frac{\pi}{2} + i\mu z_0(2\varepsilon - \cos 2\sigma)\right\}.$$

Therefore, the resultant relation of transmitted waves between amplitudes and frequencies forms sequences of trough shape compared with the incident wave spectrum, whose mean frequencies are ω_n, depths $(\varepsilon_n/2)(x_0/c_0)\omega_n$ and widths $\varepsilon_n\omega_n$: the higher the frequency, the wider and deeper becomes the apparent attenuation. If a wave with a certain spectrum is incident, the magnitude of this incident one is reduced in the resultant spectrum of the transmitted wave by the amount

$$\operatorname{sech}\left(\frac{1}{2}\,\varepsilon_n\frac{x_0}{c_0}\,\omega_n\sin 2\sigma\right), \tag{22}$$

at frequencies $\omega=\omega_n\left(1+\frac{1}{2}\,\varepsilon_n\cos 2\sigma\right)$, for each n, where σ ranges from 0 to $\pi/2$. As to the reflected waves, the disturbance gives rise to the same frequencies as the transmitted ones, but their mangitudes are smaller than $\mu' z_0$. At a frequency off that to the maximum attenuation, the reflection coefficient is dependent upon the fluctuation of every component of the structure.

Next, the following phase relation is discussed;

$$-nz_0\left\{1-(2\varepsilon-\cos 2\sigma)\frac{\varepsilon_n}{2}\frac{\tanh\mu' z_0}{\mu' z_0}\right\}. \tag{23}$$

The second term in the brackets is of the same order as the square of the fluctuation, because $(\tanh\mu' z_0)/(\mu' z_0)$ takes the maximum unity, at $\mu' z_0=0$ (or $\sin 2\sigma=0$), and with increase of $\mu' z_0$, it vanishes rapidly. From these discussions, it is concluded that the phase is transmitted approximately with the velocity c_0, and that the dispersion due to heterogeneity will be undetectable. However, the phase nz_0, equal to $n\pi$, will not be detectable, so that the phase shift

$$\left(\frac{\varepsilon_n}{4}+2\sum_{r\neq n}\frac{\varepsilon_r r^2}{n^2-r^2}-\cos 2\sigma\right)\frac{\varepsilon_n}{2}\frac{x_0}{c_0}\omega_n \tag{24}$$

cannot be neglected though small. As $r^2/(n^2-r^2)$ becomes large only when r approaches to n, it is sufficient to take a few components close to the n-th into account.

In the assumption, r_0 is not large; in other words, heterogeneity of the medium includes no components with steep variation. However, the dissipation of the wave with a certain frequency depends on the specified component of the structure, so that the wave affected by the components of the large r will be of the high frequencies. Substituting $m\omega_1$ for the frequency ω, where ω_1 is the fundamental frequency, the differential equation is written in the form

$$\frac{d^2\varphi}{dz^2}+\left(m^2\frac{\omega_1}{\gamma^2 c_0{}^2}+2m^2\sum_r \varepsilon_r \frac{r^2}{m^2}\cos 2rz\right)\varphi=0\,. \tag{25}$$

If m is large, $\varepsilon_r(r^2/m^2)\cos 2rz$ is significant only for large r. Substituting $\varepsilon_r{}'$ for $\varepsilon_r(r^2/m^2)$, $\varepsilon_r{}'$ may be regaded as of the order equivalent to ε_r, and the unstable condition for the above equation is given, referring to ω_1 equal to γc_0, by the relations

$$\mu=\frac{m^2}{n}\left\{\frac{1}{2}\varepsilon_n{}'+\left(\frac{m}{n}\right)^2 O(\varepsilon_n{}'^2)\right\}\,,$$

and (26)

$$m^2=n^2\left\{1+\left(\frac{m}{n}\right)^4 O(\varepsilon_n{}'^2)\right\}\,.$$

From the latter condition, m should be taken as equal to n, and then μ is written as $n\varepsilon_n/2$, which is the same expression as that obtained under the restriction that r_0 is not large. As to the phase shift, the above-stated discussions are described in the same way, but its magnitude is greater.

