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22. *Tsunami in an L-shaped Canal* [II].

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Abstract

This article is a continuation of a study on the tsunami in an L-shaped canal. The development of the theory is made under the approximation (III) described in the introduction of this paper.

As the theory is developed from that of the initial study on this problem, it is desirable that the directions of our method be modified a little so as to suit the new situation. Such directions are as written hereunder:—

(1) an expansion of sinusoidal functions by power series,

(2) an elimination of the expressions in the buffer domain, and as a sub-direction of (2), on the occasion of the elimination, it is preferred not to use the equations with the factors $k^{(m)}d$ ($m=1, 2, 3, \cdots$).

The last statement of the above directions is detailed in the last part of this paper.

Owing to the complexity of the reduction, the only case reviewed in the present purview is that where the widths of two canals are equal to each other.

According to the theory developed in this article, the expressions of the zeroth mode are more related with those of the higher modes as the generalization of the approximation proceeds.

The final solutions are given in (46), (47) and (56)-(59) in section 3. The numerical analysis is retained in a future study.

1. Introduction

In the papers already published[1),2)], the author has treated the propagation of tsunamis in the L-shaped canal. The first[1)] and the second[2)] papers will be referred to as paper I and II respectively. In paper I, the treatment is confined to the range of the approximation such that

1) T. MOMOI, *Bull. Earthq. Res. Inst.*, **40** (1962), 719.
2) T. MOMOI, *Bull. Earthq. Res. Inst.*, **41** (1963), 581.

$$\left.\begin{array}{l}\cos kd \simeq 1 \\ \sin kd \simeq kd\end{array}\right\},\qquad \text{(I)}$$

while, in paper II, it is extended to the range

$$\left.\begin{array}{l}\cos kd \simeq 1-\dfrac{1}{2!}(kd)^2 \\ \sin kd \simeq kd-\dfrac{1}{3!}(kd)^3\end{array}\right\},\qquad \text{(II)}$$

where k and d stand for the wave number of the incident wave and the width of the canal respectively.

In the present purview, these approximations are more generalized such that

$$\left.\begin{array}{l}\cos kd \simeq \sum_{n=0}^{2}(-1)^n\dfrac{(kd)^{2n}}{(2n)!} \\ \sin kd \simeq \sum_{n=0}^{2}(-1)^n\dfrac{(kd)^{2n+1}}{(2n+1)!}\end{array}\right\}.\qquad \text{(III)}$$

In this paper, a theory is developed by use of an expression of the long wave, although, in the foregoing papers[1),2)], the expression of the potential was employed.

As a step to the development of the theory, the case where the widths of the two canals are equal to each other only is discussed in this article.

As our theory develops from an embryo stage of paper I, we now feel a need to modify the direction of our method. In the previous paper[3)], our method was outlined as follows:

(1) the first reduction is an application of the long wave approximation,

(2) the second one is to eliminate the "buffer domain" by the relation of the wave number components.

In the first reduction described above, the expression of "the long wave approximation" is inappropriate for discussion of the theory to the approximation (III) in the introduction of this paper. If the approximation (I) (in the introduction) is used, such an expression is not so unreasonable. We now develop the theory up to the approximation (III) and intend to make the approximation more generalized. Hence it

3) T. MOMOI, *Bull. Earthq. Res. Inst.*, **41** (1963), 375.

is desirable that the directions of our method be refined as follows:

(1) *an expansion of sinusoidal functions by power series*

and

(2) *an elimination of the expression in the buffer domain.*

The second reduction is exactly the same as that in the previous paper.

2. General Theory

Let the fluid be ideal (no viscosity), incompressible and the motion irrotational. When the co-ordinate axes are taken in the conventional manner, with the (x, y) plane at the undisturbed water surface, and the wave of small wave height (ζ being the elevation of the water) is assumed, the governing equations for long waves are

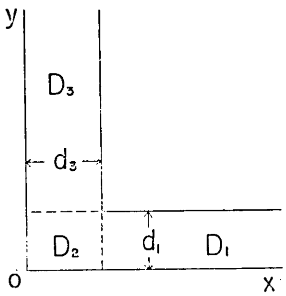

Fig. 1.

$$u_t + g\zeta_x = 0 , \tag{1}$$

$$v_t + g\zeta_y = 0 , \tag{2}$$

and

$$\zeta_t + (uh)_x + (vh)_y = 0 , \tag{3}$$

where u and v are the velocity components in the x and y directions respectively, g the acceleration due to gravity, h the depth of water, and the letter subscripts attached to dependent variables denote differentiation with respect to the denoted independent variables, unless otherwise stated.

If we set h as constant, all the subsequent treatments will be reduced to an ordinary discussion of the two-dimensional propagation of waves where the effect of the varying depth is not taken into account. The wave equation derived from (1), (2) and (3) is

$$\frac{1}{c^2}\zeta_{tt} = \zeta_{xx} + \zeta_{yy} , \tag{4}$$

where c is the velocity of long waves, *i.e.*, $\sqrt{gh}$.

If we assume ζ to be periodic with respect to time, (4) may be written as

$$\frac{\omega^2}{c^2}\zeta' + \zeta'_{xx} + \zeta'_{yy} = 0 , \tag{5}$$

where $\zeta = \zeta' \exp(-i\omega t)$ (ω: angular frequency). In later discussions, the prime in ζ' is omitted for simplicity.

