BULLETIN OF THE EARTHQUAKE
RESEARCH INSTITUTE
Vol. 42 (1964), pp. 289-368

16. *Statical Elastic Dislocations in an Infinite and Semi-Infinite Medium.**

By Takuo MARUYAMA,†

Graduate School, the University of Tokyo.

(Read January 22, 1963 and April 23, 1963.—Received March 29, 1964.)

Contents

Introduction

Crustal deformations associated with seismic activities have been reported since old times. However, we have not yet fully succeeded in explaining consistently in detail the deformation due to an earthquake from concrete causes in the crust or the mantle. For the present, it may be useful to have the stock of knowledge of deformation fields which correspond to various deformation sources for more complete knowledge of the mechanism of earthquake occurrence.

We make mention of some models considered in a semi-infinite elastic body. The actual focus of an earthquake may not be considered to be elastic, but if we limit ourselves to the outside region of the focus and to the short time scale in the process, the use of the elasticity theory should be justified.

The deformation of the surface of a semi-infinite elastic body due

* Communicated by S. MIYAMURA.

† Present address; Earthquake Research Institute.

to internal nuclei of strain has been calculated by many authors, such as K. Sezawa (1929), F. J. W. Whipple (1936), K. Soeda (1944) and K. Yamakawa (1955). K. Sezawa's theories should be mentioned first, but his solutions are related to nuclei of somewhat special form. F. J. W. Whipple (1936) presented formulae for the displacements of the surface due to an internal nucleus, with the intention of improving the investigation by H. Honda and T. Miura (1935) which was made for external forces applied to the bounding plane. K. Soeda (1944) and N. Yamakawa (1955) calculated the cases when certain stress distributions expressed in spherical harmonics were applied at the interior sherical cavity.

As for the theoretical models of slip faults, those developed by K. Kasahara (1957, 1859) and L. Knopoff (1958) should first be mentioned. These models are two-dimensional. K. Kasahara (1957) calculated the displacement fields around a vertical strike slip fault by assuming a hypothetical stress distribution in depth, while L. Knopoff (1958) treated a model of slip fault with the electric-elastic analogy. Kasahara (1959) dealt with a model of strike-slip faults with dip angles not equal to 90° and solved the problem by the relaxation method and with model experiments based on the electric-elastic analogy.

Three-dimentional models of slip faults are first introduced by J. A. Steketee (1958 a, b). Steketee suggested that Volterra's theory of dislocations might be the proper technique for a quantitative description of fractures. The geophysical questions he had in mind were (i) problems connected with fault-plane studies of earthquakes and with such phenomena as the San Andreas faults, (ii) problems connected with fracture zones in the crust and mantle which are believed to play a significant role in the structure of island arcs and certain mountain ranges.

According to the theory introduced by Steketee, the field around a dislocation in a semi-infinite medium can be expressed by means of six sets of Green's functions. He calculated one set of Green's function which is necessary for the field corresponding to a vertical strike-slip fault. This field is the same as that obtained before by Whipple (1936). K. Sezawa, in his paper (1929), touched on his intention of considering the field due to the sheet or the assemblage of nuclei in order to make a comparison with the observational data of the actual deformation, which intention does not seem to have been realized during his life-time. In Steketee's paper we see that the sheet of nuclei, if it is of some special form, corresponds to a dislocation surface.

As a mathematical model of a fault Steketee used a displacement dislocation surface, i. e., a surface across which there is a discontinuity in the displacement vector. M. A. Chinnery (1961, 1963) calculated the displacement field around a vertical rectangular transcurrent fault by integrating the nuclei over the surface.

In the field of the crystal dislocation theory, Burgers' representation is fundamental.

F. R. N. Nabarro (1951) interpreted the displacement field due to a dislocation as resulting from a combination of double forces by applying Burgers' formula to an infinitesimal loop and by expressing the field as a double integral over the dislocation surface. In the same paper Nabarro obtained some expressions corresponding to a moving dislocation by replacing the static double forces by dynamic double forces with step function time dependence.

E. H. Yoffe (1961) calculated the stress field of a dislocation line meeting a free surface of an elastic body for any angle of incidence and any Burgers vector, by a method different from Steketee's method. She states that Steketee's solution involves six sets of Green's functions and is not easy to apply in practice. However, the concept of line dislocation does not seem to have many applications in the theory of the crustal deformation or in seismology.

The concept of elastic dislocation has also been employed in the theory of earthquake mechanism.

A. V. Vvedenskaya (1956) found a system of forces which may be equivalent to a rupture accompanied by slipping in the above-mentioned Nabarro's paper (1951). She has developed her method on the consideration that a rupture accompanied by slipping is the most probable form of movement in the earthquake foci under the conditions which occur in the earth's crust and in the upper part of the mantle, in which stresses may be supposed to have a considerable duration.

Vvedenskaya (1959) obtained formulae corresponding to the case of a sudden formation of general Volterra dislocations. However, she did not work from the basis of the thory of general dynamic dislocations.

L. Knopoff and F. Gilbert (1959, 1960) considered the elasto-dynamic radiation resulting from the sudden occurrence of an earthquake by assuming a sudden occurrence of displacement discontinuity or of strain discontinuity across a surface in an infinite medium. Starting from formulae previously obtained by one of the authors (Knopoff 1956), they examined the first motions (the high-frequency solution) from the impulsive excitation of the fault surface.

T. Maruyama (1963) presented fundamental formulae for general dynamic dislocations, derived in a straight-forward manner from well-known relations, and considered the force system equivalent to a dynamic dislocation not neglecting low-frequency terms.

In Section 1 of Chapter 1 of this paper we discuss the general theory in some detail while Section 2 deals with the problem of representation of the displacement field due to a general Volterra dislocation by means of a line integral along the dislocation line. We also correct Steketee's statement on this problem a little. In the last part of Section 2, we show the derivation of the displacement field due to an edge or screw dislocation from the general surface integral expression given in Section 1, which can seldom be found in ordinary text books on the subject of crystal dislocations.

In Section 1 of Chapter 2, we calculate all the sets of Green's functions necessary for the displacement and stress fields around dislocations in a semi-infinite medium. Though some of the results may be easily obtained by an interchange of coordinates, we have arrived at such expressions by independent computation, this serving the purpose of avoiding errors in calculations. We find that the expressions of displacement on the free surface can be written in remarkably simple and refined forms. In Section 2 of Chapter 2, some examples of displacement fields on the free surface due to rectangular dislocation surfaces with constant discontinuity in displacement are given. As Chinnery employed the rectangular dislocation surface intersecting the free surface at right angles as the model of a strike slip fault, we may employ our results as the model of a dip slip fault or as other slip faults with various dip angles.

Chapter 1. Statical Elastic Dislocations in an Infinite Medium.

1-1. General.

In the elasticity theory of dislocations we consider a situation in which an infinite or finite elastic body, which may be unstrained and at rest, has the following largely imaginary process performed upon it. Imagine an open surface Σ which may be situated entirely in the interior of the body, being cut and the two faces of the cut deformed in different ways by applying some force distributions to them. If these force distributions are in static equilibrium the deformed state will also be in equilibrium. A situation created in such a manner we call a dislocation over Σ. We call the surface Σ the dislocation surface and the edge σ of Σ the dislocation line.

If the body is already strained to begin with, the process here described has to be extended in such a way that, while the cut is being made, forces equal to the original forces acting across Σ have to be introduced on the two faces of the cut to maintain the original equilibrium state (Steketee 1958 b).

For the sake of a definite description we first choose the positive direction of the dislocation line σ. If once we have chosen (arbitrarily) the positive direction of σ, then the positive sense of a closed curve linking the dislocation line, the positive sense of the outward normal $\boldsymbol{\nu}$ to Σ and the front and reverse sides of Σ are determined by the right-handed screw relation: the positive sense of a linking curve is the direction of rotation of a right-handed screw advancing the dislocation line σ and the positive sense of the normal to Σ, which goes from the reverse side of Σ to the front side, is the positive sense of the linking curve passing at the point on Σ (Fig. 1). Corresponding to the front side and the reverse side of Σ, the two faces of the cut made over Σ are specified as Σ^+ and Σ^- respectively.

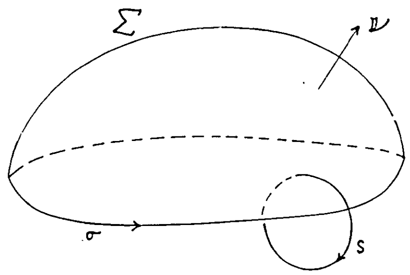

Fig. 1. Dislocation surface Σ with normal $\boldsymbol{\nu}$, dislocation line σ and a linking curve s.

We denote the components of the elastic displacement vector by u_k ($k=1, 2, 3$) and the components of the stress tensor by τ_{kl} where

$$\tau_{kl}=\lambda\delta_{kl}u_{n,n}+\mu(u_{k,l}+u_{l,k}) , \tag{1.1}$$

λ and μ are Lamé constants, δ_{kl} is the Kronecker delta, and $u_{k,l}=\partial u_k/\partial\xi_l$ where ξ_l ($l=1, 2, 3$) is a Cartesian coordinate.

The dislocation is then determined by the shape of the surface Σ and by the discontinuity Δu_k in the components of the displacement vector across the cut, that is

$$\Delta u_k=u_k^+-u_k^- , \tag{1.2}$$

where u_k^+ is the displacement vector for a point P^+, originally being at P on Σ but now situated on Σ^+, and u_k^- is the displacement vector for the corresponding point P^-, which is now on Σ^-. It is clear that the edge σ of Σ will be a singularity in general; Δu_k is not necessarily zero on the edge, and in that case the displacement components are not uniquely determined there.

We shall, following J. A. Steketee and others, distinguish between two types of dislocations:

(i) Volterra dislocations
(also called dislocations of Volterra-Weingarten),

(ii) Somigliana dislocations.

In the first case the discontinuity in the displacement components is prescribed to be

$$\left.\begin{aligned} \Delta u_k = u_k^+ - u_k^- = b_k + \Omega_{kj}\xi_j \ , \\ \Omega_{kj} = -\Omega_{jk} \ , \end{aligned}\right\} \tag{1.3}$$

where b_k and Ω_{kj} are constants and ξ_j is the coordinate of P on Σ. The relation (1.3), which is the well-known Weingarten relation, states that the discontinuity Δu_k across Σ should be of a rigid body displacement type. In solid state physics one usually considers the case $\Omega_{kj}=0$, and b_k is then called the Burgers vector.

In the case of a Somigliana dislocation Δu_k can have any form, provided the forces which maintain the dislocation do not violate the relation

$$\tau_{kl}^+\nu_l^+ + \tau_{kl}^-\nu_l^- = 0 \ , \tag{1.4}$$

where superscripts + and − correspond to the values at P^+ and P^- respectively; ν_l^+ is a component of the outward normal to the surface element $d\Sigma^+$ at P^+ and ν_l^- a component to the surface element $d\Sigma^-$ at P^-. Accordingly $\tau_{kl}^+\nu_l^+$ and $\tau_{kl}^-\nu_l^-$ in equation (1.4) are the forces per unit area on the surface elements $d\Sigma^+$ at P^+ and $d\Sigma^-$ at P^- respectively.

It is natural that the coordinate system of reference should be taken as the body-fixed coordinate system which may be fixed in the body before the dislocation is made. Therefore ν_l^+ and ν_l^- are equal and opposite to each other and are connected with the above-defined ν_l by the relation

$$-\nu_l^+ = \nu_l^- = \nu_l \ , \tag{1.5}$$

so that we may write in place of equation (1.4)

$$\tau_{kl}^+ - \tau_{kl}^- = 0. \tag{1.6}$$

For a Volterra dislocation, the relation (1.4) or (1.6) is automatically satisfied since a rigid body displacement does not generate any strains and stresses.

In case the body was unstrained to begin with, it will be strained once the dislocation is made, as there is a force distribution acting upon Σ^+ and Σ^- to keep the faces in their deformed position.

V. Volterra (1907) obtained the following theorem concerning the displacement field due to a dislocation of type (1.3) in an infinite medium: the m-component of displacement vector at an arbitrary point Q (x_1, x_2, x_3), $u_m(Q)$, is determined by the formula

$$u_m(Q)=\iint_{\Sigma} \Delta u_k(P) T^m_{kl}(P, Q)\nu_l(P)d\Sigma \ , \tag{1.7}$$

where P is a point on Σ over which the integral is taken and $T^m_{kl}(P, Q)$ is the (kl)-component of the stress tensor at P due to a unit body force in the m-direction located at Q. In Appendix 1, following Steketee (1958 a, b) and others, we will show that equation (1.7) is a relation valid to a general Somigliana dislocation.

A body force in the m-direction at Q generates a displacement field at P, the k-component of which is determined by the well-known formula

$$U^m_k(P, Q)=\frac{1}{8\pi\mu}(\delta_{km}r,_{nn}-\alpha r,_{mk}) \ , \tag{1.8}$$

where

$$\alpha=\frac{\lambda+\mu}{\lambda+2\mu} \ , \tag{1.9}$$

r is the distance from $P(\xi_1, \xi_2, \xi_3)$ to $Q(x_1, x_2, x_3)$, $r_{,n}=\partial r/\partial\xi_n$, $r_{,mk}=\partial^2 r/\partial\xi_m\partial\xi_k$ and the summation convention applies. From the definition

$$r=\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2}$$

we easily arrive at

$$r^{,m}=-r_{,m} \ , \tag{1.10}$$

where $r^{,m}=\partial r/\partial x_m$.

The expression (1.8) dose not change if k and m are interchanged:

$$U^m_k(P, Q)=U^k_m(P, Q) \ , \tag{1.11}$$

hence equation (1.8) may be considered as the m-component of displacement at P due to a unit body force at Q. Further in view of equation (1.10) we have

$$U_k^m(P, Q) = U_k^m(Q, P)\ , \tag{1.12}$$

which shows that equation (1.8) may also be considered as the displacement field at Q due to a unit body force located at P.