If two periodicities of heterogeneity are predominant, the structural wave numbers corresponding to them are given by γ_1 and γ_2, respectively, and any linear combinations of them

$$\gamma=m\gamma_1+n\gamma_2 \qquad (m,\, n=0,\, \pm 1,\, \pm 2, \cdots)$$

also are the structural wave numbers of heterogeneity. As a consequence of Fourier analysis of the structure, γ_2/γ_1 is confined to that which is real and rational, so that it can be taken as being equal to q/p, where p and q are any mutually prime integers. In this case, there is a pair of integers m and n which satisfy the condition that $mp+nq=1$ and then the wave number γ_0 combined by the relation

$$\gamma_0=\gamma_1\left(m+n\frac{q}{p}\right)=\frac{\gamma_1}{p}=\frac{\gamma_2}{q} \tag{27}$$

is regarded as the fundamental one, from which two wave numbers γ_1 and γ_2 can be determined.

Developing these discussions, we can ascertain the behaviour of the transmitted and reflected waves in a heterogeneous medium.

Since hyperbolic secant functions are approximated as

$$\operatorname{sech} x = \exp(-x^2/2)\{1+O(x^4)\} \quad \text{for small } x$$

and (28)

$$\operatorname{sech} x = 2e^{-x}\{1+O(e^{-2x})\} \quad \text{for large } x\,,$$

the apparent attenuation resulting from heterogeneity of the medium seems to be proportional to the frequency for the high frequency and to the square of the frequency for the low frequency.

5. Summary

In a previous paper[17], the wave propagation in a heterogeneous medium with the alternately stratified structures consisting of a homogeneous and a heterogeneous media was investigated, and it was suggested that the total effect of the apparent attenuation is approximately additive. In this paper, the heterogeneity is expressed by Fourier cosine series. The wave equation in such a medium is given by Hill's equation after some substitution. It follows from the condition of the most unstable solution that the wave amplitude with the wave length λ_n is affected by the structural component with the wave length $\lambda_n/2$ and is independent of the other components.

The expressions of these solutions obtained are of the standing waves, so that some modification and reformation are taken in order to obtain one of the progressive waves—the transmission coefficient. The modulus of the transmission coefficient is of hyperbolic secant function but not of exponential, and that of the reflection coefficient is of hyperbolic tangent. This type of wave reflection follows from the apparent attenuation of waves passing through a heterogeneous medium.

From the phase transmission it is concluded that the dependence of the frequency on the velocity is negligibly small, i. e., the dispersion with respect to the velocity is hardly anticipated. However, the phase shift cannot be neglected though small. In addition, the phase of reflected waves is shifted by $-\pi/2$. The same phase shift has been found in the internal reflection of elastic waves by H. Jeffreys[18], and others.

If n is specified, the modulus of transmitted and reflected waves

17) I. ONDA, "Effect of the Intermediate Dissipative Medium on the Transmission of Elastic Waves through a Heterogeneous Medium having Periodic Structures," *Bull. Earthq. Res. Inst.*, **42** (1964), 11-18.

18) H. JEFFREYS, "Elastic Waves in a Continuously Stratified Medium," *Mon. Not. Roy. Astr. Soc., Geophys. Suppl.*, **7** (1957), 332-337.

and the constant phase shift of reflected waves agree with Yoshiyama's deductions[19] from solutions on a medium with a sinusoidal fluctuation, while the phase shift resulting from heterogeneity is slightly different from that. These agreements are suggested by the superposition of Yoshiyama's solution because the treated equations are both linear.