Provided that k is the wave number of the incident waves $(=\omega/c)$, and the definitions of the domains D_j $(j=1, 2, 3)$ and the widths d_j $(j=1, 3)$ of the canals are as illustrated in Fig. 1, the solutions of (5) for the periodic waves, under the condition of "no flux" at the rigid boundaries, are given as follows (these solutions having already been given in paper I by use of the expression of potential):

in the domain D_1,

$$\zeta_1=\zeta_0 e^{-ikx}+\sum_{m=0}^{\infty}\zeta_1^{(m)}\cdot\cos\frac{m\pi}{d_1}y\cdot e^{+ik_1^{(m)}x}\,; \tag{6}$$

in the domain D_2,

$$\zeta_2=\sum_{f_2}A_2(f_2)\cdot\cos k_2^{(x)}x\cdot\cos k_2^{(y)}y\,; \tag{7}$$

in the domain D_3,

$$\zeta_3=\sum_{m=0}^{\infty}\zeta_3^{(m)}\cdot\cos\frac{m\pi}{d_3}x\cdot e^{+ik_3^{(m)}y}\,; \tag{8}$$

where ζ_0 stands for the amplitude of the incident waves; $\zeta_1^{(m)}$ and $\zeta_3^{(m)}$ $(m=0, 1, 2, \cdots)$ the unknowns to be determined by the boundary conditions; $k_1^{(m)}=\sqrt{k^2-(m\pi/d_1)^2}$ and $k_3^{(m)}=\sqrt{k^2-(m\pi/d_3)^2}$; $A_2(f_2)$ the unknowns in the domain D_2, which is a function of $k_2^{(x)}$ and $k_2^{(y)}$; f_2 denotes the relation $k^2=(k_2^{(x)})^2+(k_2^{(y)})^2$; $\sum_{f_2}$ the integration over the range f_2; ζ_j $(j=1, 2, 3)$ the wave heights in the domain D_j $(j=1, 2, 3)$.

In a manner similar to paper I, though the potential theory was employed, the available conditions for determining the arbitrary constants are given as follows:

$$\left.\begin{aligned}&\zeta_2=\zeta_1\\&\frac{\partial\zeta_2}{\partial x}=\frac{\partial\zeta_1}{\partial x}\end{aligned}\right\}\ (x=d_3)\,, \tag{9}$$

and

$$\left.\begin{aligned}&\zeta_3=\zeta_2\\&\frac{\partial\zeta_3}{\partial y}=\frac{\partial\zeta_2}{\partial y}\end{aligned}\right\}\ (y=d_1)\,. \tag{10}$$

Substituting (6)–(8) into (9) and (10), and applying the operators:

$$\int_0^{d_1} \cos \frac{m\pi}{d_1} y dy$$

and

$$\int_0^{d_3} \cos \frac{m\pi}{d_3} x dx \quad (m=0, 1, 2, \cdots),$$

to (9) and (10) respectively, we obtain the following relations (see paper I):

$$\zeta_0 e^{-ikd_3} \cdot d_1 + \zeta_1^{(0)} e^{+ikd_3} \cdot d_1 = \sum_{f_2} A_2(f_2) \cdot \cos k_2^{(x)} d_3 \cdot \frac{1}{k_2^{(y)}} \cdot \sin k_2^{(y)} d_1 , \tag{11}$$

$$\zeta_3^{(0)} e^{+ikd_1} \cdot d_3 = \sum_{f_2} A_2(f_2) \cdot \frac{1}{k_2^{(x)}} \cdot \sin k_2^{(x)} d_3 \cdot \cos k_2^{(y)} d_1 , \tag{12}$$

$$\begin{aligned}(-ikd_1)\zeta_0 e^{-ikd_3} + (+ikd_1)\zeta_1^{(0)} e^{+ikd_3} \\ = \sum_{f_2} A_2(f_2) \cdot k_2^{(x)} \cdot (-1) \cdot \sin k_2^{(x)} d_3 \cdot \frac{1}{k_2^{(y)}} \cdot \sin k_2^{(y)} d_1 ,\end{aligned} \tag{13}$$

$$(+ikd_3)\zeta_3^{(0)} e^{+ikd_1} = \sum_{f_2} A_2(f_2) \cdot \frac{1}{k_2^{(x)}} \cdot \sin k_2^{(x)} d_3 \cdot k_2^{(y)} \cdot (-1) \cdot \sin k_2^{(y)} d_1 , \tag{14}$$

$$\begin{aligned}\zeta_1^{(m)} \cdot \frac{1}{2} d_1 \cdot e^{+ik_1^{(m)} d_3} = \sum_{f_2} A_2(f_2) \cdot \cos k_2^{(x)} d_3 \cdot (-1)^m \\ \cdot \sin k_2^{(y)} d_1 \cdot \frac{(k_2^{(y)} d_1) d_1}{(k_2^{(y)} d_1)^2 - (m\pi)^2} \quad (m=1, 2, \cdots) .\end{aligned} \tag{15}$$

$$\begin{aligned}\zeta_3^{(m)} \cdot \frac{1}{2} d_3 \cdot e^{+ik_3^{(m)} d_1} = \sum_{f_2} A_2(f_2) \cdot \cos k_2^{(y)} d_1 \cdot (-1)^m \\ \cdot \sin k_2^{(x)} d_3 \cdot \frac{(k_2^{(x)} d_3) d_3}{(k_2^{(x)} d_3)^2 - (m\pi)^2} \quad (m=1, 2, \cdots) ,\end{aligned} \tag{16}$$