If we write simply U_k^m in place of $U_k^m(P, Q)$ in equation (1.8), above-defined $T_{kl}^m(P, Q)$ is expressed as

$$T_{kl}^m(P, Q) = \lambda\delta_{kl}U_{n,n}^m + \mu(U_{k,l}^m + U_{l,k}^m)\ , \tag{1.13}$$

which may be written in view of equations (1.8), (1.10) and (1.11) as

$$T_{kl}^m(P, Q) = -\lambda\delta_{kl}U_m^{n,n} - \mu(U_m^{k,l} + U_m^{l,k})\ . \tag{1.14}$$

By means of equation (1.11) and (1.12) the right-hand side of (1.14) can be interpreted as a linear combination of the first derivatives with respect to the coordinates of Q of the m–components of displacement fields at Q caused by unit body forces at P in the x_1-, x_2-, x_3-, x_k- and x_l-directions. For instance $U_m^{1,1}$ is by definition

$$U_m^{1,1} = \lim_{\Delta x_1 \to 0}\left\{\frac{1}{\Delta x_1}U_m^1(x_1+\Delta x_1, \cdots) - \frac{1}{\Delta x_1}U_m^1(x_1, \cdots)\right\}\ .$$

This means the m-component of displacement at Q generated by a singularity which is obtained by passing to the limit by supposing that Δx_1 is diminished indefinitely while two opposite forces in the x_1- and $(-x_1)$-directions of equal magnitude $(\Delta x_1)^{-1}$ at a distance of Δx_1 are acting in the neighborhood of P, and which is called a nucleus of strain or a double force by Love. If $k=l$, $U_m^{k,l}$ is referred to as a double force without moment and for $k \neq l$ it is a double force with moment. Hence $T_{kl}^m(P, Q)$ may be considered as m-component of the displacement vector at Q generated by a combination of strain nuclei at P; in the case $k=l$ this is a combination of a center of dilatation and an additional double force without moment, A-nucleus after Steketee; in the case $k \neq l$ it is a combination of twe coplanar, mutually perpendicular double forces with moment, B-nucleus after Steketee. If we denote by a sphere a center of dilatation which is a combination of three equal, mutually perpendicular double forces without moment, the two combinations may be schematically represented as in Fig. 2. It follows that the displace-

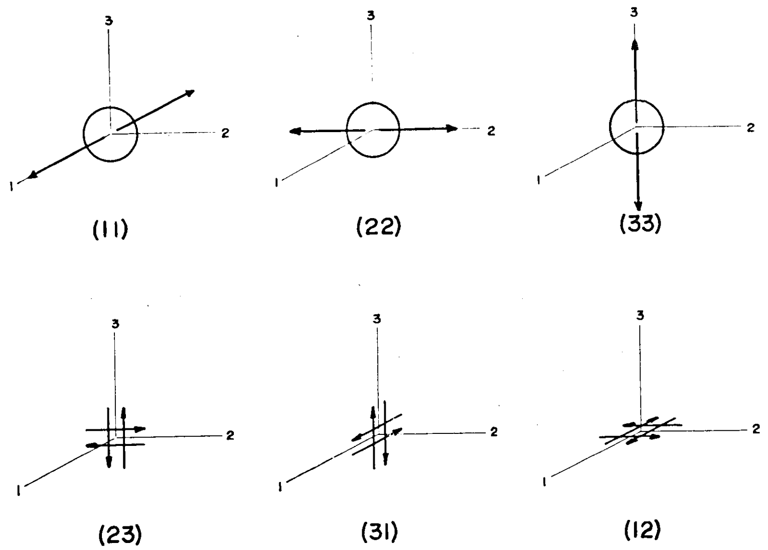

Fig. 2. (kl) is a combination of double forces at P which generates the displacement field T^m_{kl} at Q. (11), (22) and (33) are A-nuclei; (23), (31) and (12) are B-unclei.

ment component $u_m(Q)$ in (1.7) may be considered as the resultant effect of a distribution of A- and B-nuclei over Σ. The magnitude of the contribution of a particular combination of nuclei at a point P on Σ to the displacement components at Q depends both on the local values of Δu_k and the orientation of the surface element $d\Sigma$ with respect to Q and the direction considered at Q. If we consider a surface element $d\Sigma$ of Σ with normal in the x_3-direction, its contribution to the displacement at Q is

$$du_m(Q) = (\Delta u_1 T^m_{13} + \Delta u_2 T^m_{23} + \Delta u_3 T^m_{33}) d\Sigma \, .$$

It is clear that Δu_1 and Δu_2, which are in directions perpendicular to the normal may be called the slip, while Δu_3 represents the discontinuity in displacement of the two faces $d\Sigma^+$ and $d\Sigma^-$ in the direction of normal. The effect of the slip can be described by nuclei of type B while the effect of the normal discontinuity is described by a nucleus A.

Once the displacement field $u_m(Q)$ has been obtained as in equation (1.7), the stresses at the point $Q(x_k)$ can be found by differentiating with respect to x_k similar to (1.1). If the point $Q(x_k)$ is not on Σ, these differentiations can be performed in general under the integration sign where the variables $x_k-\xi_k$ appear in T^m_{kl}. We then obtain for points not on Σ

$$\tau_{mn}(Q)=\iint \Delta u_k(P)G^{mn}_{kl}(P, Q)\nu_l(P)d\Sigma\ , \tag{1.15}$$

where

$$G^{mn}_{kl}(P, Q)=\lambda\delta_{mn}T^{h,h}_{kl}+\mu(T^{m,n}_{kl}+T^{n,m}_{kl})\ . \tag{1.16}$$

We can work out $T^m_{kl}(P, Q)$ and $G^{mn}_{kl}(P, Q)$ from (1.8), (1.14) and (1.16) in explicit expressions. They can be written as follows

$$T^m_{kl}(P, Q)=\frac{1}{4\pi}\left\{(1-\alpha)\left(-\delta_{kl}\frac{r_m}{r^3}+\delta_{mk}\frac{r_l}{r^3}+\delta_{lm}\frac{r_k}{r^3}\right)+3\alpha\frac{r_kr_lr_m}{r^5}\right\}, \tag{1.14$'$}$$

$$\begin{aligned} G^{mn}_{kl}(P, Q)=\frac{\mu}{4\pi}\Big\{&-2(2-3\alpha)\delta_{kl}\delta_{mn}\frac{1}{r^3}\\ &+2(1-\alpha)(\delta_{km}\delta_{ln}+\delta_{lm}\delta_{kn})\frac{1}{r^3}\\ &+6(1-\alpha)(\delta_{kl}r_mr_n+\delta_{mn}r_kr_l)\frac{1}{r^5}\\ &-3(1-2\alpha)(\delta_{km}r_lr_n+\delta_{lm}r_kr_n+\delta_{kn}r_lr_m+\delta_{ln}r_kr_m)\frac{1}{r^5}\\ &-30\alpha(r_kr_lr_mr_n)\frac{1}{r^7}\Big\}\ , \end{aligned} \tag{1.16$'$}$$

where

$$r_k=x_k-\xi_k \qquad (k=1,\ 2,\ 3)$$

for $P(\xi_1,\ \xi_2,\ \xi_3)$ and $Q(x_1,\ x_2,\ x_3)$.

1-2. On Volterra Dislocations.

As we have seen above in equations (1.3) a Volterra dislocation is characterized by a simple type of discontinuity in displacement on the dislocation surface Σ. Owing to the simplicity it can easily be handled and possesses some prominent properties which might offer aid in the comprehension of the singularity of a dislocation in general.

Now we show that the displacement field due to a Volterra dislocation (1.3) does not depend on the actual shape of Σ but only on the shape of the edge σ of Σ*.

* Steketee remarks that this theorem is valid if the $3\Omega_{kj}$'s in (1.3) vanish, however, it is valid in general.

In equation (1.7) T^m_{kl} is the (kl)-component of the stress tensor on the surface element $d\Sigma$ containing the point P due to a unit body force $\boldsymbol{e}_m$ in the m-direction located at Q.

Let S be an arbitrarily closed surface supposed in an infinite medium, where we assume that no body forces are acting in the region D which is surrounded by the surface S, and that a unit body force $\boldsymbol{e}_m$ in the m-direction is acting at a point Q outside S. We define a vector $\boldsymbol{T}^m$ of which k-component can be written as

$$(\boldsymbol{T}^m)_k = T_k{}^m = T^m_{kl}\nu_l\ , \tag{1.17}$$

where T^m_{kl} is the (kl)-component of the stress tensor at the point on the surface element dS due to the body force $\boldsymbol{e}_m$ at Q. From the definition of the stress tensor, $\boldsymbol{T}^m$ may then be considered as the force per unit area exerted from the front side to the reverse side across the surface element dS. According to the equilibrium condition,

(i) the resultant force exerted on S vanishes;

(ii) the resultant moment of the force exerted on S vanishes.

Let $\boldsymbol{b}$ and $\boldsymbol{\omega}$ be two arbitrary constant vectors. Using the above-defined $\boldsymbol{T}^m$ the two conditions may be written as

$$\iint \boldsymbol{b}\cdot \boldsymbol{T}^m dS = 0\ , \tag{1.18}$$

$$\iint \boldsymbol{\omega}\cdot (\boldsymbol{\xi}\times \boldsymbol{T}^m) dS = 0\ , \tag{1.19}$$

where $\boldsymbol{\xi}(\xi_1, \xi_2, \xi_3)$ is the position vector of the point P on dS. The integrand of (1.18) becomes

$$\boldsymbol{b}\cdot \boldsymbol{T}^m = b_k T^m_k = b_k T^m_{kl}\nu_l\ . \tag{1.20}$$

If we introduce Ω_{kj} defined as

$$\left.\begin{aligned} \omega_1 &= -\Omega_{23} = \Omega_{32} \\ \omega_2 &= -\Omega_{31} = \Omega_{13} \\ \omega_3 &= -\Omega_{12} = \Omega_{21} \\ \Omega_{kk} &= 0 \end{aligned}\right\} \quad \text{or} \quad \omega_i = -\frac{1}{2}\Omega_{jk}\varepsilon_{ijk}\ , \tag{1.21}$$

where

$\varepsilon_{ijk} = 1$ when i, j, k is an even permutation of the number 1, 2, 3;

$\quad = 0$ when any two of the indices are equal;

$=-1$ when i, j, k is an odd permutation of the number 1, 2, 3, (1.22)

from the definition of vector product we have

$$
\begin{aligned}
(\boldsymbol{\omega}\times\xi)_k &= \omega_i\xi_j\varepsilon_{ijk} \\
&= -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{ihl}\Omega_{hl}\xi_j \\
&= \Omega_{kj}\xi_j\,,
\end{aligned}
\tag{1.23}
$$

where we employed the following relation ,

$$\varepsilon_{ijk}\varepsilon_{ihl}=\delta_{jh}\delta_{kl}-\delta_{jl}\delta_{kh}\,. \tag{1.24}$$

Using (1.23) the integrand of (1.19) becomes

$$
\begin{aligned}
\omega\cdot(\xi\times \boldsymbol{T}^m) &= (\boldsymbol{\omega}\times\xi)\cdot \boldsymbol{T}^m \\
&= (\boldsymbol{\omega}\times\xi)_k T^m_k \\
&= (\Omega_{kj}\xi_j)T^m_{kl}\nu_l\,.
\end{aligned}
\tag{1.25}
$$

It follows that for an arbitrary constant vector b_k and an arbitrary constant anti-symmetric tensor Ω_{kj} we have the relation

$$\iint(b_k+\Omega_{kj}\xi_j)T^m_{kl}\nu_l=0\,. \tag{1.26}$$

If the closed suface S is divided into two open surfaces S_1 and S_2 by a closed curve s on S and if the positive direction of the normal to S_1 and S_2 are taken consistently according to the positive direction of s, we get the following formula

$$\iint_{S_1}(b_k+\Omega_{kj}\xi_j)T^m_{kl}\nu_l dS=\iint_{S_2}(b_k+\Omega_{kj}\xi_j)T^m_{kl}\nu_l dS\,. \tag{1.27}$$

Fixing the closed curve s, the constant vector b_k and the constant anti-symmetric tensor Ω_{kj} and repeating the above derivation for an arbitrary close surface S including s, we can obtain the above formula. Accordingly in view of the definition of a Volterra dislocation and equation (1.7) we arrive at the conclusion that the displacement field due to a Volterra dislocation does not depend on the actual shape of S_1 but only on the shape of the edge s of S_1.

So far we have assumed that Q is outside S. If Q is within S, according to the equilibrium condition with respect to the portion, in place of (1.18) and (1.19), we have

$$\iint b \cdot \boldsymbol{T}^m dS + \boldsymbol{b} \cdot \boldsymbol{e}_m = 0\,, \tag{1.28}$$

$$\iint \boldsymbol{\omega} \cdot (\xi \times \boldsymbol{T}^m) dS + \boldsymbol{\omega} \cdot (\boldsymbol{x} \times \boldsymbol{e}_m) = 0\,, \tag{1.29}$$

where $\boldsymbol{x}(x_1, x_2, x_3)$ is the position vector of the point Q. Using these equations we can examine the discontinuity in displacement supposed on the dislocation surface Σ.