Since the thickness of the heterogeneous medium is taken as equal to half a wave length of the frequency which can be regarded as the fundamental one, the quantities connected with the n-th term correspond to those of the n-th higher mode. The wave equation in this problem becomes inaccurate with the increase of the mode number. Under the assumption that the short wave lengths of the structure affect waves with higher modes only, the conclusions stated-above are established except for the magnitude of the phase shift: the short waves are more complicated in the phase spectrum.

Dependence of the apparent attenuation on the frequency and the distance travelled is proportional to those for the short waves and to the square of those for the long waves, as a consequence of behaviour of hyperbolic secant functions.

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Appendix; General Solutions of Hill's Equation

Hill's equation is written in the form

$$\frac{d^2\varphi}{dz^2}+(\theta_0+2\sum_r \theta_r \cos 2rz)\varphi=0\,, \tag{A-1}$$

solutions of which have been obtained in the various forms. According to Floquet's theorem, one solution is written in the form

$$\varphi=e^{\mu z}y\,, \tag{A-2}$$

where μ is a constant and y is a periodic function.

Strictly speaking, the factor μ is obtained by solving an infinite

19) R. YOSHIYAMA, *loc. cit.*, 13).

determinant. E. T. Whittaker[1], however, introduced the expression in terms of the series expansion convenient to the numerical calculation of Mathieu's equation, and E. L. Ince[2] who was interested in the problems of the lunar perigee, generalized this method to Hill's equation, in which quantity θ_0 approaches values 0, 1 and 4 but not the others. In our investigation, quantity θ_0 can approach any values particularly squares of any integers, and it is not confined that $\theta_1 > \theta_2 > \cdots > \theta_r > \theta_{r+1} > \cdots$. The solutions of Hill's equation in which θ_0 approaches an arbitrary value are obtained in this Appendix, according to their method.

Substituting expression (A-2) into equation (A-1), we have the differential equation

$$\frac{d^2y}{dz^2} + 2\mu\frac{dy}{dz} + (\theta_0 + \mu^2 + 2\sum\theta_r \cos 2rz)y = 0\,. \tag{A-3}$$

Now, let the factor μ and the solution y be expanded, respectively, in the series of θ_r's, using a new parameter σ;

$$\mu = \sum_k \theta_k p_k(\sigma) + \sum_{k,l} \theta_k\theta_l p_{kl}(\sigma) + \sum_{k,l,m} \theta_k\theta_l\theta_m p_{klm}(\sigma) + \cdots\,, \tag{A-4}$$

and

$$y = \sin(\nu z - \sigma) + \sum_k \theta_k A_k(z,\sigma) + \sum_{k,l} \theta_k\theta_l A_{kl}(z,\sigma) + \sum_{k,l,m} \theta_k\theta_l\theta_m A_{klm}(z,\sigma) + \cdots\,, \tag{A-5}$$

σ being determined by the relation

$$\theta_0 = \nu^2 + \sum_k \theta_k q_k(\sigma) + \sum_{k,l} \theta_k\theta_l q_{kl}(\sigma) + \sum_{k,l,m} \theta_k\theta_l\theta_m q_{klm}(\sigma) + \cdots\,. \tag{A-6}$$

Substituting these relations into equation (A-3), it is expressed by a form of power series of θ, and the condition under which this solution must identically satisfy all the values of θ is given by equating coefficients of each term to zero. In addition, the solution y and therefore $A_k(z,\sigma)$, $A_{kl}(z,\sigma)$, etc. must be periodic functions, respectively.

As a result, the following expressions are obtained:

Terms in θ_k;

$$\frac{d^2A_k}{dz^2} + \nu^2 A_k + 2\nu p_k \cos(\nu z - \sigma) + q_k \sin(\nu z - \sigma) + 2\cos 2kz \sin(\nu z - \sigma) = 0\,.$$

1) E. T. WHITTAKER, "On the General Solution of Mathieu's Equation," *Proc. Edin. Math. Soc.*, **32** (1914), 75-80.