$$\begin{aligned}(+ik_1^{(m)})\zeta_1^{(m)} \cdot \frac{1}{2} d_1 \cdot e^{+ik_1^{(m)} d_3} = \sum_{f_2} A_2(f_2) \cdot k_2^{(x)} \cdot (-1)^{m+1} \cdot \sin k_2^{(x)} d_3 \\ \cdot \sin k_2^{(y)} d_1 \cdot \frac{(k_2^{(y)} d_1) d_1}{(k_2^{(y)} d_1)^2 - (m\pi)^2} \quad (m=1, 2, \cdots) ,\end{aligned} \tag{17}$$

$$\begin{aligned}(+ik_3^{(m)})\zeta_3^{(m)} \cdot \frac{1}{2} d_3 \cdot e^{+ik_3^{(m)} d_1} = \sum_{f_2} A_2(f_2) \cdot k_2^{(y)} \cdot (-1)^{m+1} \cdot \sin k_2^{(y)} d_1 \\ \cdot \sin k_2^{(x)} d_3 \cdot \frac{(k_2^{(x)} d_3) d_3}{(k_2^{(x)} d_3)^2 - (m\pi)^2} \quad (m=1, 2, \cdots) .\end{aligned} \tag{18}$$

The above relations (11)–(18) are completely identical with those in paper I, except that the latter are derived from the use of the potential theory.

Up to this stage of the reduction, no approximation has been employed. In the next section, the solutions in the canal of the *uniform width* will be discussed under the *approximation* (*III*) in the introduction.

3. Particular Case

Let $d_1=d_3=d$ and hence $k_1^{(m)}=k_3^{(m)}=k^{(m)}$ (*we treat the problem in the canal of uniform width* as a step in the course of the development of our theory). Then the relations (11)–(18) are reduced into the following forms:

$$(\zeta_0 e^{-ikd}+\zeta_1^{(0)}e^{+ikd})\cdot d=\sum_{f_2} A_2(f_2)\cdot \cos k_2^{(x)}d\cdot \frac{1}{k_2^{(y)}}\cdot \sin k_2^{(y)}d\,, \tag{19}$$

$$\zeta_3^{(0)}e^{+ikd}\cdot d=\sum_{f_2} A_2(f_2)\cdot \frac{1}{k_2^{(x)}}\cdot k_2^{(x)}d\cdot \cos k_2^{(y)}d\,, \tag{20}$$

$$(-ikd)\zeta_0 e^{-ikd}+(+ikd)\zeta_1^{(0)}e^{+ikd}$$
$$=\sum_{f_2} A_2(f_2)\cdot k_2^{(x)}\cdot(-1)\cdot \sin k_2^{(x)}d\cdot \frac{1}{k_2^{(y)}}\cdot \sin k_2^{(y)}d\,, \tag{21}$$

$$(+ikd)\zeta_3^{(0)}e^{+ikd}=\sum_{f_2} A_2(f_2)\cdot \frac{1}{k_2^{(x)}}\cdot \sin k_2^{(x)}d\cdot k_2^{(y)}\cdot(-1)\cdot \sin k_2^{(y)}d\,, \tag{22}$$

$$\zeta_1^{(m)}\cdot \frac{1}{2}d\cdot e^{+ik^{(m)}d}=\sum_{f_2} A_2(f_2)\cdot \cos k_2^{(x)}d\cdot(-1)^m$$
$$\cdot \sin k_2^{(y)}d\cdot \frac{(k_2^{(y)}d)d}{(k_2^{(y)}d)^2-(m\pi)^2}\quad (m=1, 2, \cdots)\,, \tag{23}$$

$$\zeta_3^{(m)}\cdot \frac{1}{2}d\cdot e^{+ik^{(m)}d}=\sum_{f_2} A_2(f_2)\cdot \cos k_2^{(y)}d\cdot(-1)^m$$
$$\cdot \sin k_2^{(x)}d\cdot \frac{(k_2^{(x)}d)d}{(k_2^{(x)}d)^2-(m\pi)^2}\quad (m=1, 2, \cdots)\,, \tag{24}$$

$$(+ik^{(m)}d)\zeta_1^{(m)}\cdot \frac{1}{2}\cdot e^{+ik^{(m)}d}=\sum_{f_2} A_2(f_2)\cdot k_2^{(x)}\cdot(-1)^{m+1}\cdot \sin k_2^{(x)}d$$
$$\cdot \sin k_2^{(y)}d\cdot \frac{(k_2^{(y)}d)d}{(k_2^{(y)}d)^2-(m\pi)^2}\quad (m=1, 2, \cdots)\,. \tag{25}$$

$$(+ik^{(m)}d)\zeta_3^{(m)}\cdot\frac{1}{2}\cdot e^{+ik^{(m)}d}=\sum_{f_2}A_2(f_2)\cdot k_2^{(y)}\cdot(-1)^{m+1}\cdot\sin k_2^{(y)}d$$

$$\cdot\sin k_2^{(x)}d\cdot\frac{(k_2^{(x)}d)d}{(k_2^{(x)}d)^2-(m\pi)^2}\quad(m=1,2,\cdots)\,,\tag{26}$$