For an open surface Σ in an infinite medium by adding another open surface Σ', we suppose a closed surface $(\Sigma+\Sigma')$ (Fig. 3), and consider two points close to Σ: Q^+ outside $(\Sigma+\Sigma')$ and Q^- inside $(\Sigma+\Sigma)'$. If a unit body force in the m-direction $\boldsymbol{e}_m$ is acting at Q^+, from equations (1.18) and (1.19), we have

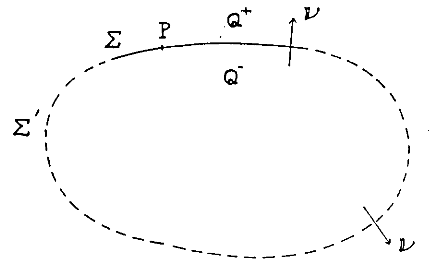

Fig. 3. An open surface Σ and another surface Σ' make a closed surface $(\Sigma+\Sigma')$.

$$\iint_{\Sigma+\Sigma'} (\boldsymbol{b} + \boldsymbol{\omega} \times \xi) \boldsymbol{T}^{m+} d\Sigma = 0\,,$$

while if $\boldsymbol{e}_m$ is acting at Q^-, from equations (1.28) and (1.29), we have

$$\iint_{\Sigma+\Sigma'} (\boldsymbol{b} + \boldsymbol{\omega} \times \xi) \boldsymbol{T}^{m-} d\Sigma + (\boldsymbol{b} + \boldsymbol{\omega} \times \boldsymbol{x}^-) \cdot \boldsymbol{e}_m = 0\,,$$

where superscripts $+$ and $-$ correspond to Q^+ and Q^- respectively. From these equations,

$$\iint_{\Sigma} (\boldsymbol{b} + \boldsymbol{\omega} \times \xi)\, \boldsymbol{T}^{m+}\, d\Sigma - \iint_{\Sigma} (\boldsymbol{b} + \boldsymbol{\omega} \times \xi) \boldsymbol{T}^{m-} d\Sigma$$
$$= (\boldsymbol{b} + \boldsymbol{\omega} \times \boldsymbol{x}^-) \boldsymbol{e}_m + \left[-\iint_{\Sigma'} (\boldsymbol{b} + \boldsymbol{\omega} \times \xi)\, \boldsymbol{T}^{m+}\, d\Sigma + \iint_{\Sigma'} (\boldsymbol{b} + \boldsymbol{\omega} \times \xi)\, \boldsymbol{T}^{m-} d\Sigma \right]. \tag{1.30}$$

By equation (1.7) the left-hand side in (1.30) represents the difference in the m-component of the displacements at Q^+ and at Q^-, that is $u_m(Q^+) - u_m(Q^-)$, caused by a Volterra dislocation specified by $\boldsymbol{b}$ and $\boldsymbol{\omega}$

on Σ. If we pass to the limit so that Q^+ and Q^- come to a point Q on Σ, on the right-hand side in (1.30) $\boldsymbol{x}^-$ bocomes $\boldsymbol{x}$, the position vector of Q, and the expressions in square brackets vanish, hence the relative displacement $\Delta\boldsymbol{u}(Q)$ of neighboring points on either side of Σ is given by an expression of the form

$$\Delta\boldsymbol{u}(Q)=\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{x}\,. \tag{1.31}$$

As we have seen above, the displacement field due to a Volterra dislocation (1.3) does not depend on the actual shape of Σ but only on the edge σ of Σ, hence, though the discontinuity in displacement (1.3) is first defined on Σ, there appears no singularity except on the edge σ. This corresponds to the displacement field which is many-valued and continuous in the region, then in order to make the displacement field single-valued, if we suppose an arbitrary surface Σ' which possesses σ for the edge, the discontinuity in diplacement across Σ' can be expressed by equation (1.31). It follows that the surface integral along Σ in equation (1.7) can be replaced by a line integral along σ in the case of a Volterra dislocation.

Here we recall Weingarten theorem related to the displacement in an elastic body occupying a multiply-connected region, where the displacement may be regarded as many-valued and continuous in the region, or as single-valued and discontinuous at the barriers:

Let the multiply-connected region occupied by the body be reduced to a simply-connected region by means of a system of barriers, physically by means of a system of cuts in the body. If the strains in the body are everywhere finite and continuously differentiable for twice, the disturbance of the cut faces cannot be arbitrary but their relative displacement must be of one type which is a possible displacement for a rigid body as in equations (1.3).

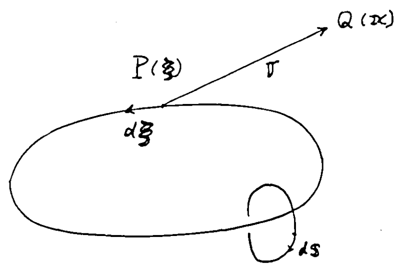

Fig. 4. Line element $d\xi$ of dislocation line and line element ds of a linking curve s.

We can apply this theorem to the case easily. If we remove the material inside a closed thin tude surrounding the dislocation line σ, the region becomes doubly-connected. Then we observe that if the dislocation Δu_k on the surface Σ is in any form which does not reduce to a rigid displacement, that is if it is not a Volterra dislocation, the strains are not everywhere finite and continuously differentiable for twice in the region. Accordingly in the case of a general Somigliana dislocation the singularity will remain over Σ and the strains will not be continuously differentiable for twice over Σ.

For a Volterra dislocation $\Delta \boldsymbol{u}=\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{\xi}$, the formula (1.7) can be written by means of line integrals along the loop σ in Fig. 4. (the derivation is given in Appendix 2) as

$$\begin{aligned}\boldsymbol{u}(Q)=&\frac{1}{4\pi}(\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{x})\Omega\\&+\frac{1}{4\pi}\int\frac{1}{r}(\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{\xi})\times d\boldsymbol{\xi}\\&+\frac{\alpha}{4\pi}\ \mathrm{grad}_Q\int\frac{1}{r}[(\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{\xi})\times\boldsymbol{r}]\cdot d\boldsymbol{\xi}\\&+\frac{1}{4\pi}\int\frac{1}{r}\boldsymbol{\omega}\times(\boldsymbol{r}\times d\boldsymbol{\xi})\\&-\frac{\alpha}{4\pi}\int\frac{1}{r}(\boldsymbol{\omega}\times\boldsymbol{r})\times d\boldsymbol{\xi}\,,\end{aligned}\tag{1.32}$$

where $d\boldsymbol{\xi}(d\xi_1,\ d\xi_2,\ d\xi_3)$ is an element of the line σ and the integrals are taken once round the dislocation line in the positive sense and Ω is the solid angle subtended by the loop σ at the point Q and the suffix Q means the derivative with respect to the coordinates of Q.

In equation (1.32) we observe that the multi-valued character of the displacement components is expressed by the first term of the right-hand member. If the point Q is not on the loop σ, the second term and those that follow are finite and single-valued functions which depend on the shape of the loop σ. Further we observe that, in order to exclude multi-valued character of the displacement, if we suppose an open surface bounded by the loop σ, the discontinuity of the displacement across the surface is

$$\Delta \boldsymbol{u}=\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{x}\,,$$

where $\boldsymbol{x}$ is the position vector of the point through which we suppose the barrier surface.

We can also obtain this discontinuity by the line integration along a linking curve s in the positive direction as

$$\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{x}=-\int\frac{\partial\boldsymbol{u}}{\partial s}ds\,,$$

where the integral is taken once round the linking curve. When $\omega=0$, this relation is often used to define the Burgers vector.

Since the derivatives of solid angle are single-valued we observe that the stress field due to a Volterra dislocation is single-valued everywhere except on the dislocation line σ.

As for a special case in which $\omega=0$, equation (1.32) can be written as

$$\boldsymbol{u}(Q)=\frac{1}{4\pi}\boldsymbol{b}\Omega+\frac{1}{4\pi}\boldsymbol{b}\times\int\frac{1}{r}d\boldsymbol{\xi}+\frac{\alpha}{4\pi}\operatorname{grad}_Q\int\frac{1}{r}[\boldsymbol{b}\times\boldsymbol{r}]\cdot d\boldsymbol{\xi}\,, \tag{1.33}$$

which is the formula obtained by Burgers (1939).

According to Nabarro (1952), the first application of dislocations to the theory of plastic deformation was made by Prandtl in a lecture course in 1921–2, in which he developed a model to explain the plastic properties of crystals by assuming that in regions of imperfection there were molecules which could change their allegiance from one relatively perfect lattice to another. Prandtl (1928), however, did not indicate the connection of these imperfections with the dislocations of the theory of elasticity. The foundations of the modern theory of crystal dislocation were laid by Taylor, Orowan and Polanyi in 1934.

Taylor (1934) investigated the characteristic properties of an elementary, two-dimensional type of dislocation which is now called 'edge' dislocation. He pointed out the connection of this dislocation with the elastic dislocation of Volterra.

The two-dimensional type of dislocation must extend in a straight line through the lattice from one boundary surface of a crystal to the opposite boundary. J. M. Burgers (1939) considered that dislocations characterized by disturbances of a more general, three-dimensional type, which may be confined to a region of finite extent, might lead to a more adequate picture of what is to be found in an actual crystal. Burgers presented the formula (1.33) with ingenious interpretations of each term and was led to consider a so-called 'screw' dislocation. Concerning the application of the elasticity theory to the problem of crystal, Burgers affirms that although the components of the displacement in reality are defined only for the (enumerable) set of lattice points where atoms are to be found, they can be considered as being determined by functions of the coordinates which in general are everywhere continuous and finite.

F. R. N. Nabarro (1951), contrary to our derivation, by applying

Burgers' formula to an infinitessimal loop and by expressing the field as a double integral over the dislocation surface, interpreted the dislocation field due to a dislocation as resulting from a combination of double forces.

Here we shall show some examples of the employment of equation (1.14)$'$ to obtain the displacement field due to an edge and a screw dislocation. Consider a dislocation surface of plane in (x_1x_3)-plane. If the dislocation surface is extended to a half-plane and the singularity is left along a straight line, which we take here as x_1-axis or x_3-axis, we arrive at two simple cases of slip dislocation. The Burgers vector $\boldsymbol{b}$ is taken to be parallel to x_3-axis; if the dislocation line is x_1-axis, the case is edge dislocation; if the dislocation line is x_3-axis, the case is screw dislocation. This type of computation is after F. R. N. Nabarro (1951).

In equations (2.7) and (2.3), puting

$$(b_1,\ b_2,\ b_3)=(0,\ 0,\ b)\ ,$$

$$(\nu_1,\ \nu_2,\ \nu_3)=(0,\ 1,\ 0)\ ,$$

we have

$$u_m=b\int_0^\infty d\xi_3\int_{-\infty}^\infty T_{23}^m d\xi_1\ ,$$

for an edge dislocation lying along x_1-axis and

$$u_m=b\int_0^\infty d\xi_1\int_{-\infty}^\infty T_{23}^m d\xi_3\ ,$$

for a screw dislocation lying along x_3-axis, where

$$T_{23}^1=\frac{3\alpha}{4\pi}\frac{(x_1-\xi_1)x_2(x_3-\xi_3)}{r^5}\ ,$$

$$T_{23}^2=\frac{(1-\alpha)}{4\pi}\frac{(x_3-\xi_3)}{r^3}+\frac{3\alpha}{4\pi}\frac{x_2^2(x_3-\xi_3)}{r^5}\ ,$$

$$T_{23}^3=\frac{(1-\alpha)}{4\pi}\frac{x_2}{r^3}+\frac{3\alpha}{4\pi}\frac{x_2(x_3-\xi_3)^2}{r^5}\ .$$

We obtain the displacement field due to the edge dislocation along x_1-axis as

$$u_1=0\ ,$$

$$u_2=b\left\{\frac{\alpha}{2\pi}\frac{x_2^2}{x_2^2+x_3^2}-\frac{(1-\alpha)}{4\pi}\log(x_2^2+x_3^2)\right\}\ ,$$

$$u_3 = b\left\{\frac{\alpha}{2\pi}\frac{x_2 x_3}{x_2^2+x_3^2}+\frac{1}{2\pi}\arctan\left(\frac{x_2}{x_3}\right)\right\} ,$$

and that due to the screw dislocation along x_3-axis as

$$u_1=0 ,$$

$$u_2=0 ,$$

$$u_3=b\left\{-\frac{1}{2\pi}\arctan\frac{x_2}{x_1}\right\} .$$

Chapter 2. Statical Elastic Dislocations in a Semi-Infinite Medium.

2-1. General.

Let the surface S of the semi-infinite elastic medium be the plane $x_3=0$ where the positive x_3-axis penetrates the medium. The boundary conditions we have to satisfy are those for each point Q in the plane $x_3=0$,

$$\tau_{31}(Q)=\tau_{32}(Q)=\tau_{33}(Q)=0 , \tag{2.1}$$

while we shall in general also require the stresses and displacements to be continuous and differentiable everywhere, with the possible exception of Σ, and moreover to vanish at infinity.

In order to obtain the displacement field due to a dislocation in a semi-infinite medium we may employ the solution to the infinite medium.

As we have already seen stresses due to a dislocation in an infinite medium are constructed by integration of a function including $G_{kl}^{mn}(P, Q)$ along the dislocation surface and for fixed k and l $G_{kl}^{mn}(P, Q)$ in itself may be considered as a stress component (mn) at Q due to a combination of double forces at P.

In equation (1.16) we observe that in case G_{kl}^{31} and G_{kl}^{32} are odd (even) in $r_3=x_3-\xi_3$, the function G_{kl}^{33} is even (odd) in r_3. Then if we put the same (opposite) combinations of double forces at the image point $P'(\xi_1, \xi_2, -\xi_3)$ of P with respect to the plane $x_3=0$, we obtain by superposition of the original and the image fields a stress field which satisfies the relation

$$G_{kl}^{31}(P, Q)\pm G_{kl}^{31}(P', Q)=0 ,$$

$$G_{kl}^{32}(P, Q)\pm G_{kl}^{32}(P', Q)=0 ,$$

at each point Q on the plane $x_3=0$. Since the normal stress is doubled for $x_3=0$, we have to consider the solution to the normal load as equal and opposite to the ones obtained from the double forces at P and P'.