2) E. L. INCE, "On a General Solution of Hill's Equation," *Mon. Not. Roy. Astr. Soc.*, **75** (1915), 436-446; **76** (1916), 431-442 and **78** (1917), 141-147.

i) $\nu=k$. $p_k=\dfrac{\sin 2\sigma}{2k}$, $q_k=\cos 2\sigma$,

$$A_k=\frac{\sin(3kz-\sigma)}{8k^2}.$$

ii) $\nu\neq k$. $p_k=q_k=0$,

$$A_k=\frac{\sin\{(\nu+2k)z-\sigma\}}{4k(\nu+k)}-\frac{\sin\{(\nu-2k)z-\sigma\}}{4k(\nu-k)}.$$

Terms in θ_k^2;

$$\frac{d^2A_{kk}}{dz^2}+\nu^2A_{kk}+2\nu p_{kk}\cos(\nu z-\sigma)+2p_k\frac{dA_k}{dz}$$
$$+(q_{kk}+p_k^2)\sin(\nu z-\sigma)+(q_k+2\cos 2kz)A_k=0.$$

i) $\nu=k$. $p_{kk}=0$, $q_{kk}=\dfrac{\cos 4\sigma-2}{8k^2}$,

$$A_{kk}=\frac{\sin(5kz-\sigma)}{192k^4}+\frac{3\sin 2\sigma\cos(3kz-\sigma)}{64k^4}+\frac{\cos 2\sigma\sin(3kz-\sigma)}{64k^4}.$$

ii) $\nu=2k$. $p_{kk}=-\dfrac{\sin 2\sigma}{16k^3}$, $q_{kk}=\dfrac{2-3\cos 2\sigma}{12k^2}$,

$$A_{kk}=\frac{\sin(6kz-\sigma)}{384k^4}.$$

iii) $\nu\neq k,\ 2k$. $p_{kk}=0$, $q_{kk}=\dfrac{1}{2(\nu^2-k^2)}$,

$$A_{kk}=\frac{\sin\{(\nu+4k)z-\sigma\}}{32k^2(\nu+k)(\nu+2k)}+\frac{\sin\{(\nu-4k)z-\sigma\}}{32k^2(\nu-k)(\nu-2k)}.$$

Terms in $\theta_k\theta_l$;

$$\frac{d^2A_{kl}}{dz^2}+\nu^2A_{kl}+2\nu p_{kl}\cos(\nu z-\sigma)+2p_k\frac{dA_l}{dz}+2p_l\frac{dA_k}{dz}$$
$$+(q_{kl}+2p_kp_l)\sin(\nu z-\sigma)+q_kA_l+q_lA_k+2\cos 2kzA_l+2\cos 2lzA_k=0.$$

i) $\nu=k$, $l=2k$. $p_{kl}=\dfrac{\sin 2\sigma}{8k^3}$, $q_{kl}=\dfrac{\cos 2\sigma}{4k^2}$,

$$A_{kl}=\frac{\sin(7kz-\sigma)}{288k^4}+\frac{\sin 2\sigma\cos(5kz-\sigma)}{288k^4}-\frac{\cos 2\sigma\sin(5kz-\sigma)}{288k^4}$$
$$-\frac{\sin 4\sigma\cos(3kz-\sigma)}{32k^4}+\frac{(2-3\cos 4\sigma)\sin(3kz-\sigma)}{96k^4}.$$

ii) $\nu=k$, $l\neq 2k$. $p_{kl}=q_{kl}=0$,

$$A_{kl}=\frac{(2k^2+kl+l^2)\sin\{(3k+2l)z-\sigma\}}{32k^2l(k+l)^2(2k+l)}+\frac{(k^2+kl-l^2)\sin 2\sigma\cos\{(k+2l)z-\sigma\}}{8(k-l)kl^2(k+l)^2}$$