Applying the approximation (III) in the introduction to the right-hand sides of the relations (19)-(22) and retaining the terms up to the fourth order of $k_2^{(z)}d$ ($z=x$ or y), we have:

$$\zeta_0e^{-ikd}+\zeta_1^{(0)}e^{+ikd}=\sum_{f_2}A_2(f_2)\cdot\Big\{1-\frac{1}{2!}(k_2^{(x)}d)^2-\frac{1}{3!}(k_2^{(y)}d)^2$$

$$+\frac{1}{2!}\cdot\frac{1}{3!}\cdot(k_2^{(x)}d)^2(k_2^{(y)}d)^2+\frac{1}{4!}(k_2^{(x)}d)^4+\frac{1}{5!}(k_2^{(y)}d)^4\Big\}\,,\tag{19'}$$

$$\zeta_3^{(0)}e^{+ikd}=\sum_{f_2}A_2(f_2)\cdot\Big\{1-\frac{1}{3!}(k_2^{(x)}d)^2-\frac{1}{2!}(k_2^{(y)}d)^2$$

$$+\frac{1}{3!}\cdot\frac{1}{2!}\cdot(k_2^{(x)}d)^2(k_2^{(y)}d)^2+\frac{1}{5!}(k_2^{(x)}d)^4+\frac{1}{4!}(k_2^{(y)}d)^4\Big\}\,,\tag{20'}$$

$$(-ikd)\zeta_0e^{-ikd}+(+ikd)\zeta_1^{(0)}e^{+ikd}$$

$$=\sum_{f_2}A_2(f_2)\cdot(-1)\cdot\Big\{(k_2^{(x)}d)^2-\frac{1}{3!}(k_2^{(x)}d)^2(k_2^{(y)}d)^2-\frac{1}{3!}(k_2^{(x)}d)^4\Big\}\,,\tag{21'}$$

$$(+ikd)\zeta_3^{(0)}e^{+ikd}$$

$$=\sum_{f_2}A_2(f_2)\cdot(-1)\cdot\Big\{(k_2^{(y)}d)^2-\frac{1}{3!}(k_2^{(x)}d)^2(k_2^{(y)}d)^2-\frac{1}{3!}(k_2^{(y)}d)^4\Big\}\,.\tag{22'}$$

Although the reductions for the zeroth mode of waves are made in the foregoing manner, some care is needed for reducing the right-hand sides of (23)-(26) (the higher modes of waves), which has already been noted in the previous paper[4].

When the application of the theory is confined to the range

$$kd<\pi\,,\tag{27}$$

the following expansion is possible:

$$\frac{1}{(k_2^{(z)}d)^2-(m\pi)^2}=\frac{-1}{(m\pi)^2}\cdot\Big\{1+\Big(\frac{k_2^{(z)}d}{m\pi}\Big)^2+\Big(\frac{k_2^{(z)}d}{m\pi}\Big)^4+\cdots\Big\}$$

$$(z=x\text{ or }y\,;\ m=1,2,\cdots)\,,\tag{28}$$

4) T. MOMOI, *Bull. Earthq. Res. Inst.*, **41** (1963), 708.

If kd is no less than π, the above expansion is not always permissible. But, *in the present purview, the development of the theory is limited to the range of (27).*

Using the expression (28) and the approximation (III), with the terms up to the fourth order of $k_2^{(z)}d$ ($z=x$ or y), the relation of the higher modes (23)-(26) become as follows:

$$\zeta_1^{(m)}\cdot\frac{1}{2}\cdot e^{+ik^{(m)}d}=\frac{(-1)^{m+1}}{(m\pi)^2}\cdot\sum_{f_2}A_2(f_2)\cdot\Big[(k_2^{(y)}d)^2+\left\{\frac{1}{(m\pi)^2}-\frac{1}{3!}\right\}\cdot(k_2^{(y)}d)^4-\frac{1}{2!}(k_2^{(x)}d)^2(k_2^{(y)}d)^2\Big],\qquad(23')$$

$$\zeta_3^{(m)}\cdot\frac{1}{2}\cdot e^{+ik^{(m)}d}=\frac{(-1)^{m+1}}{(m\pi)^2}\cdot\sum_{f_2}A_2(f_2)\cdot\Big[(k_2^{(x)}d)^2+\left\{\frac{1}{(m\pi)^2}-\frac{1}{3!}\right\}\cdot(k_2^{(x)}d)^4-\frac{1}{2!}(k_2^{(x)}d)^2(k_2^{(y)}d)^2\Big],\qquad(24')$$

$$(+ik^{(m)}d)\zeta_1^{(m)}\cdot\frac{1}{2}\cdot e^{+ik^{(m)}d}=\frac{(-1)^m}{(m\pi)^2}\sum_{f_2}A_2(f_2)(k_2^{(x)}d)^2(k_2^{(y)}d)^2,\qquad(25')$$

$$(+ik^{(m)}d)\zeta_3^{(m)}\cdot\frac{1}{2}\cdot e^{+ik^{(m)}d}=\frac{(-1)^m}{(m\pi)^2}\sum_{f_2}A_2(f_2)(k_2^{(x)}d)^2(k_2^{(y)}d)^2,\qquad(26')$$