Thus for the displacement field due to a dislocation in a semi-infinite medium, we have to superpose three displacement fields, which may be cited for fixed k and l as

(i) double force (kl) at P,

(ii) double force $(kl)'$ at P',

(iii) normal load $(-2G_{kl}^{33})$ on the plane $x_3=0$.

It is clear that the resultant displacement field satisfies the boundary condition (2.1) and that it vanishes at infinity if proper solutions to (iii) are selected. By the force systems of (ii) and (iii) no singularities are added in the half-space where the dislocation is situated, and we shall have almost the same field as in an infinite medium in the neighborhood of the dislocation where the effect of the boundary can be neglected. Hence we arrive at the solution: if we denote by W_{kl}^m the m-component of the resultant displacement due to (i), (ii) and (iii) for fixed k and l, we get the displacement field due to an arbitrary dislocation in a semi-infinite medium by employing W_{kl}^m in place of T_{kl}^m in the infinite case.

To obtain the displacement field due to (iii) is often called the Boussinesq problem. Here we shall solve the problem by making use of Galerkin vectors after Steketee.

When there are no body forces the equation of equilibrium in terms of displacements is

$$(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u}+\mu\nabla^2\boldsymbol{u}=0\ ,$$

or (2.2)

$$(\lambda+\mu)u_{n;}^{\ ;nk}+\mu\, u_{k;}^{\ ;nn}=0\ .$$

The Galerkin vector $\boldsymbol{\Gamma}(\Gamma_1, \Gamma_2, \Gamma_3)$ is defined as a vector from which a displacement field $\boldsymbol{u}(x_1, x_2, x_3)$, satisfying equation (2.2), is obtained by differentiation according to a linear second-order operator, that is

$$\boldsymbol{u}=(\nabla^2-\alpha\,\mathrm{grad}\,\mathrm{div})\boldsymbol{\Gamma}\ ,$$

or (2.3)

$$u_k=\Gamma_{k}^{;nn}-\alpha\Gamma_{n}^{;nk}\ ,$$

where $\alpha=(\lambda+\mu)/(\lambda+2\mu)$ as before. If the Galerkin vector is known for a problem, the problem is solved.

By substituting (2.3) in equation (2.2) we can show that each component of the Galerkin vector is biharmonic,

$$\nabla^2\nabla^2\boldsymbol{\Gamma}=0 ,$$

or (2.4)

$$\Gamma_{k}^{,mmnn}=0 .$$

As for the displacement field at Q, U_m^k, due to a point body force at P in the x_k-direction, we can write it as follows,

$$\begin{aligned} U_m^k &= \frac{1}{8\pi\mu}\left[\delta_{mk}r^{,nn}-\alpha r^{,mk}\right] \\ &= \frac{1}{8\pi\mu}\left[(\delta_{mk}r)^{,nn}-\alpha(\delta_{kl}r)^{,lm}\right] . \end{aligned} \tag{2.5}$$

Hence the m-component of the Galerkin vector which leads the displacement field U_m^k may be considered as

$$\Gamma_m^k=\frac{1}{8\pi\mu}\delta_{mk}r . \tag{2.6}$$

Using (2.6) we can easily obtain the Galerkin vector Γ_{kl}^m which gives the displacement field T_{kl}^m with k and l fixed. Corresponding to

$$T_{kl}^m=-\lambda\delta_{kl}U_m^{n,n}-\mu\{U_m^{k,l}+U_m^{l,k}\} ,$$

we have

$$\begin{aligned} 8\pi\mu\Gamma_{kl}^m &= -\lambda\delta_{kl}(\delta_{mn}r)^{,n}-\mu[(\delta_{mk}r)^{,l}+(\delta_{ml}r)^{,k}] \\ &= -\lambda\,\delta_{kl}r^{,m}-\mu\,[\delta_{mk}r^{,l}+\delta_{ml}r^{,k}] . \end{aligned} \tag{2.7}$$

The solution to the Boussinesq problem is given by a Galerkin vector of the form (Steketee 1958 a, b)

$$\boldsymbol{\Gamma}=(0, 0, \Gamma) . \tag{2.8}$$

Let the double Fourier transform of Γ be denoted as $\bar{\Gamma}$, then by definition

$$\begin{aligned} \Gamma(x_1, x_2, x_3) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty}\bar{\Gamma}(k_1, k_2, x_3)e^{i(k_1x_1+k_2x_2)}dk_1dk_2 , \\ \bar{\Gamma}(k_1, k_2, x_3) &= \frac{1}{2\pi}\iint_{-\infty}^{\infty}\Gamma(x_1, x_2, x_3)e^{-i(k_1x_1+k_2x_2)}dx_1dx_2 . \end{aligned} \tag{2.9}$$

Concerning the Fourier transforms of the derivatives of a function $\varphi(x_1, x_2, x_3)$, under proper conditions of derivatives of φ when $x_1, x_2 \rightarrow \pm\infty$, we get

$$\overline{\left(\frac{\partial^r \varphi}{\partial x_1^r}\right)} = (ik_1)^r \overline{\varphi} , \quad \overline{\left(\frac{\partial^r \varphi}{\partial x_2^r}\right)} = (ik_2)^r \overline{\varphi} . \tag{2.10}$$

Operating $\nabla^4 = (\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2)^2$ to the first equation of (2.9), we have from (2.4)

$$\left(\frac{d^2}{dx_3^2} - k^2\right)^2 \overline{\Gamma} = 0 , \tag{2.11}$$

with the general solution

$$\overline{\Gamma} = (A + Bkx_3)e^{-kx_3} + (C + Dkx_3)e^{kx_3} ,$$

where

$$k = \sqrt{k_1^2 + k_2^2} . \tag{2.12}$$

Since we have chosen the positive x_3-axis to penetrate the medium, owing to the boundary conditions at infinity $C = D = 0$ and we are left with

$$\overline{\Gamma}(k_1, k_2, x_3) = (A + Bkx_3)^{-kx_3} . \tag{2.13}$$

From equation (2.3) and (2.8) the displacement field and stress field derived from Γ are expressed as

$$u_k = \delta_{3k} \Gamma^{,nn} - \alpha \Gamma^{,3k} ,$$

$$\tau_{kl} = \delta_{kl} \lambda (1 - \alpha) \Gamma^{,nn3} + \mu (\delta_{3k} \Gamma^{,lnn} + \delta_{3l} \Gamma^{,knn}) - 2\alpha\mu \Gamma^{,3kl} .$$

Using (2.10), we have

$$\left.\begin{aligned}
\overline{u}_1 &= -\alpha (ik_1) \frac{d}{dx_3} \overline{\Gamma} , \\
\overline{u}_2 &= -\alpha (ik_2) \frac{d}{dx_3} \overline{\Gamma} , \\
\overline{u}_3 &= \left[(1 - \alpha) \frac{d^2}{dx_3^2} - k^2 \right] \overline{\Gamma} , \\
\overline{\tau}_{11} &= \mu \frac{d}{dx_3} \left[(2\alpha - 1) \left(\frac{d^2}{dx_3^2} - k^2 \right) + 2\alpha k_1^2 \right] \overline{\Gamma} , \\
\overline{\tau}_{22} &= \mu \frac{d}{dx_3} \left[(2\alpha - 1) \left(\frac{d^2}{dx_3^2} - k^2 \right) + 2\alpha k_2^2 \right] \overline{\Gamma} ,
\end{aligned}\right\} \tag{2.14}$$

$$\bar{\tau}_{33}=\mu\frac{d}{dx_3}\left[\frac{d^2}{dx_3^{\ 2}}-(1+2\alpha)k^2\right]\bar{\Gamma},$$

$$\bar{\tau}_{23}=\mu(ik_2)\left[(1-2\alpha)\frac{d^2}{dx_3^2}-k^2\right]\bar{\Gamma},$$

$$\bar{\tau}_{31}=\mu(ik_1)\left[(1-2\alpha)\frac{d^2}{dx_3^2}-k^2\right]\bar{\Gamma},$$

$$\bar{\tau}_{12}=2\alpha\mu\, k_1k_2\frac{d}{dx_3}\bar{\Gamma}.$$

Substituting (2.13) in these equations we obtain

$$\bar{\tau}_{31}=2\mu(ik_1)k^2[-\alpha A+(2\alpha-1)B-\alpha Bkx_3]e^{-kx_3},$$

$$\bar{\tau}_{32}=2\mu(ik_2)k^2[-\alpha A+(2\alpha-1)B-\alpha Bkx_3]e^{-kx_3},$$

$$\bar{\tau}_{33}=2\mu k^3[\alpha A+(1-\alpha)B+\alpha Bkx_3]e^{-kx_3}.$$

Since the shearing stress and their transforms have to vanish at $x_3=0$, we have

$$A=\left(2-\frac{1}{\alpha}\right)B,$$

which yields

$$\bar{\tau}_{33}=2\mu\alpha Bk^3(1+kx_3)e^{-kx_3}.$$

Hence, if the normal load distribution on $x_3=0$ is denoted as $p(x_1, x_2)$,

$$\tau_{33}(x_1, x_2, 0)=p(x_1, x_2), \tag{2.15}$$

and the Fourier transforms of p as $\bar{p}$, B is determined by the relation

$$B(k_1, k_2)=\frac{1}{2\alpha}\frac{1}{k^3}\frac{\bar{p}(k_1, k_2)}{\mu}.$$

Thus, the Boussinesq problem is solved by the Galerkin vector $\boldsymbol{\Gamma}(0, 0, \Gamma)$, where the transform of Γ is given as

$$\bar{\Gamma}=\frac{1}{2\alpha}\frac{\bar{p}}{\mu}\left\{(2-\frac{1}{\alpha})k^{-3}+x_3k^{-2}\right\}e^{-kx_3}. \tag{2.16}$$

Now we compute the displacement field W_{kl}^m and the stress field F_{kl}^{mn} which corresponds to G_{kl}^{mn} in the infinite case.

We verify from equation (1.16)′ that G_{kl}^{13} and G_{kl}^{23} are even functions of r_3 and G^{33}_{kl} is an odd function of r_3 when

(Case I): $(kl)=(11),\ (22),\ (33),\ (12),$

while G^{13}_{kl} and G^{23}_{kl} are odd functions of r_3 and G^{33}_{kl} is an even function of r_3 when

(Case II): $(kl)=(23),\ (31).$

For the sake of simplicity we replace r_1 and r_2 by x_1 and x_2 respectively for the present, or we may say that we compute for a point P with the coordinate $(0, 0, \xi_3)$. For an arbitrary point $P(\xi_1, \xi_2, \xi_3)$, we can get the corresponding results by replacing x_1 and x_2 by $r_1=(x_1-\xi_1)$ and $r_2=(x_2-\xi_2)$ respectively in the final expressions. Then if we put $x_3=0$ in equation (1.16)$'$ for $P(0, 0, \xi_3)$, we see that G^{31}_{kl} and G^{32}_{kl} are even functions of ξ_3 in Case I, while they are odd functions of ξ_3 in Case II. Therefore we have to put the same combination of double forces at the image point P' of P with respect to the plane $x_3=0$ in Case I, while the opposite combination of double forces in Case II, in order to cancel the tangential stresses on the boundary $x_3=0$ due to the force systems at P and P'.

Using equation (2.7), we can easily make a list of the components of Galerkin vectors which give the fields due to force systems at P and P'. They are as shown concretely in equations (2.17), though they are not always indispensable for the following calculations.

Case I:—

$$\left.\begin{array}{ll}
\left\{\begin{aligned}
\Gamma^1_{11} &= -\frac{1}{8\pi}\left(\frac{\lambda+2\mu}{\mu}\right)\left(\frac{x_1}{R}+\frac{x_1}{S}\right), \\
\Gamma^2_{11} &= -\frac{1}{8\pi}\left(\frac{\lambda}{\mu}\right)\left(\frac{x_2}{R}+\frac{x_2}{S}\right), \\
\Gamma^3_{11} &= -\frac{1}{8\pi}\left(\frac{\lambda}{\mu}\right)\left(\frac{x_3-\xi_3}{R}+\frac{x_3+\xi_3}{S}\right),
\end{aligned}\right. &
\left\{\begin{aligned}
\Gamma^1_{12} &= -\frac{1}{8\pi}\left(\frac{x_2}{R}+\frac{x_2}{S}\right), \\
\Gamma^2_{12} &= -\frac{1}{8\pi}\left(\frac{x_1}{R}+\frac{x_1}{S}\right), \\
\Gamma^3_{12} &= 0,
\end{aligned}\right. \\
\left\{\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{8\pi}\left(\frac{\lambda}{\mu}\right)\left(\frac{x_1}{R}+\frac{x_1}{S}\right), \\
\Gamma^2_{22} &= -\frac{1}{8\pi}\left(\frac{\lambda+2\mu}{\mu}\right)\left(\frac{x_2}{R}+\frac{x_2}{S}\right), \\
\Gamma^3_{22} &= -\frac{1}{8\pi}\left(\frac{\lambda}{\mu}\right)\left(\frac{x_3-\xi_3}{R}+\frac{x_3+\xi_3}{S}\right),
\end{aligned}\right. &
\end{array}\right\} \quad (2.17)$$

$$\left\{\begin{aligned}
\Gamma^1_{33}&=-\frac{1}{8\pi}\left(\frac{\lambda}{\mu}\right)\left(\frac{x_1}{R}+\frac{x_1}{S}\right),\\
\Gamma^2_{33}&=-\frac{1}{8\pi}\left(\frac{\lambda}{\mu}\right)\left(\frac{x_2}{R}+\frac{x_2}{S}\right),\\
\Gamma^3_{33}&=-\frac{1}{8\pi}\left(\frac{\lambda+2\mu}{\mu}\right)\left(\frac{x_3-\xi_3}{R}+\frac{x_3+\xi_3}{S}\right),
\end{aligned}\right.$$

Case II :—

$$\left\{\begin{aligned}
\Gamma^1_{23}&=0\ ,\\
\Gamma^2_{23}&=-\frac{1}{8\pi}\left(\frac{x_3-\xi_3}{R}-\frac{x_3+\xi_3}{S}\right),\\
\Gamma^3_{23}&=-\frac{1}{8\pi}\left(\frac{x_2}{R}-\frac{x_2}{S}\right),
\end{aligned}\right.\qquad
\left\{\begin{aligned}
\Gamma^1_{31}&=-\frac{1}{8\pi}\left(\frac{x_3-\xi_3}{R}-\frac{x_3+\xi_3}{S}\right),\\
\Gamma^2_{31}&=0\ ,\\
\Gamma^3_{31}&=-\frac{1}{8\pi}\left(\frac{x_1}{R}-\frac{x_1}{S}\right).
\end{aligned}\right.$$

In (2.17), Γ^m_{kl} is the m-component of Galerkin vector which gives the field at $Q(x_1, x_2, x_3)$ due to force systems at $P(0, 0, \xi_3)$ and $P'(0, 0, -\xi_3)$ and $R=\overline{QP}=\sqrt{x_1^2+x_2^2+(x_3-\xi_3)^2}$, $S=\overline{QP'}=\sqrt{x_1^2+x_2^2+(x_3+\xi_3)^2}$.