$$+\frac{k\cos 2\sigma\sin\{(k+2l)z-\sigma\}}{8(k-l)l^2(k+l)^2}+\frac{(k^2-kl-l^2)\sin 2\sigma\cos\{(k-2l)z-\sigma\}}{8(k-l)^2kl^2(k+l)}$$

$$+\frac{k\cos 2\sigma\sin\{(k-2l)z-\sigma\}}{8(k-l)^2l^2(k+l)}-\frac{(2k^2-kl+l^2)\sin\{(3k-2l)z-\sigma\}}{32k^2l(k-l)^2(2k-l)}.$$

iii) $\nu=k-l$, $\nu\neq k, l$. $p_{kl}=\dfrac{\sin 2\sigma}{4kl(k-l)}$, $q_{kl}=\dfrac{\cos 2\sigma}{2kl}$,

$$A_{kl}=\frac{\sin\{(3k+l)z-\sigma\}}{16kl(k+l)(2k-l)}+\frac{\sin\{(k+3l)z+\sigma\}}{16kl(k+l)(k-2l)}$$

$$-\frac{(k^2-kl+l^2)\sin\{3(k-l)z-\sigma\}}{16kl(k-2l)(2k-l)(k-l)^2}.$$

iv) $\nu=k+l$. $p_{kl}=\dfrac{\sin 2\sigma}{4kl(k+l)}$, $q_{kl}=\dfrac{\cos 2\sigma}{2kl}$,

$$A_{kl}=\frac{(k^2+kl+l^2)\sin\{3(k+l)z-\sigma\}}{16kl(k+l)^2(k+2l)(2k+l)}$$

$$+\frac{\sin\{(-k+3l)z-\sigma\}}{16kl(k-l)(k+2l)}-\frac{\sin\{(3k-l)z-\sigma\}}{16kl(k-l)(2k+l)}.$$

v) $\nu\neq k, l$, $k+l$, $k-l$. $p_{kl}=q_{kl}=0$,

$$A_{kl}=\frac{\{\nu(k+l)+k^2+l^2\}\sin\{(\nu+2k+2l)z-\sigma\}}{16kl(k+l)(\nu+k)(\nu+l)(\nu+k+l)}$$

$$-\frac{\{\nu(k-l)-(k^2+l^2)\}\sin\{(\nu-2k+2l)z-\sigma\}}{16kl(k-l)(\nu-k)(\nu+l)(\nu-k+l)}$$

$$-\frac{\{\nu(k-l)+(k^2+l^2)\}\sin\{(\nu+2k-2l)z-\sigma\}}{16kl(k-l)(\nu+k)(\nu-l)(\nu+k-l)}$$

$$+\frac{\{\nu(k+l)-(k^2+l^2)\}\sin\{(\nu-2k-2l)z-\sigma\}}{16kl(k+l)(\nu-k)(\nu-l)(\nu-k-l)}.$$

Terms in θ_k^3;

$$\frac{d^2A_{kkk}}{dz^2}+\nu^2A_{kkk}+2\nu p_{kkk}\cos(\nu z-\sigma)+2p_{kk}\frac{dA_k}{dz}+2p_k\frac{dA_{kk}}{dz}+q_{kkk}\sin(\nu z-\sigma)$$

$$+2p_kp_{kk}\sin(\nu z-\sigma)+(q_{kk}+p_k^2)A_k+q_kA_{kk}+2\cos 2kzA_{kk}=0.$$

i) $\nu=k$. $p_{kkk}=-\dfrac{3\sin 2\sigma}{128k^5}$, $q_{kkk}=-\dfrac{\cos 2\sigma}{64k^4}$,

$$A_{kkk}=\frac{\sin(7kz-\sigma)}{9216k^6}+\frac{7\sin 2\sigma\cos(5kz-\sigma)}{2304k^6}+\frac{\cos 2\sigma\sin(5kz-\sigma)}{1152k^6}$$
$$+\frac{3\sin 4\sigma\cos(3kz-\sigma)}{512k^6}+\frac{(15\cos 4\sigma-14)\sin(3kz-\sigma)}{1536k^6}.$$