Adding (20′) to (19′) and subtracting (20′) from (19′) respectively, we have

$$\zeta_0e^{-ikd}+\zeta_1^{(0)}e^{+ikd}+\zeta_3^{(0)}e^{+ikd}=\sum_{f_2}A_2(f_2)\cdot\Big[2-\left(\frac{1}{2!}+\frac{1}{3!}\right)\cdot\left\{(k_2^{(x)}d)^2+(k_2^{(y)}d)^2\right\}+\frac{1}{3!}(k_2^{(x)}d)^2(k_2^{(y)}d)^2+\left(\frac{1}{4!}+\frac{1}{5!}\right)\cdot\left\{(k_2^{(x)}d)^4+(k_2^{(y)}d)^4\right\}\Big],\qquad(29)$$

$$\zeta_0e^{-ikd}+\zeta_1^{(0)}e^{+ikd}-\zeta_3^{(0)}e^{+ikd}=\sum_{f_2}A_2(f_2)\cdot\Big[\left(-\frac{1}{2!}+\frac{1}{3!}\right)\cdot\left\{(k_2^{(x)}d)^2-(k_2^{(y)}d)^2\right\}+\left(\frac{1}{4!}-\frac{1}{5!}\right)\cdot\left\{(k_2^{(x)}d)^4-(k_2^{(y)}d)^4\right\}\Big].\qquad(30)$$

Likewise, adding (22′) to (21′) and subtracting (22′) from (21′), we have

$$(-ikd)\zeta_0 e^{-ikd}+(+ikd)\zeta_1^{(0)}e^{+ikd}+(+ikd)\zeta_3^{(0)}e^{+ikd}$$
$$=\sum_{f_2} A_2(f_2)\cdot(-1)\cdot\Big[\{(k_2^{(x)}d)^2+(k_2^{(y)}d)^2\}$$
$$-\frac{2}{3!}(k_2^{(x)}d)^2(k_2^{(y)}d)^2-\frac{1}{3!}\{(k_2^{(x)}d)^4+(k_2^{(y)}d)^4\}\Big]\,, \tag{31}$$

$$(-ikd)\zeta_0 e^{-ikd}+(+ikd)\zeta_1^{(0)}e^{+ikd}-(+ikd)\zeta_3^{(0)}e^{+ikd}$$
$$=\sum_{f_2} A_2(f_2)\cdot(-1)\cdot\Big[\{(k_2^{(x)}d)^2-(k_2^{(y)}d)^2\}-\frac{1}{3!}\{(k_2^{(x)}d)^4-(k_2^{(y)}d)^4\}\Big]. \tag{32}$$

As a result of a separation of the variable, the following relation yields, *i. e.*

$$(k_2^{(x)})^2+(k_2^{(y)})^2=k^2\,. \tag{33}$$

Squaring both sides of the above expression, we get

$$(k_2^{(x)})^4+(k_2^{(y)})^4=k^4-2\cdot(k_2^{(x)})^2\cdot(k_2^{(y)})^2\,. \tag{34}$$

Making use of (33) and (34), and after some reductions, (29)–(32) become:

$$\zeta_0 e^{-ikd}+\zeta_1^{(0)}e^{+ikd}+\zeta_3^{(0)}e^{+ikd}$$
$$=\sum_{f_2} A_2(f_2)\cdot\left\{2-\frac{4}{3!}(kd)^2+\frac{6}{5!}(kd)^4+\frac{8}{5!}(k_2^{(x)}d)^2(k_2^{(y)}d)^2\right\}\,, \tag{29'}$$

$$\zeta_0 e^{-ikd}+\zeta_1^{(0)}e^{+ikd}-\zeta_3^{(0)}e^{+ikd}$$
$$=\left\{-\frac{2}{3!}+\frac{4}{5!}(kd)^2\right\}\cdot\sum_{f_2} A_2(f_2)\cdot\{(k_2^{(x)}d)^2-(k_2^{(x)}d)^2\}\,, \tag{30'}$$

$$(-ikd)\zeta_0 e^{-ikd}+(+ikd)\zeta_1^{(0)}e^{+ikd}+(+ikd)\zeta_3^{(0)}e^{+ikd}$$
$$=\sum_{f_2} A_2(f_2)\cdot(-1)\cdot\left\{(kd)^2-\frac{1}{3!}(kd)^4\right\}\,, \tag{31'}$$

$$(-ikd)\zeta_0 e^{-ikd}+(+ikd)\zeta_1^{(0)}e^{+ikd}-(+ikd)\zeta_3^{(0)}e^{+ikd}$$
$$=\left\{-1+\frac{1}{3!}(kd)^2\right\}\cdot\sum_{f_2} A_2(f_2)\cdot\{(k_2^{(x)}d)^2-(k_2^{(y)}d)^2\}\,. \tag{32'}$$

On the other hand, for the higher modes of waves, the addition of (23′) and (24′) and the subtraction of the former from the latter yield:

$$\zeta_1^{(m)} \cdot \frac{1}{2} \cdot e^{+ik^{(m)}d} + \zeta_3^{(m)} \cdot \frac{1}{2} \cdot e^{+ik^{(m)}d}$$

$$= \frac{(-1)^{m+1}}{(m\pi)^2} \cdot \sum_{f_2} A_2(f_2) \cdot \Big[\{(k_2^{(x)}d)^2 + (k_2^{(y)}d)^2\}$$

$$+ \left\{\frac{1}{(m\pi)^2} - \frac{1}{3!}\right\} \cdot \{(k_2^{(x)}d)^4 + (k_2^{(y)}d)^4\} - (k_2^{(x)}d)^2 (k_2^{(y)}d)^2 \Big], \qquad (35)$$

$$-\zeta_1^{(m)} \cdot \frac{1}{2} \cdot e^{+ik^{(m)}d} + \zeta_3^{(m)} \cdot \frac{1}{2} \cdot e^{+ik^{(m)}d}$$

$$= \frac{(-1)^{m+1}}{(m\pi)^2} \cdot \sum_{f_2} A_2(f_2) \cdot \Big[\{(k_2^{(x)}d)^2 - (k_2^{(y)}d)^2\}$$

$$+ \left\{\frac{1}{(m\pi)^2} - \frac{1}{3!}\right\} \cdot \{(k_2^{(x)}d)^4 - (k_2^{(y)}d)^4\}. \qquad (36)$$