The normal load which must further be added to the above-mentioned force systems for the boundary condition on the plane $x_3=0$ is

$$p_{kl}(x_1, x_2)=-2G^{33}_{kl}(x_1, x_2, 0)$$

for each combination of (kl). From equation (1.16)$'$ we have

$$\left.\begin{aligned}
\frac{p_{11}}{\mu}&=\frac{1}{\pi}\left\{(2-3\alpha)\frac{1}{\rho^3}-3(1-\alpha)\frac{x_1^2+\xi_3^2}{\rho^5}+15\alpha\frac{x_1^2\xi_3^2}{\rho^7}\right\},\\
\frac{p_{22}}{\mu}&=\frac{1}{\pi}\left\{(2-3\alpha)\frac{1}{\rho^3}-3(1-\alpha)\frac{x_2^2+\xi_3^2}{\rho^5}+15\alpha\frac{x_2^2\xi_3^2}{\rho^7}\right\},\\
\frac{p_{33}}{\mu}&=\frac{1}{\pi}\left\{-\alpha\frac{1}{\rho^3}-6\alpha\frac{\xi_3^2}{\rho^5}+15\alpha\frac{\xi_3^4}{\rho^7}\right\},\\
\frac{p_{23}}{\mu}&=\frac{1}{\pi}\left\{3\alpha\frac{x_2\xi_3}{\rho^5}-15\alpha\frac{x_2\xi_3^3}{\rho^7}\right\},\\
\frac{p_{31}}{\mu}&=\frac{1}{\pi}\left\{3\alpha\frac{x_1\xi_3}{\rho^5}-15\alpha\frac{x_1\xi_3^3}{\rho^7}\right\},\\
\frac{p_{12}}{\mu}&=\frac{1}{\pi}\left\{-3(1-\alpha)\frac{x_1x_2}{\rho^5}+15\alpha\frac{x_1x_2\xi_3^2}{\rho^7}\right\},
\end{aligned}\right\}\qquad(2.18)$$

where

$$\rho=\sqrt{x_1^2+x_2^2+\xi_3^2}\,.$$

The first step in solving the Boussinesq problems for these normal stresses is to calculate their Fourier transforms.

Let us now consider the Fourier transform $\bar{f}(k_1, k_2)$ of a simple function $f(x_1, x_2)=1/\sqrt{x_1^2+x_2^2+c^2}$,

$$\bar{f}(k_1, k_2)=\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{1}{\sqrt{x_1^2+x_2^2+c^2}}e^{-i(k_1x_1+k_2x_2)}\,dx_1dx_2\,.$$

If in this integral we make the substitutions

$$x_1=r\cos\theta\,,\quad x_2=r\sin\theta\,,\quad k_1=k\cos\varphi\,,\quad k_2=k\sin\varphi\,, \tag{2.19}$$

then, by means of Hansen's integral representation of Bessel function $J_n(kr)$,

$$J_n(kr)=\frac{(-i)^n}{2\pi}\int_0^{2\pi}e^{ikr\cos\theta}\cos n\theta\,d\theta\,, \tag{2.20}$$

we find that $\bar{f}(k_1, k_2)$ is a function of k and may be written

$$\bar{f}=\int_0^{\infty}\frac{1}{\sqrt{r^2+c^2}}J_0(kr)r\,dr\,.$$

From the equation

$$\int_0^{\infty}e^{-ck}J_n(kr)dk=\frac{(\sqrt{r^2+c^2}-c)^n}{r^n\sqrt{r^2+c^2}}\,,\quad (c>0) \tag{2.21}$$

which is found in textbooks (e.g. Watson 1922) or may be easily established if we replace $J_n(kr)$ by (2.20) and then change the order of integrations, we can evaluate $\bar{f}$ by making use of the Hankel inversion theorem in the case $n=0$. Thus we have

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{1}{\sqrt{x_1^2+x_2^2+c^2}}e^{-i(k_1x_1+k_2x_2)}\,dx_1dx_2=\frac{1}{k}e^{-ck}\,,\ (k=\sqrt{k_1^2+k_2^2})\,. \tag{2.22}$$

If we differentiate equation (2.22) with respect to k_1, k_2, or c, taking the derivative on the left side under the integral sign, we find Fourier transforms for various functions necessary for $\bar{p}_{kl}$ as shown in Table 1.

Table 1. Fourier transforms.

$f(x_1, x_2)=\dfrac{1}{2\pi}\displaystyle\iint_{-\infty}^{\infty}\bar{f}(k_1, k_2)e^{i(k_1x_1+k_2x_2)}dk_1dk_2$	$\bar{f}(k_1, k_2)=\dfrac{1}{2\pi}\displaystyle\iint_{-\infty}^{\infty}f(x_1, x_2)e^{-i(k_1x_1+k_2x_2)}dx_1dx_2$
$\dfrac{1}{\rho}$	$\dfrac{1}{k}e^{-ck}$
$\dfrac{1}{\rho^3}$	$\dfrac{1}{c}e^{-ck}$
$\dfrac{1}{\rho^5}$	$\dfrac{1}{3c^3}(1+ck)e^{-ck}$
$\dfrac{1}{\rho^7}$	$\dfrac{1}{15c^5}(3+3ck+c^2k^2)e^{-ck}$
$\dfrac{x_1}{\rho^3}$	$-i\dfrac{k_1}{k}c^{-ck}$
$\dfrac{x_1}{\rho^5}$	$-i\dfrac{1}{3c}k_1e^{-ck}$
$\dfrac{x_1}{\rho^7}$	$-i\dfrac{1}{15c^3}(1+ck)k_1e^{-ck}$
$\dfrac{x_1x_2}{\rho^3}$	$-(1+ck)\dfrac{k_1k_2}{k^3}e^{-ck}$
$\dfrac{x_1x_2}{\rho^5}$	$-\dfrac{1}{3}\dfrac{k_1k_2}{k}e^{-ck}$
$\dfrac{x_1x_2}{\rho^7}$	$-\dfrac{1}{15c}k_1k_2e^{-ck}$
$\dfrac{x_1^2}{\rho^3}$	$[k^2-(1+ck)k_1^2]\dfrac{1}{k^3}e^{-ck}$
$\dfrac{x_1^2}{\rho^5}$	$\dfrac{1}{3c}(k-ck_1^2)\dfrac{1}{k}e^{-ck}$
$\dfrac{x_1^2}{\rho^7}$	$\dfrac{1}{15c^3}[(1+ck)-c^2k_1^2]e^{-ck}$

$(\rho=\sqrt{x_1^2+x_2^2+c^2},\quad k=\sqrt{k_1^2+k_2^2},\quad c>0)$

Using Table 1, we obtain Fourier transforms $\bar{p}_{kl}$ as follows.

$$\left.\begin{aligned}
\frac{\bar{p}_{11}}{\mu}&=\frac{1}{\pi}\left\{\alpha\frac{k_1^2}{k}+(2\alpha-1)\frac{k_2^2}{k}-\alpha\xi_3k_1^2\right\}e^{-\xi_3k}\ ,\\
\frac{\bar{p}_{22}}{\mu}&=\frac{1}{\pi}\left\{(2\alpha-1)\frac{k_1^2}{k}+\alpha\frac{k_2^2}{k}-\alpha\xi_3k_2^2\right\}e^{-\xi_3k}\ ,\\
\frac{\bar{p}_{33}}{\mu}&=\frac{1}{\pi}\alpha\{k+\xi_3k^2\}e^{-\xi_3k}\ ,
\end{aligned}\right\}\tag{2.23}$$

$$\left.\begin{aligned}
\frac{\bar{p}_{23}}{\mu}&=\frac{1}{\pi}\{i\alpha\xi_3 k_2 k\}e^{-\xi_3 k}\ ,\\
\frac{\bar{p}_{31}}{\mu}&=\frac{1}{\pi}\{i\alpha\xi_3 k_1 k\}e^{-\xi_3 k}\ ,\\
\frac{\bar{p}_{12}}{\mu}&=\frac{1}{\pi}\left\{(1-\alpha)\frac{k_1k_2}{k}-\alpha\xi_3 k_1 k_2\right\}e^{-\xi_3 k}\ .
\end{aligned}\right\}$$

Substitution of (2.23) into (2.17) gives

$$\left.\begin{aligned}
\overline{\Gamma}_{11}&=\frac{1}{2\pi}\left[\left(2-\frac{1}{\alpha}\right)\frac{k_1^2}{k^4}+\left(2-\frac{1}{\alpha}\right)^2\frac{k_2^2}{k^4}+\left\{-\left(2-\frac{1}{\alpha}\right)\xi_3+x_3\right\}\frac{k_1^2}{k^3}\right.\\
&\qquad\left.+\left(2-\frac{1}{\alpha}\right)x_3\frac{k_2^2}{k^3}-\xi_3x_3\frac{k_1^2}{k^2}\right]e^{-pk}\ ,\\
\overline{\Gamma}_{22}&=\frac{1}{2\pi}\left[\left(2-\frac{1}{\alpha}\right)^2\frac{k_1^2}{k^4}+\left(2-\frac{1}{\alpha}\right)\frac{k_2^2}{k^4}+\left(2-\frac{1}{\alpha}\right)x_3\frac{k_1^2}{k^3}\right.\\
&\qquad\left.+\left\{-\left(2-\frac{1}{\alpha}\right)\xi_3+x_3\right\}\frac{k_2^2}{k^3}-\xi_3x_3\frac{k_2^2}{k^2}\right]e^{-pk}\ ,\\
\overline{\Gamma}_{33}&=\frac{1}{2\pi}\left[\left(2-\frac{1}{\alpha}\right)\frac{1}{k^2}+\left\{\left(2-\frac{1}{\alpha}\right)\xi_3+x_3\right\}\frac{1}{k}+\xi_3x_3\right]e^{-pk}\ ,\\
\overline{\Gamma}_{23}&=\frac{i}{2\pi}\left[\left(2-\frac{1}{\alpha}\right)\frac{\xi_3k_2}{k^2}+\frac{k_2}{k}\xi_3x_3\right]e^{-pk}\ ,\\
\overline{\Gamma}_{31}&=\frac{i}{2\pi}\left[\left(2-\frac{1}{\alpha}\right)\frac{\xi_3k_1}{k^2}+\frac{k_1}{k}\xi_3x_3\right]e^{-pk}\ ,\\
\overline{\Gamma}_{12}&=\frac{1}{2\pi}\left[-\left(2-\frac{1}{\alpha}\right)\left(1-\frac{1}{\alpha}\right)\frac{k_1k_2}{k^4}-\left\{\left(2-\frac{1}{\alpha}\right)\xi_3+\left(1-\frac{1}{\alpha}\right)x_3\right\}\frac{k_1k_2}{k^3}\right.\\
&\qquad\left.-\xi_3x_3\frac{k_1k_2}{k^2}\right]e^{-pk}\ ,
\end{aligned}\right\}\quad(2.24)$$

where

$$p=\xi_3+x_3\ .$$

Now we have to calculate the inversions of $\overline{\Gamma}_{kl}$. Substitutions (2.19) reduce two-dimensional Fourier transforms to zero-order Hankel transforms. By equation (2.21) we have $\int_0^\infty e^{-pk}J_n(kr)dk$, from which we can evaluate $\int_0^\infty e^{-pk}J_n(kr)k\,dk$ by differentiating with respect to c, and $\int_0^\infty e^{-pk}J_2(kr)k^{-1}dk$ with the help of a recurrence formula of Bessel functions. In some inversions of $\bar{\Gamma}_{kl}$ we encounter an integral, $\int_0^\infty e^{-pk}J_0(kr)k^{-1}dk$, where the integrand tends to infinity at $k=0$ and this improper integral has no limit. In these cases we can not get the inversion Γ of $\bar{\Gamma}$.