ii) $\nu=2k$. $p_{kkk}=q_{kkk}=0$,

$$A_{kkk}=\frac{\sin(8kz-\sigma)}{23040k^6}-\frac{\sin 2\sigma\cos(4kz-\sigma)}{288k^6}$$
$$+\frac{(19-24\cos 2\sigma)\sin(4kz-\sigma)}{13824k^6}-\frac{(2-3\cos 2\sigma)\sin\sigma}{192k^6}.$$

iii) $\nu=3k$. $p_{kkk}=\dfrac{\sin 2\sigma}{384k^5}$, $q_{kkk}=\dfrac{\cos 2\sigma}{64k^4}$,

$$A_{kkk}=\frac{\sin(9kz-\sigma)}{46080k^6}+\frac{7\sin(5kz-\sigma)}{20480k^6}-\frac{\sin(kz-\sigma)}{1024k^6}.$$

iv) $\nu\neq k, 2k, 3k$. $p_{kkk}=q_{kkk}=0$,

$$A_{kkk}=\frac{\sin\{(\nu+6k)z-\sigma\}}{384k^3(\nu+k)(\nu+2k)(\nu+3k)}+\frac{(\nu^2+4\nu k+7k^2)\sin\{(\nu+2k)z-\sigma\}}{128k^3(\nu+k)^3(\nu-k)(\nu+2k)}$$
$$-\frac{(\nu^2-4\nu k+7k^2)\sin\{(\nu-2k)z-\sigma\}}{128k^3(\nu+k)(\nu-k)^3(\nu-2k)}-\frac{\sin\{(\nu-6k)z-\sigma\}}{384k^3(\nu-k)(\nu-2k)(\nu-3k)}.$$

The other and higher terms can be obtained in the same way, if necessary, though laborious, without great difficulty.

Summarizing these results, the characteristic value and the solution are obtained as follows:

$$\theta_0=n^2+\theta_n\cos 2\sigma+\theta_n^2\frac{\cos 4\sigma-2}{8n^2}+\alpha\theta_{n/2}^2\frac{2-3\cos 2\sigma}{3n^2}+\sum_{k\neq n,n/2}\frac{\theta_k^2}{2(n^2-k^2)}$$
$$+\sum_{k\neq n,n/2}\theta_k\theta_{|n-k|}\frac{\cos 2\sigma}{2k|n-k|}$$
$$-\theta_n^3\frac{\cos 2\sigma}{64n^4}+\alpha\theta_{n/3}^3\frac{81\cos 2\sigma}{64n^4}+\theta_n\theta_{2n}^2\frac{48-64\cos 2\sigma+9\cos 4\sigma}{36n^4}$$
$$-\theta_{2n}\theta_n^2\frac{1+\cos 2\sigma+3\cos 4\sigma}{192n^4}+\sum_{k\neq n,\,n/2}\theta_k^2\theta_{|n-2k|}\frac{|2n-3k|\cos 2\sigma}{32k^2(n-2k)(n-k)^2}$$
$$-\frac{3}{4}\sum_{k\neq n,\,2n}\frac{\theta_k^2\theta_{2k}}{(n^2-k^2)(n^2-4k^2)}-\alpha\theta_{n/4}^2\theta_{n/2}\frac{48-200\cos 2\sigma}{45n^4}+O(\theta^4),$$

$$\mu=\sin 2\sigma\left\{\frac{\theta_n}{2n}-\alpha\frac{\theta_{n/2}^2}{2n^3}+\sum_{k\neq n,n/2}\frac{\theta_k\theta_{|n-k|}}{4nk|n-k|}-\frac{3\theta_n^3}{128n^5}+\alpha\frac{81\theta_{n/3}^3}{128n^5}\right.$$
$$\left.+\alpha\frac{20\theta_{n/4}^2\theta_{n/2}}{9n^5}+\sum_{k\neq n,n/2}\frac{|2n-3k|\theta_k^2\theta_{|n-2k|}}{16k^2(n-2k)(n-k)^2n}\right\}$$
$$-\theta_n\theta_{2n}^2\frac{16\sin 2\sigma-9\sin 4\sigma}{36n^5}+\theta_n^2\theta_{2n}\frac{\sin 2\sigma+3\sin 4\sigma}{384n^5}+O(\theta^4)\,.$$