Using the relations of the wave number (33) and (34), and after some reduction, (35) and (36) become

$$\frac{1}{2} \cdot \zeta_1^{(m)} e^{+ik^{(m)}d} + \frac{1}{2} \cdot \zeta_3^{(m)} \cdot e^{+ik^{(m)}d}$$

$$= \frac{(-1)^{m+1}}{(m\pi)^2} \cdot \sum_{f_2} A_2(f_2) \cdot \Big[(kd)^2 + \left\{\frac{1}{(m\pi)^2} - \frac{1}{3!}\right\} \cdot (kd)^4$$

$$-2 \cdot \left\{\frac{1}{(m\pi)^2} + \frac{2}{3!}\right\} \cdot (k_2^{(x)}d)^2 (k_2^{(y)}d)^2 \Big], \qquad (35')$$

$$-\frac{1}{2} \cdot \zeta_1^{(m)} \cdot e^{+ik^{(m)}d} + \frac{1}{2} \cdot \zeta_3^{(m)} \cdot e^{+ik^{(m)}d}$$

$$= \frac{(-1)^{m+1}}{(m\pi)^2} \cdot \left[1 + \left\{\frac{1}{(m\pi)^2} - \frac{1}{3!}\right\} \cdot (kd)^2\right]$$

$$\cdot \sum_{f_2} A_2(f_2) \cdot \{(k_2^{(x)}d)^2 - (k_2^{(y)}d)^2\}. \qquad (36').$$

When the approximation of the higher order is used, the higher modes (the first mode in this case) come into the reduction and the unknowns are increased up to a certain number (four unknowns in the present study, *i.e.* $\zeta_1^{(0)}$, $\zeta_1^{(1)}$, $\zeta_3^{(0)}$ and $\zeta_3^{(1)}$) to suffice for the elimination of the expressions of the *buffer domain* D_2. Therefore, we need a number of equations sufficient to solve the unknowns (four equations in this case).

Since our consideration is limited to the range (27), it is appropriate to consider that the relation for eliminating the expressions in the *buffer domain* is furnished by the first mode ($m=1$) in (25) or (26).

Among the four relations of the zeroth modes, *i. e.*, (29′) to (32′) there exists a term of the form

$$\sum_{f_2} A_2(f_2)\cdot(k_2^{(x)}d)^2(k_2^{(y)}d)^2$$

only in (29′), so we cannot eliminate the above expression by use of (29′)–(32′). The above expression results from the relations of the first mode, *viz.*, (25′) or (26′). When the approximations of the lower order are employed, such a term does not appear (refer to papers I and II).

As the approximation proceeds, the expressions of the zeroth mode are more related with those of the higher modes. This is very plausible from physical considerations.

Now let us return to the reduction of the equations.

Eliminating the factor

$$\sum_{f_2} A_2(f_2)\cdot\{(k_2^{(x)}d)^2-(k_2^{(y)}d)^2\}\,,$$

from (30′) and (32′), we have

$$\zeta_3^{(0)}-\zeta_1^{(0)}=\frac{P+Q\cdot(+ikd)}{P-Q\cdot(+ikd)}\cdot\zeta_0\cdot e^{-i\cdot 2kd}\,, \tag{37}$$

where

$$\left.\begin{aligned} P&=1-\frac{1}{3!}(kd)^2 \\ Q&=\frac{2}{3!}-\frac{4}{5!}(kd)^2\,. \end{aligned}\right\} \tag{38}$$

and

From the first modes of (25′) and (26′),

$$\sum_{f_2} A_2(f_2)\cdot(k_2^{(x)}d)^2(k_2^{(y)}d)^2=-\frac{\pi^2}{4}\cdot(+ik^{(1)}d)\cdot(\zeta_1^{(1)}+\zeta_3^{(1)})e^{+ik^{(1)}d}\,, \tag{39}$$

Substituting (31′) and (39) into (29′), we have

$$\begin{aligned}\{A+(-ikd)\cdot B\}\cdot e^{-ikd}\zeta_0+\{A+(+ikd)\cdot B\}\cdot e^{+ikd}\cdot(\zeta_1^{(0)}+\zeta_3^{(0)})\\ =-\frac{2}{5!}\cdot A\cdot\pi^2\cdot(+ik^{(1)}d)\cdot e^{+ik^{(1)}d}\cdot(\zeta_1^{(1)}+\zeta_3^{(1)})\,,\end{aligned} \tag{40}$$

where

and

$$\left.\begin{aligned} A &= (kd)^2 - \frac{1}{3!}(kd)^4 \\ B &= 2 - \frac{4}{3!}(kd)^2 + \frac{6}{5!}(kd)^4 . \end{aligned}\right\} \quad (41)$$

Likewise, substituting (31′) and (39) into (35′), the following is obtained:

$$W \cdot \{(-ikd) \cdot \zeta_0 \cdot e^{-ikd} + (+ikd) \cdot \zeta_1^{(0)} \cdot e^{+ikd} + (+ikd) \cdot \zeta_3^{(0)} \cdot e^{+ikd}\}$$

$$= -\frac{1}{2} \cdot P \cdot V \cdot \pi^2 \cdot e^{+ik^{(1)}d} \cdot (\zeta_1^{(1)} + \zeta_3^{(1)}) , \quad (42)$$

where

$$\left.\begin{aligned} V &= 1 - \left(\frac{1}{\pi^2} + \frac{2}{3!}\right) \cdot (+ik^{(1)}d) , \\ W &= 1 + \left(\frac{1}{\pi^2} - \frac{1}{3!}\right) \cdot (kd)^2 , \end{aligned}\right\} \quad (43)$$

and where P has been already given in (38).