Here we look back upon the process of solving the Boussinesq problem. If there exist displacement components (u_1, u_2, u_3) and their two-dimensional Fourier transforms $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, then from equation (2.2) we have the following relation in terms of $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$,

$$\alpha(ik_1)\left\{(ik_1)\bar{u}_1+(ik_2)\bar{u}_2+\frac{d}{dx_3}\bar{u}_3\right\}+(1-\alpha)\left(\frac{d^2}{dx_3^2}-k^2\right)\bar{u}_1=0\ ,$$

$$\alpha(ik_2)\left\{(ik_1)\bar{u}_1+(ik_2)\bar{u}_2+\frac{d}{dx_3}\bar{u}_3\right\}+(1-\alpha)\left(\frac{d^2}{dx_3^2}-k^2\right)\bar{u}_2=0\ ,$$

$$\alpha\frac{d}{dx_3}\left\{(ik_1)\bar{u}_1+(ik_2)\bar{u}_2+\frac{d}{dx_3}\bar{u}_3\right\}+(1-\alpha)\left(\frac{d^2}{dx_3^2}-k^2\right)\bar{u}_3=0\ .$$

These equations are satisfied by $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in equation (2.14) provided that equation (2.11) holds. In the Boussinesq problem $\bar{\Gamma}$ is determined in equation (2.16) such that equation (2.11) and boundary conditions at infinity and those on the plane $x_3=0$ are satisfied. Therefore we can conclude that if there exist the inversions of $\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{\tau}_{11}, \cdots, \bar{\tau}_{12}$ in (2.14) with $\bar{\Gamma}$ in (2.16), they satisfy the equation of motion and boundary conditions, and form the solution. Thus we do not always require Γ itself. As for Γ, we may take any form for it provided that $\dfrac{\partial^2\Gamma}{\partial x_1\partial x_3}$, $\dfrac{\partial^2\Gamma}{\partial x_1\partial x_3}$ and $\nabla^2\Gamma$ are derived from it in the same forms as the inversions of $(ik_1)\dfrac{d\bar{\Gamma}}{dx_3}$, $(ik_2)\dfrac{d\bar{\Gamma}}{dx_3}$ and $\left(\dfrac{d^2}{dx_3^2}-k^2\right)\bar{\Gamma}$ respectively. In this way, for convenience, we define a function $f(r, p)$ which corresponds to $\iint_{-\infty}^\infty e^{-pk}k^{-2}e^{i(k_1x_1+k_2x_2)}dk_1dk_2$ as

$$\left.\begin{aligned}\frac{\partial}{\partial x_3}f(r, p)&=\frac{1}{2\pi}\iint_{-\infty}^\infty\frac{d}{dx_3}(e^{-pk}k^{-2})e^{i(k_1x_1+k_2x_2)}dk_1dk_2\\&=\int_0^\infty J_0(kr)e^{-pk}dk\ ,\\ \nabla^2f(r, p)&=\frac{1}{2\pi}\iint\left(\frac{d^2}{dx_3^2}-k^2\right)(e^{-pk}k^{-2})e^{i(k_1x_1+k_2x_2)}dk_1dk_2\\&=0\ ,\\ &\text{where}\\ r=\sqrt{x_1^2+x_2^2}\ ,\ S&=\sqrt{x_1^2+x_2^2+p^2}\ ,\ \ k=\sqrt{k_1^2+k_2^2}\ ,\ p=x_3+\xi_3>0\ .\end{aligned}\right\}\qquad(2.25)$$

Using $f(r, p)$, the inversions of functions necessary for Γ_{kl} are shown in Table 2.

Table 2. Fourier transforms.

$f(x_1, x_2)=\frac{1}{2\pi}\iint_{-\infty}^{\infty}\bar{f}(k_1, k_2)e^{i(k_1x_1+k_2x_2)}dk_1dk_2$	$\bar{f}(k_1, k_2)=\frac{1}{2\pi}\iint_{-\infty}^{\infty}f(x_1, x_2)e^{-i(k_1x_1+k_2x_2)}dx_1dx_2$
$\frac{p}{S^3}$	e^{-pk}
$\frac{1}{S}$	$\frac{1}{k}e^{-pk}$
$f(r, p)$	$\frac{1}{k^2}e^{-pk}$
$i\frac{x_1}{S^3}$	$\frac{k_1}{k}e^{-pk}$
$i\frac{x_1}{r^2}\frac{(S-p)}{S}$	$\frac{k_1}{k^2}e^{-pk}$
$\frac{1}{2}\frac{p}{S^3}+\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_1^2}{r^4}\right)\frac{(S-p)^2(2S+p)}{S^3}$	$\frac{k_1^2}{k^2}e^{-pk}$
$\frac{1}{2}\frac{1}{S}+\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_1^2}{r^4}\right)\frac{(S-p)^2}{S}$	$\frac{k_1^2}{k^3}e^{-pk}$
$\frac{1}{2}f(r, p)+\frac{1}{2}\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_1^2}{r^4}\right)(S-p)^2$	$\frac{k_1^2}{k^4}e^{-pk}$
$-\frac{x_1x_2}{r^4}\frac{(S-p)^2(2S+p)}{S^3}$	$\frac{k_1k_2}{k^2}e^{-pk}$
$-\frac{x_1x_2}{r^4}\frac{(S-p)^2}{S}$	$\frac{k_1k_2}{k^3}e^{-pk}$
$-\frac{1}{2}\frac{x_1x_2}{r^4}(S-p)^2$	$\frac{k_1k_2}{k^4}e^{-pk}$

$(r=\sqrt{x_1^2+x_2^2},\quad S=\sqrt{x_1^2+x_2^2+p^2},\quad k=\sqrt{k_1^2+k_2^2},\quad p>0)$

With the aid of Table 2 we obtain Γ_{kl} as follows.

$$
\begin{aligned}
\Gamma_{11}=\frac{1}{2\pi}\Big[&\left(2-\frac{1}{\alpha}\right)\left(3-\frac{1}{\alpha}\right)\frac{1}{2}f(r, p)\\
&+\left\{-\left(2-\frac{1}{\alpha}\right)\xi_3+\left(3-\frac{1}{\alpha}\right)x_3\right\}\frac{1}{2}\frac{1}{S}\\
&-\left(2-\frac{1}{\alpha}\right)\left(1-\frac{1}{\alpha}\right)\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_1^2}{r^4}\right)\frac{1}{2}(S-p)^2\\
&+\left\{-\left(2-\frac{1}{\alpha}\right)\xi_3-\left(1-\frac{1}{\alpha}\right)x_3\right\}\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_1^2}{r^4}\right)\frac{(S-p)^2}{S}
\end{aligned}
$$

$$
\left.
\begin{aligned}
&-\xi_3 x_3\left\{\frac{1}{2}\frac{p}{S^3}+\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_1^2}{r^4}\right)\frac{(S-p)^2(2S+p)}{S^3}\right\}\Bigg],\\
\Gamma_{22}=&\frac{1}{2\pi}\Bigg[\left(2-\frac{1}{\alpha}\right)\left(3-\frac{1}{\alpha}\right)\frac{1}{2}f(r,p)\\
&+\left\{-\left(2-\frac{1}{\alpha}\right)\xi_3+\left(3-\frac{1}{\alpha}\right)x_3\right\}\frac{1}{2}\frac{1}{S}\\
&-\left(2-\frac{1}{\alpha}\right)\left(1-\frac{1}{\alpha}\right)\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_2^2}{r^4}\right)\frac{1}{2}(S-p)^2\\
&+\left\{-\left(2-\frac{1}{\alpha}\right)\xi_3-\left(1-\frac{1}{\alpha}\right)x_3\right\}\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_2^2}{r^4}\right)\frac{(S-p)^2}{S}\\
&-\xi_3 x_3\left\{\frac{1}{2}\frac{p}{S^3}+\left(\frac{1}{2}\frac{1}{r^2}-\frac{x_2^2}{r^4}\right)\frac{(S-p)^2(2S+p)}{S^3}\right\}\Bigg],\\
\Gamma_{33}=&\frac{1}{2\pi}\left[\left(2-\frac{1}{\alpha}\right)f(r,p)+\left\{\left(2-\frac{1}{\alpha}\right)\xi_3+x_3\right\}\frac{1}{S}+\xi_3x_3\frac{p}{S^3}\right],\\
\Gamma_{23}=&\frac{1}{2\pi}\xi_3\left[-\left(2-\frac{1}{\alpha}\right)\frac{x_2}{r^2}+\left(2-\frac{1}{\alpha}\right)\frac{x_2}{r^2}\frac{p}{S}-\frac{x_2x_3}{S^3}\right],\\
\Gamma_{31}=&\frac{1}{2\pi}\xi_3\left[-\left(2-\frac{1}{\alpha}\right)\frac{x_1}{r^2}+\left(2-\frac{1}{\alpha}\right)\frac{x_1}{r^2}\frac{p}{S}-\frac{x_1x_3}{S^3}\right],\\
\Gamma_{12}=&\frac{1}{2\pi}\frac{x_1x_2}{r^4}(S-p)^2\Bigg[\frac{1}{2}\left(2-\frac{1}{\alpha}\right)\left(1-\frac{1}{\alpha}\right)+\left\{\left(2-\frac{1}{\alpha}\right)\xi_3\right.\\
&\left.-\left(1-\frac{1}{\alpha}\right)x_3\right\}\frac{1}{S}+\xi_3x_3\frac{(2S+p)}{S^3}\Bigg].
\end{aligned}
\right\}
\qquad (2.26)
$$

Thus for each combination of (kl), the displacement field and the stress field may be derived from the resultant Galerkin vector, the x_1- and x_2- components of which are given in equation (2.17), while the x_3-component is the sum of the value in (2.17) and in (2.26).

In order to simplify our further calculations we make the assumption $\lambda=\mu$, which is not unusual in applications of elasticity to geophysics. It follows that $\alpha=2/3$. Then Γ_{kl}'s in (2.26) are given as follows.

$$
\begin{aligned}
\Gamma_{11}=&\frac{1}{4\pi}\Bigg[\frac{3}{4}f(r,p)+\frac{1}{8}\frac{1}{r^2}(S^2+14Sp-7p^2-8S^{-1}p^3)\\
&+\frac{1}{4}\frac{x_1^2}{r^4}(-S^2-2Sp+7p^2-4S^{-1}p^3)\\
&+\xi_3(-3S^{-1}+0+2x_1^2S^{-3})\\
&+\xi_3^2\left\{2\frac{1}{r^2}(1-S^{-1}p)+2\frac{x_1^2}{r^4}(-2+3S^{-1}p+0-S^{-3}p^3)\right\}\Bigg],
\end{aligned}
$$

$$\left.\begin{aligned}
\Gamma_{22}=&\frac{1}{4\pi}\Big[\frac{3}{4}f(r,\,p)+\frac{1}{8}\frac{1}{r^2}(S^2+14Sp-7p^2-8S^{-1}p^3)\\
&+\frac{1}{4}\frac{x_2^2}{r^4}(-S^2-2Sp+7p^2-4S^{-1}p^3)\\
&+\xi_3(-3S^{-1}+0+2x_2^2S^{-3})\\
&+\xi_3^2\Big\{2\frac{1}{r^2}(1-S^{-1}p)+2\frac{x_2^2}{r^4}(-2+3S^{-1}p+0-S^{-3}p^3)\Big\}\Big]\,,\\
\Gamma_{33}=&\frac{1}{4\pi}[f(r,\,p)+2S^{-1}p+\xi_3(-S^{-1}+0+2S^{-3}p^2)+\xi_3^2(-2S^{-3}p)]\,,\\
\Gamma_{23}=&\frac{1}{4\pi}\xi_3x_2\Big[-\frac{1}{r^2}(1-S^{-1}p)-2x_3S^{-3}]\,,\\
\Gamma_{31}=&\frac{1}{4\pi}\xi_3x_1\Big[-\frac{1}{r^2}(1-S^{-1}p)-2x_3S^{-3}]\,,\\
\Gamma_{12}=&\frac{1}{4\pi}\frac{x_1x_2}{r^4}\Big[\Big(-\frac{1}{4}S^2-\frac{1}{2}Sp+\frac{7}{4}p^2-S^{-1}p^3)\\
&+2\xi_3(S+0-2S^{-1}p^2+0+S^{-3}p^4)\\
&+2\xi_3^2(-2+3S^{-1}p+0-S^{-3}p^3)\Big]\,.
\end{aligned}\right\}\qquad(2.27)$$

The displacement field is calculated by means of the general relation (2.3) from Galerkin vector.

The displacement components for each (kl) derived from equation (2.17), which we denote as w_{kl}^m, may also be obtained without differentiation from equation (1.14)′ considering the contribution from the image point. They are given as follows.