$$y=\sin(nz-\sigma)+\theta_n\frac{\sin(3nz-\sigma)}{8n^2}+\sum_{k\neq n}\theta_k\left[\frac{\sin\{(n+2k)z-\sigma\}}{2k(n+k)}\right.$$
$$\left.-\frac{\sin\{(n-2k)z-\sigma\}}{2k(n-k)}\right]\,, \tag{A-7}$$

where α is unity if the suffix is an integer and zero if other, and the notations O and o are due to Landau[3]. In the stable regions, as σ is imaginary and $\cos m\sigma$ and $\sin m\sigma$ are involved in the coefficient of θ^m, the n should be selected to take the modulus of 2σ as small as possible to avoid any divergence of solutions. The most unstable conditions of these solutions are given by taking $\pi/4$ for σ, similarly to those of Mathieu's equation within the assumed approximation.

The fundamental system of solutions of Hill's equation is written, as is well known, in the form

$$\varphi(z)=Ae^{\mu z}y(z,\sigma)+Be^{-\mu z}y(z,-\sigma)\,. \tag{A-8}$$

The conditions $\mu=0$ or $\sigma=0$ and $\pi/2$ in (A-7) are used to determine the boundaries between the stable and unstable regions as a function of the value θ_0 in terms of the given values of $\theta_1, \theta_2, \cdots, \theta_k, \cdots$.[4]

3) e.g., E. T. WHITTAKER and G. N. WATSON, *A Course of Modern Analysis*, 4th ed., (Cambridge, 1935), p. 11.

4) For Mathieu's equations, see J. J. STOKER, *Nonlinear Vibrations in Mechanical and Electrical Systems*, (Interscience Publishers, 1950), pp. 208-213.

21. ある周期構造を有する不均質媒質を伝わる弾性波

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速度の分布が $c_0(1+\sum_{r=1}^{r_0}\varepsilon_r\cos 2r\gamma x)$ で表わされる不均質媒質を伝わる波動の伝搬を調べた．伝搬径路の不均質性をフーリエ解析するとこのような速度分布に置きかえることが可能であろう．また，同様な問題は平らでない地形を横切る表面波の伝搬と結びつけて考えることもできる．このような媒質における波動方程式はヒルの方程式で書かれ，その解の安定性を議論するために Whittaker や Ince らによつて与えられた解を少し一般化して用いた．しかし，その解は定常波の表現であるので，進行波として波の性質を調べるためにこの不均質媒質を中に挟んで，両側が均質一様な媒質の一方から入射する波を考えて，その透過および反射の波の振幅を計算した．この時，不均質媒質の厚さの半分の長さをもつ波を見かけの基本波とみなすならば，透過波のスペクトルは入射波のそれに比べて，r 次モードの周波数の附近で $\mathrm{sech}(\varepsilon_r k_r x_0/2)$ 倍となつている．ここで k_r は r 次の波数，x_0 は不均質媒質の厚さである．従つて，透過波のスペクトルには飛び飛びの谷が現われることが期待される．次に透過波の位相をみると，それはわずかながらずれが生じており，その量は構造の多くの成分の変動量 (ε_r) に依存している．透過波と入射波の位相のずれから，位相速度の分散は見かけ上非常に小さい (ε^2 の大きさ) ことが導かれた．一方，反射波の振幅は構造の変動量の大きさ (ε_r) であり，その位相には $\pi/2$ のずれが現われる．