Here we have the same factor $(\zeta_1^{(1)} + \zeta_3^{(1)})$ in the right-hand sides of (40) and (42). Hence, substituting $(\zeta_1^{(1)} + \zeta_3^{(1)})$ of (40) into that of (42) and after some reductions, the next equation is obtained:

$$\zeta_3^{(0)} + \zeta_1^{(0)} = \frac{R + P \cdot (+ikd)}{R - P \cdot (+ikd)} \cdot \zeta_0 \cdot e^{-i \cdot 2kd} , \quad (44)$$

where

$$\left.\begin{aligned} R &= S - T , \\ S &= 2 - \frac{4}{3!} \cdot (kd)^2 + \frac{6}{5!} \cdot (kd)^4 , \\ T &= \frac{4}{5!} \cdot (kd)^2 \cdot (+ik^{(1)}d) \cdot \frac{W}{V} \end{aligned}\right\} \quad (45)$$

(W and V are described in (43)).

Now we have two equations (37) and (44), which suffice to obtain the unknowns of the zeroth mode ($\zeta_1^{(0)}$ and $\zeta_3^{(0)}$) in the straight parts of the canals.

Solving the two equations mentioned above, the expressions of the zeroth mode become:
from a sum of two equations,

$$\zeta_3^{(0)} = \frac{P \cdot \{R+(kd)^2 \cdot Q\}}{\{P-(+ikd) \cdot Q\} \cdot \{R-(+ikd) \cdot P\}} \cdot \zeta_0 e^{-i \cdot 2kd}, \tag{46}$$

from a difference of two equations,

$$\zeta_1^{(0)} = \frac{(+ikd) \cdot (P^2-QR)}{\{P-(+ikd) \cdot Q\} \cdot \{R-(+ikd) \cdot P\}} \cdot \zeta_0 e^{-i \cdot 2kd}, \tag{47}$$

where the expressions of P, Q, and R have already been described in the fore-going paragraphs.

Substituting (46) and (47) into (30′) and after some reductions, we obtain the expression:

$$\sum_{f_2} A_2(f_2) \cdot \{(k_2^{(x)}d)^2-(k_2^{(y)}d)^2\} = \frac{(+ikd)}{P-(+ikd) \cdot Q} \cdot 2\zeta_0 e^{-ikd}. \tag{48}$$

Likewise, substituting (46) and (47) into (31′), the following is obtained:

$$\sum_{f_2} A_2(f_2) = \frac{1}{R-(+ikd) \cdot P} \cdot 2\zeta_0 e^{-ikd}, \tag{49}$$

where R is expressed in (45).

If the substitution of (46) and (47) into (32′) is made, the relation to be obtained is identically the same as in (48). This is due to the fact that the solutions (46) and (47) are obtained from the equations derived from (30′) and (32′).

Substituting the solutions (46) and (47) into the left-hand side of (29′), the left-hand side of the equation becomes:

$$\zeta_0 e^{-ikd} + \zeta_1^{(0)} e^{+ikd} + \zeta_3^{(0)} e^{+ikd} = \frac{2R}{R-(+ikd) \cdot P} \cdot \zeta_0 e^{-ikd}. \tag{50}$$

Using the expressions (50) and (49), we can obtain the following from (29′), *i.e.*,

$$\frac{4}{5!} \sum_{f_2} A_2(f_2) \cdot (k_2^{(x)}d)^2 (k_2^{(y)}d)^2 = \frac{-T}{R-(+ikd) \cdot P} \cdot \zeta_0 e^{-ikd}. \tag{51}$$

Now, the substitution of (49) and (51) into (35′) yields:

$$\zeta_1^{(1)} + \zeta_3^{(1)} = \frac{4}{\pi^2} \cdot \frac{W}{V} \cdot \frac{(kd)^2}{R-(+ikd) \cdot P} \cdot \zeta_0 e^{-ik^{(1)}d-ikd}, \tag{52}$$

for the first mode of waves;

$$\zeta_1^{(m)}+\zeta_3^{(m)}=(-1)^{m+1}\cdot\frac{4}{(m\pi)^2}\cdot\frac{(kd)^2}{R-(+ikd)\cdot P}$$
$$\cdot\left[W_m+\left\{\frac{1}{(m\pi)^2}+\frac{2}{3!}\right\}\cdot(+ik^{(1)}d)\cdot\frac{W}{V}\right]\cdot\zeta_0 e^{-ik^{(m)}d-ikd}\,, \tag{53}$$

for the modes excepting the first one,
where

$$\left.\begin{aligned}&W_m=1+\left\{\frac{1}{(m\pi)^2}-\frac{1}{3!}\right\}\cdot(kd)^2\quad(m=2,3,\cdots)\,,\\&\text{and the following identity exists, } i.e.,\\&W_1=W\,.\end{aligned}\right\} \tag{54}$$

On the other hand, the substitution of (48) into (36′) yields:

$$-\zeta_1^{(m)}+\zeta_3^{(m)}=(-1)^{m+1}\cdot\frac{4}{(m\pi)^2}\cdot\frac{(+ikd)}{P-(+ikd)\cdot Q}\cdot W_m\cdot\zeta_0 e^{-ik^{(m)}d-ikd}\,, \tag{55}$$

where W_m is given in (54).