$$\left\{\begin{aligned}
w_{11}^1&=\frac{1}{4\pi}\Big[\frac{1}{3}\Big(\frac{1}{R^3}+\frac{1}{S^3}\Big)x_1+2\Big(\frac{1}{R^5}+\frac{1}{S^5}\Big)x_1^3\Big]\,,\\
w_{11}^2&=\frac{1}{4\pi}\Big[-\frac{1}{3}\Big(\frac{1}{R^3}+\frac{1}{S^3}\Big)x_2+2\Big(\frac{1}{R^5}+\frac{1}{S^5}\Big)x_1^2x_2\Big]\,,\\
w_{11}^3&=\frac{1}{4\pi}\Big[-\frac{1}{3}\Big(\frac{x_3-\xi_3}{R^3}+\frac{x_3+\xi_3}{S^3}\Big)+2\Big(\frac{x_3-\xi_3}{R^5}+\frac{x_3+\xi_3}{S^5}\Big)x_1^2\Big]\,,
\end{aligned}\right.$$

$$\left\{\begin{aligned}
w_{22}^1&=\frac{1}{4\pi}\Big[-\frac{1}{3}\Big(\frac{1}{R^3}+\frac{1}{S^3}\Big)x_1+2\Big(\frac{1}{R^5}+\frac{1}{S^5}\Big)x_1x_2^2\Big]\,,\\
w_{22}^2&=\frac{1}{4\pi}\Big[\frac{1}{3}\Big(\frac{1}{R^3}+\frac{1}{S^3}\Big)x_2+2\Big(\frac{1}{R^5}+\frac{1}{S^5}\Big)x_2^3\Big]\,,\\
w_{22}^3&=\frac{1}{4\pi}\Big[-\frac{1}{3}\Big(\frac{x_3-\xi_3}{R^3}+\frac{x_3+\xi_3}{S^3}\Big)+2\Big(\frac{x_3-\xi_3}{R^5}+\frac{x_3+\xi_3}{S^5}\Big)x_2^2\Big]\,,
\end{aligned}\right.$$

$$\left\{\begin{aligned}
w_{33}^1 &= \frac{1}{4\pi}\left[-\frac{1}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)+2\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)x_1\right],\\
w_{33}^2 &= \frac{1}{4\pi}\left[-\frac{1}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)+2\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)x_2\right],\\
w_{33}^3 &= \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{x_3-\xi_3}{R^3}+\frac{x_3+\xi_3}{S^3}\right)+2\left(\frac{(x_3-\xi_3)^3}{R^5}+\frac{(x_3+\xi_3)^3}{S^5}\right)\right],
\end{aligned}\right.$$

$$\left\{\begin{aligned}
w_{23}^1 &= \frac{1}{4\pi}\left[2\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)x_1x_2\right],\\
w_{23}^2 &= \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{x_3-\xi_3}{R^3}-\frac{x_3+\xi_3}{S^3}\right)+2\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)x_2^2\right],\\
w_{23}^3 &= \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{1}{R^3}-\frac{1}{S^3}\right)x_2+2\left(\frac{(x_3-\xi_3)^2}{R^5}-\frac{(x_3-\xi_3)^2}{S^5}\right)x_2\right],
\end{aligned}\right.$$

$$\left\{\begin{aligned}
w_{31}^1 &= \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{x_3-\xi_3}{R^3}-\frac{x_3+\xi_3}{S^3}\right)+2\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)x_1^2\right],\\
w_{31}^2 &= \frac{1}{4\pi}\left[2\left(\frac{x_3-\xi_3}{R^5}-\frac{x_3+\xi_3}{S^5}\right)x_1x_2\right],\\
w_{31}^3 &= \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{1}{R^3}-\frac{1}{S^3}\right)x_1+2\left(\frac{(x_3-\xi_3)^2}{R^5}-\frac{(x_3+\xi_3)^2}{S^5}\right)x_1\right],
\end{aligned}\right.$$

$$\left\{\begin{aligned}
w_{12}^1 &= \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)x_2+2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)x_1^2x_2\right],\\
w_{12}^2 &= \frac{1}{4\pi}\left[\frac{1}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)x_1+2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)x_1x_2^2\right],\\
w_{12}^3 &= \frac{1}{4\pi}\left[2\left(\frac{(x_3-\xi_3)}{R^5}+\frac{(x_3+\xi_3)}{S^5}\right)x_1x_2\right].
\end{aligned}\right.$$

The displacement components for each (kl) derived from equations (2.27), which we denote as ω_{kl}^m, are given as follows.

$$\left\{\begin{aligned}
\omega_{11}^1 = \frac{1}{4\pi}\Bigg[&\frac{1}{3}\frac{x_1}{r^4}\left(4S-18p+13\frac{p^2}{S}+7\frac{p^4}{S^3}-6\frac{p^6}{S^5}\right)\\
&+\frac{x_1^3}{r^6}\left(-S+8p-12\frac{p^2}{S}+7\frac{p^4}{S^3}-2\frac{p^6}{S^5}\right)\\
&+2\xi_3x_1\left(7\frac{p}{S^5}-10x_1^2\frac{p}{S^7}\right)\\
&+4\xi_3^2x_1\left(-3\frac{1}{S^5}+5x_1^2\frac{1}{S^7}\right)\Bigg],
\end{aligned}\right.$$

$$
\left\{
\begin{aligned}
\omega_{11}^{2} = \frac{1}{4\pi}\Big[& \frac{1}{3}\frac{x_2}{r^4}\Big(2S-6p-\frac{p^2}{S}+11\frac{p^4}{S^3}-6\frac{p^6}{S^5}\Big) \\
& +\frac{x_1^2x_2}{r^6}\Big(-S+8p-12\frac{p^2}{S}+7\frac{p^4}{S^3}-2\frac{p^6}{S^5}\Big) \\
& +2\xi_3x_2\Big(3\frac{p}{S^5}-10x_1^2\frac{p}{S^7}\Big)+4\xi_3^2x_2\Big(-\frac{1}{S^5}+5x_1^2\frac{1}{S^7}\Big)\Big]\,, \\
\omega_{11}^{3} = \frac{1}{4\pi}\Big[& \frac{1}{r^2}\Big(-1-\frac{p^3}{S^3}+2\frac{p^5}{S^7}\Big)+\frac{x_1^2}{r^4}\Big(2-\frac{p}{S}-3\frac{p^3}{S^3}+2\frac{p^5}{S^5}\Big) \\
& +2\xi_3\Big(\frac{1}{S^3}+3\frac{p^2}{S^5}-4x_1^2\frac{1}{S^5}-10x_1^2\frac{p^2}{S^7}\Big)+4\xi_3^2\Big(-\frac{p}{S^5}+5x_1^2\frac{p}{S^7}\Big)\Big]\,,
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\omega_{22}^{1} = \frac{1}{4\pi}\Big[& \frac{1}{3}\frac{x_1}{r^4}\Big(2S-6p-\frac{p^2}{S}+11\frac{p^4}{S^3}-6\frac{p^6}{S^5}\Big) \\
& +\frac{x_1x_2^2}{r^6}\Big(-S+8p-12\frac{p^2}{S}+7\frac{p^4}{S^3}-2\frac{p^6}{S^5}\Big) \\
& +2\xi_3x_1\Big(3\frac{p}{S^5}-10x_2^2\frac{p}{S^7}\Big)+4\xi_3^2x_1\Big(-\frac{1}{S^5}+5x_2^2\frac{1}{S^7}\Big)\Big]\,, \\
\omega_{22}^{2} = \frac{1}{4\pi}\Big[& \frac{1}{3}\frac{x_2}{r^4}\Big(4S-18p+13\frac{p^2}{S}+7\frac{p^4}{S^3}-6\frac{p^6}{S^5}\Big) \\
& +\frac{x_2^3}{r^6}\Big(-S+8p-12\frac{p^2}{S}+7\frac{p^4}{S^3}-2\frac{p^6}{S^5}\Big) \\
& +2\xi_3x_2\Big(7\frac{p}{S^5}-10x_2^2\frac{p}{S^7}\Big)+4\xi_3^2x_2\Big(-3\frac{1}{S^5}+5x_2^2\frac{1}{S^7}\Big)\Big]\,, \\
\omega_{22}^{3} = \frac{1}{4\pi}\Big[& \frac{1}{r^2}\Big(-1-\frac{p^3}{S^3}+2\frac{p^5}{S^5}\Big)+\frac{x_2^2}{r^4}\Big(2-\frac{p}{S}-3\frac{p^3}{S^3}+2\frac{p^5}{S^5}\Big) \\
& +2\xi_3\Big(\frac{1}{S^3}+3\frac{p^2}{S^5}-4x_2^2\frac{1}{S^5}-10x_2^2\frac{p^2}{S^7}\Big)+4\xi_3^2\Big(-\frac{p}{S^5}+5x_2^2\frac{p}{S^7}\Big)\Big]\,,
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\omega_{33}^{1} = \frac{1}{4\pi}\Big[& x_1\Big\{\frac{2}{3}\Big(\frac{1}{S^3}-6\frac{p^2}{S^5}\Big) \\
& +10\xi_3\Big(\frac{p}{S^5}-2\frac{p^3}{S^7}\Big)+4\xi_3^2\Big(-\frac{1}{S^5}+5\frac{p^2}{S^7}\Big)\Big\}\Big]\,, \\
\omega_{33}^{2} = \frac{1}{4\pi}\Big[& x_2\Big\{\frac{2}{3}\Big(\frac{1}{S^3}-6\frac{p^2}{S^5}\Big)+10\xi_3\Big(\frac{p}{S^5}-2\frac{p^3}{S^7}\Big)+4\xi_3^2\Big(-\frac{1}{S^5}+5\frac{p^2}{S^7}\Big)\Big\}\Big]\,,
\end{aligned}
\right.
$$

$$\left\{\begin{aligned}\omega_{33}^{3} &= \frac{1}{4\pi}\left[\frac{2}{3}\left(-\frac{p}{S^3}-6\frac{p^3}{S^5}\right)+\frac{2}{3}\,\xi_3\left(\frac{1}{S^3}+15\frac{p^2}{S^5}-30\frac{p^4}{S^7}\right)\right. \\ &\quad \left.+\,4\xi_3^2\left(-3\frac{p}{S^5}+5\frac{p^3}{S^7}\right)\right],\end{aligned}\right.$$

$$\left\{\begin{aligned}\omega_{23}^{1} &= \frac{1}{4\pi}\left[-2\xi_3\frac{x_1x_2}{S^5}+20\xi_3\frac{x_1x_2x_3p}{S^7}\right], \\ \omega_{23}^{2} &= \frac{1}{4\pi}\left[\frac{2}{3}\,\xi_3\frac{1}{S^3}-2\xi_3\frac{x_2^2}{S^5}-4\frac{\xi_3x_3p}{S^5}+20\xi_3\frac{x_2^2x_3p}{S^7}\right], \\ \omega_{23}^{3} &= \frac{1}{4\pi}\left[2\xi_3\frac{x_2p}{S^5}+4\xi_3^2\frac{x_2}{S^5}+20\xi_3\frac{x_2x_3p^2}{S^7}\right],\end{aligned}\right.$$

$$\left\{\begin{aligned}\omega_{31}^{1} &= \frac{1}{4\pi}\left[\frac{2}{3}\,\xi_3\frac{1}{S^3}-4\xi_3\frac{x_3p}{S^5}-2\xi_3\frac{x_1^2}{S^5}+20\xi_3\frac{x_1^2x_3p}{S^7}\right], \\ \omega_{31}^{2} &= \frac{1}{4\pi}\left[-2\xi_3\frac{x_1x_2}{S^5}+20\xi_3\frac{x_1x_2x_3p}{S^7}\right], \\ \omega_{31}^{3} &= \frac{1}{4\pi}\left[2\xi_3\frac{x_1p}{S^5}+4\xi_3^2\frac{x_1}{S^5}+20\frac{x_1x_3p^2\xi_3}{S^7}\right],\end{aligned}\right.$$

$$\left\{\begin{aligned}\omega_{12}^{1} &= \frac{1}{4\pi}\left[-\frac{1}{3}\frac{x_2}{r^2}\left(\frac{5}{S}-2\frac{p^2}{S^3}\right)+2\frac{(S-p)}{r^4}x_2\right. \\ &\quad -8\frac{(S-p)}{r^6}x_1^2x_2+\frac{x_1^2x_2}{r^4}\left(7\frac{1}{S}-5\frac{p^2}{S^3}+2\frac{p^4}{S^5}\right) \\ &\quad \left.+\,4\xi_3\frac{x_2x_3}{S^5}-20\xi_3\frac{x_1^2x_2x_3}{S^7}\right], \\ \omega_{12}^{2} &= \frac{1}{4\pi}\left[-\frac{1}{3}\frac{x_1}{r^2}\left(\frac{5}{S}-2\frac{p^2}{S^3}\right)+2\frac{x_1}{r^4}(S-p)\right. \\ &\quad -8\frac{x_1x_2^2}{r^6}(S-p)+\frac{x_1x_2^2}{r^4}\left(7\frac{1}{S}-5\frac{p^2}{S^3}+2\frac{p^4}{S^5}\right) \\ &\quad \left.+\,4\xi_3\frac{x_1x_3}{S^5}-20\xi_3\frac{x_1x_2^2x_3}{S^7}\right], \\ \omega_{12}^{3} &= \frac{1}{4\pi}\left[2\frac{x_1x_2}{r^4}\left(1-\frac{p}{S}\right)+\frac{x_1x_2}{r^2}\left(\frac{p}{S^3}-2\frac{p^3}{S^5}\right)\right. \\ &\quad \left.-\,8\xi_3\frac{x_1x_2}{S^5}-20\xi_3\frac{x_1x_2x_3p}{S^7}\right],\end{aligned}\right.$$

The resultant displacement field, W_{kl}^m, for each (kl) is expressed by

$$W_{kl}^m = w_{kl}^m + \omega_{kl}^m \,. \tag{2.28}$$

As for the stress components at $Q(x_1, x_2, x_3)$, we can obtain them by differentiating with respect to x_k's as in equation (1.1). The (mn)-component of stress tensor for each (kl) can be written as

$$H_{kl}^{mn} = h_{kl}^{mn} + \eta_{kl}^{mn} \,, \tag{2.29}$$

where h_{kl}^{mn} and η_{kl}^{mn} are computed from w_{kl}^m and ω_{kl}^m respectively. The stress components h_{kl}^{mn} may also be found by means of equation (1.16)′ considering the contribution from the image point. They may be written as follows.