From (52) and the expression of the first mode ($m=1$) in (55), we have the following solutions for the first mode:

$$\zeta_1^{(1)}=\frac{2}{\pi^2}\cdot\frac{W}{V}\cdot\left\{\frac{(kd)^2}{R-(+ikd)\cdot P}-\frac{(+ikd)\cdot V}{P-(+ikd)\cdot Q}\right\}\cdot\zeta_0 e^{-ik^{(1)}d-ikd}\,, \tag{56}$$

$$\zeta_3^{(1)}=\frac{2}{\pi^2}\cdot\frac{W}{V}\cdot\left\{\frac{(kd)^2}{R-(+ikd)\cdot P}+\frac{(+ikd)\cdot V}{P-(+ikd)\cdot Q}\right\}\cdot\zeta_0 e^{-ik^{(1)}d-ikd}\,. \tag{57}$$

From (53) and (55), the solutions for $m\geqq 2$ become as follows:

$$\zeta_1^{(m)}=(-1)^{m+1}\cdot\frac{2}{(m\pi)^2}\cdot\left[W_m\cdot\left\{\frac{(kd)^2}{R-(+ikd)\cdot P}-\frac{(+ikd)}{P-(+ikd)\cdot Q}\right\}\right.$$
$$\left.+\frac{(kd)^2}{R-(+ikd)\cdot P}\cdot\left\{\frac{1}{(m\pi)^2}+\frac{2}{3!}\right\}\cdot(+ik^{(1)}d)\cdot\frac{W}{V}\right]\cdot\zeta_0 e^{-ik^{(m)}d-ikd}\,, \tag{58}$$

$$\zeta_3^{(m)}=(-1)^{m+1}\cdot\frac{2}{(m\pi)^2}\cdot\left[W_m\cdot\left\{\frac{(kd)^2}{R-(+ikd)\cdot P}+\frac{(+ikd)}{P-(+ikd)\cdot Q}\right\}\right.$$
$$\left.+\frac{(kd)^2}{R-(+ikd)\cdot P}\cdot\left\{\frac{1}{(m\pi)^2}+\frac{2}{3!}\right\}\cdot(+ik^{(1)}d)\cdot\frac{W}{V}\right]\cdot\zeta_0 e^{-ik^{(m)}d-ikd}\,. \tag{59}$$

The expressions (56) to (59) have been obtained from the relations (19′) to (22′) of the zeroth mode, and (23′) and (24′) of the higher modes. On the other hand, alternative solutions might be obtained using (25′)

and (26′) in addition to relations used in the foregiong reduction.

Equating (25′) and (26′), we have

$$(+ik^{(m)}d)\zeta_1^{(m)}\cdot\frac{1}{2}\cdot e^{+ik^{(m)}d}=(+ik^{(m)}d)\zeta_3^{(m)}\cdot\frac{1}{2}\cdot e^{+ik^{(m)}d}\,, \tag{60}$$

where

$$k^{(m)}=\sqrt{k^2-\left(\frac{m\pi}{d}\right)^2}\,. \tag{61}$$

The above equation holds only when the factors $(+ik^{(m)}d)$ exist on either side of the equation (60). Suppose that, dividing (60) in terms of $(+ik^{(m)}d)\frac{1}{2}\exp(+ik^{(m)}d)$, the equation

$$\zeta_1^{(m)}=\zeta_3^{(m)}\,, \tag{62}$$

is obtained.

When kd approaches π, that is to say the value of $k^{(m)}d$ tends to zero, the validity of the equation (62) becomes more and more ambiguous. As kd approaches π, the solutions obtained by use of the relation (62) involve a larger amount of errors than the solutions derived in the foregoing reduction. Hence, it is preferable to supplement the following expression as a sub-direction to the second direction of our method (see the introduction).

On the occasion of the reduction, the use of the equations with the factors $k^{(m)}d$ $(m=1, 2, 3\cdots)$, *i. e.* (25′) *and* (26′) *must be refrained from as much as possible.*

In this article, succeeding the previous papers which treated the same problem, our study is focused on the development of our method. Hence the numerical analysis has not been done in this purview.

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22. L 字水路における津波 [II]

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本論説は L 字水路における津波を取りあつかう方法に関しておこなつてきた理論展開の延長である．前論説においては，本論説の序説に述べてある近似 (I) および (II) の範囲で理論の展開をおこなつた．本論説では，序説の (III) の場合にまで近似を高め，理論を展開した．近似が (III) の場合まで高まると，第零次モードだけでは最早 buffer domain の表現を除去することはできなくなり，第一次モードも演算の中に入つてくる．すなわち，近似が高まるにつれて零次モードと高次モードとの関係が一層強くなつてくる．

各モード解の形式的表現は (46)，(47) および (56)-(59) で与えられる．