$$\left\{\begin{aligned}
h_{11}^{11} &= \frac{\mu}{4\pi}\left[\frac{4}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)+8x_1^2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)-20x_1^4\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right],\\
h_{11}^{22} &= \frac{\mu}{4\pi}\left[2(x_1^2+x_2^2)\left(\frac{1}{R^5}+\frac{1}{S^5}\right)-20x_1^2x_2^2\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right],\\
h_{11}^{33} &= \frac{\mu}{4\pi}\left[2x_1^2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)+2\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)\right.\\
&\qquad\left.-20x_1^2\left(\frac{(x_3-\xi_3)^2}{R^7}+\frac{(x_3+\xi_3)^2}{S^7}\right)\right],\\
h_{11}^{23} &= \frac{\mu}{4\pi}\left[2x_2\left(\frac{x_3-\xi_3}{R^5}+\frac{x_3+\xi_3}{S^5}\right)-20x_1^2x_2\left(\frac{x_3-\xi_3}{R^7}+\frac{x_3+\xi_3}{S^7}\right)\right],\\
h_{11}^{31} &= \frac{\mu}{4\pi}\left[4x_1\left(\frac{x_3-\xi_3}{R^5}+\frac{x_3+\xi_3}{S^5}\right)-20x_1^3\left(\frac{x_3-\xi_3}{R^7}+\frac{x_3+\xi_3}{S^7}\right)\right],\\
h_{11}^{12} &= \frac{\mu}{4\pi}\left[4x_1x_2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)-20x_1^3x_2\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right],
\end{aligned}\right.$$

$$\left\{\begin{aligned}
h_{22}^{11} &= \frac{\mu}{4\pi}\left[2(x_1^2+x_2^2)\left(\frac{1}{R^5}+\frac{1}{S^5}\right)-20x_1^2x_2^2\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right],\\
h_{22}^{22} &= \frac{\mu}{4\pi}\left[\frac{4}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)+8x_2^2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)-20x_2^4\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right],\\
h_{22}^{33} &= \frac{\mu}{4\pi}\left[2x_2^2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)+2\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)\right.\\
&\qquad\left.-20x_2^2\left(\frac{(x_3-\xi_3)^2}{R^7}+\frac{(x_3+\xi_3)^2}{S^7}\right)\right],
\end{aligned}\right.$$

$$
\left\{
\begin{aligned}
h_{22}^{23} &= \frac{\mu}{4\pi}\left[4x_2\left(\frac{x_3-\xi_3}{R^5}+\frac{x_3+\xi_3}{S^5}\right)-20x_2^3\left(\frac{x_3-\xi_3}{R^7}+\frac{x_3+\xi_3}{S^7}\right)\right],\\
h_{22}^{31} &= \frac{\mu}{4\pi}\left[2x_1\left(\frac{x_3-\xi_3}{R^5}+\frac{x_3+\xi_3}{S^5}\right)-20x_1x_2^2\left(\frac{x_3-\xi_3}{R^7}+\frac{x_3+\xi_3}{S^7}\right)\right],\\
h_{22}^{12} &= \frac{\mu}{4\pi}\left[4x_1x_2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)-20x_1x_2^3\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right],
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
h_{33}^{11} &= \frac{\mu}{4\pi}\left[2x_1^2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)+2\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)\right.\\
&\qquad\left.-20x_1^2\left(\frac{(x_3-\xi_3)^2}{R^7}+\frac{(x_3+\xi_3)^2}{S^7}\right)\right],\\
h_{33}^{22} &= \frac{\mu}{4\pi}\left[2x_2^2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)+2\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)\right.\\
&\qquad\left.-20x_2^2\left(\frac{(x_3-\xi_3)^2}{R^7}+\frac{(x_3+\xi_3)^2}{S^7}\right)\right],\\
h_{33}^{33} &= \frac{\mu}{4\pi}\left[\frac{4}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)+8\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)\right.\\
&\qquad\left.-20\left(\frac{(x_3-\xi_3)^4}{R^7}+\frac{(x_3+\xi_3)^4}{S^7}\right)\right],\\
h_{33}^{23} &= \frac{\mu}{4\pi}\left[4x_2\left(\frac{(x_3-\xi_3)}{R^5}+\frac{(x_3+\xi_3)}{S^5}\right)-20x_2\left(\frac{(x_3-\xi_3)^3}{R^7}+\frac{(x_3+\xi_3)^3}{S^7}\right)\right],\\
h_{33}^{31} &= \frac{\mu}{4\pi}\left[4x_1\left(\frac{(x_3-\xi_3)}{R^5}+\frac{(x_3+\xi_3)}{S^5}\right)-20x_1\left(\frac{(x_3-\xi_3)^3}{R^7}+\frac{(x_3+\xi_3)^3}{S^7}\right)\right],\\
h_{33}^{12} &= \frac{\mu}{4\pi}\left[2x_1x_2\left(\frac{1}{R^5}+\frac{1}{S^5}\right)-20x_1x_2\left(\frac{(x_3-\xi_3)^2}{R^7}+\frac{(x_3+\xi_3)^2}{S^7}\right)\right],
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
h_{23}^{11} &= \frac{\mu}{4\pi}\left[2x_2\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)-20x_1^2x_2\left(\frac{x_3-\xi_3}{R^7}-\frac{x_3+\xi_3}{S^7}\right)\right],\\
h_{23}^{22} &= \frac{\mu}{4\pi}\left[4x_2\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)-20x_2^3\left(\frac{(x_3-\xi_3)}{R^7}-\frac{(x_3+\xi_3)}{S^7}\right)\right],\\
h_{23}^{33} &= \frac{\mu}{4\pi}\left[4x_2\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)-20x_2\left(\frac{(x_3-\xi_3)^3}{R^7}-\frac{(x_3+\xi_3)^3}{S^7}\right)\right],\\
h_{23}^{23} &= \frac{\mu}{4\pi}\left[\frac{2}{3}\left(\frac{1}{R^3}-\frac{1}{S^3}\right)+x_2^2\left(\frac{1}{R^5}-\frac{1}{S^5}\right)+\left(\frac{(x_3-\xi_3)^2}{R^5}-\frac{(x_3+\xi_3)^2}{S^5}\right)\right.\\
&\qquad\left.-20x_2^2\left(\frac{(x_3-\xi_3)^2}{R^7}-\frac{(x_3+\xi_3)^2}{S^7}\right)\right],
\end{aligned}
\right.
$$

$$\left\{\begin{aligned}
h_{23}^{31} &= \frac{\mu}{4\pi}\left[x_1x_2\left(\frac{1}{R^5}-\frac{1}{S^5}\right) -20x_1x_2\left(\frac{(x_3-\xi_3)^2}{R^7}-\frac{(x_3+\xi_3)^2}{S^7}\right)\right], \\
h_{23}^{12} &= \frac{\mu}{4\pi}\left[x_1\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right) -20x_1x_2^2\left(\frac{(x_3-\xi_3)}{R^7}-\frac{(x_3+\xi_3)}{S^7}\right)\right],
\end{aligned}\right.$$

$$\left\{\begin{aligned}
h_{31}^{11} &= \frac{\mu}{4\pi}\left[4x_1\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)-20x_1^3\left(\frac{(x_3-\xi_3)}{R^7}-\frac{(x_3+\xi_3)}{S^7}\right)\right], \\
h_{31}^{22} &= \frac{\mu}{4\pi}\left[2x_1\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right) -20x_1x_2^2\left(\frac{(x_3-\xi_3)}{R^7}-\frac{(x_3+\xi_3)}{S^7}\right)\right], \\
h_{31}^{33} &= \frac{\mu}{4\pi}\left[4x_1\left(\frac{(x_3-\xi_3)}{R^5}-\frac{(x_3+\xi_3)}{S^5}\right)-20x_1\left(\frac{(x_3-\xi_3)^3}{R^7}-\frac{(x_3+\xi_3)^3}{S^7}\right)\right], \\
h_{31}^{23} &= \frac{\mu}{4\pi}\left[x_1x_2\left(\frac{1}{R^5}-\frac{1}{S^5}\right) -20x_1x_2\left(\frac{(x_3-\xi_3)^2}{R^7}-\frac{(x_3+\xi_3)^2}{S^7}\right)\right], \\
h_{31}^{31} &= \frac{\mu}{4\pi}\left[\frac{2}{3}\left(\frac{1}{R^3}-\frac{1}{S^3}\right)+x_1^2\left(\frac{1}{R^5}-\frac{1}{S^5}\right)+\left(\frac{(x_3-\xi_3)^2}{R^5}-\frac{(x_3+\xi_3)^2}{S^5}\right)\right. \\
&\qquad \left. -20x_1^2\left(\frac{(x_3-\xi_3)^2}{R^7}-\frac{(x_3+\xi_3)^2}{S^7}\right)\right], \\
h_{31}^{12} &= \frac{\mu}{4\pi}\left[x_2\left(\frac{x_3-\xi_3}{R^5}-\frac{x_3+\xi_3}{S^5}\right) -20x_1^2x_2\left(\frac{(x_3-\xi_3)}{R^7}-\frac{(x_3+\xi_3)}{S^7}\right)\right],
\end{aligned}\right.$$

$$\left\{\begin{aligned}
h_{12}^{11} &= \frac{\mu}{4\pi}\left[4x_1x_2\left(\frac{1}{R^5}+\frac{1}{S^5}\right) -20x_1^3x_2\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right], \\
h_{12}^{22} &= \frac{\mu}{4\pi}\left[4x_1x_2\left(\frac{1}{R^5}+\frac{1}{S^5}\right) -20x_1x_2^3\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right], \\
h_{12}^{33} &= \frac{\mu}{4\pi}\left[2x_1x_2\left(\frac{1}{R^5}+\frac{1}{S^5}\right) -20x_1x_2\left(\frac{(x_3-\xi_3)^2}{R^7}+\frac{(x_3+\xi_3)^2}{S^7}\right)\right], \\
h_{12}^{23} &= \frac{\mu}{4\pi}\left[x_1\left(\frac{(x_3-\xi_3)}{R^5}+\frac{(x_3+\xi_3)}{S^5}\right) -20x_1x_2^2\left(\frac{(x_3-\xi_3)}{R^7}+\frac{(x_3+\xi_3)}{S^7}\right)\right], \\
h_{12}^{31} &= \frac{\mu}{4\pi}\left[x_2\left(\frac{(x_3-\xi_3)}{R^5}+\frac{(x_3+\xi_3)}{S^5}\right) -20x_1^2x_2\left(\frac{(x_3-\xi_3)}{R^7}+\frac{(x_3+\xi_3)}{S^7}\right)\right], \\
h_{12}^{12} &= \frac{\mu}{4\pi}\left[\frac{5}{3}\left(\frac{1}{R^3}+\frac{1}{S^3}\right)-\left(\frac{(x_3-\xi_3)^2}{R^5}+\frac{(x_3+\xi_3)^2}{S^5}\right)\right. \\
&\qquad \left. -\, 20x_1^2x_2^2\left(\frac{1}{R^7}+\frac{1}{S^7}\right)\right].
\end{aligned}\right.$$

The stress components η_{kl}^{mn} may be obtained as follows.

$$
\left\{
\begin{aligned}
\eta_{11}^{11}=\frac{\mu}{4\pi}\Bigg[&\frac{2}{3}\frac{1}{r^2}\left(-14\frac{1}{S}-\frac{p^2}{S^3}+6\frac{p^4}{S^5}\right)+12\frac{1}{r^4}(S-p)\\
&+2\frac{x_1^2}{r^4}\left(41\frac{1}{S}-18\frac{p^2}{S^3}-9\frac{p^4}{S^5}+10\frac{p^6}{S^7}\right)-96\frac{x_1^2}{r^6}(S-p)\\
&+2\frac{x_1^4}{r^6}\left(-43\frac{1}{S}+42\frac{p^2}{S^3}-33\frac{p^4}{S^5}+10\frac{p^6}{S^7}\right)+96\frac{x_1^4}{r^8}(S-p)\\
&+2\frac{p^2}{S^5}+24\frac{x_3\xi_3}{S^5}-2\frac{x_1^2}{S^5}-240\frac{x_1^2x_3\xi_3}{S^7}+280\frac{x_1^4x_3\xi_3}{S^9}\Bigg],\\[2ex]
\eta_{11}^{22}=\frac{\mu}{4\pi}\Bigg[&\frac{2}{3}\frac{1}{r^2}\left(14\frac{1}{S}+16\frac{p^2}{S^3}-51\frac{p^4}{S^5}+30\frac{p^6}{S^7}\right)-12\frac{1}{r^4}(S-p)\\
&+2\frac{x_1^2x_2^2}{r^6}\left(-43\frac{1}{S}+42\frac{p^2}{S^3}-33\frac{p^4}{S^5}+10\frac{p^6}{S^7}\right)+96\frac{x_1^2x_2^2}{r^8}(S-p)\\
&+2\frac{p^2}{S^5}+8\frac{x_3\xi_3}{S^5}-20\frac{x_1^2x_3^2}{S^7}-20\frac{x_2^2\xi_3^2}{S^7}\\
&-60\frac{x_1^2x_3\xi_3}{S^7}-60\frac{x_2^2x_3\xi_3}{S^7}+280\frac{x_1^2x_2^2x_3\xi_3}{S^9}\Bigg],\\[2ex]
\eta_{11}^{33}=\frac{\mu}{4\pi}\Bigg[&-4\frac{x_1^2}{S^5}-4\frac{x_3^2}{S^5}-4\frac{\xi_3^2}{S^5}\\
&+20\frac{p^2x_3^2}{S^7}-20\frac{p^2x_3\xi_3}{S^7}-20\frac{x_1^2x_3p}{S^7}+40\frac{x_1^2\xi_3^2}{S^7}+280\frac{x_1^2x_3p^2\xi_3}{S^9}\Bigg],\\[2ex]
\eta_{11}^{23}=\frac{\mu}{4\pi}\Bigg[&20\frac{x_2p^3}{S^7}-20\frac{x_2p\xi_3^2}{S^7}-20\frac{x_1^2x_2x_3}{S^7}-60\frac{x_2x_3p\xi_3}{S^7}+280\frac{x_1^2x_2x_3p\xi_3}{S^9}\Bigg],\\[2ex]
\eta_{11}^{31}=\frac{\mu}{4\pi}\Bigg[&8\frac{x_1x_3}{S^5}-20\frac{x_1^3x_3}{S^7}+20\frac{x_1p^3}{S^7}-20\frac{x_1p\xi_3^2}{S^7}\\
&-140\frac{x_1x_3p\xi_3}{S^7}+280\frac{x_1^3x_3p\xi_3}{S^9}\Bigg],\\[2ex]
\eta_{11}^{12}=\frac{\mu}{4\pi}\Bigg[&2\frac{x_1x_2}{r^4}\left(20\frac{1}{S}-3\frac{p^2}{S^3}-15\frac{p^4}{S^5}+10\frac{p^6}{S^7}\right)-48\frac{x_1x_2}{r^6}(S-p)\\
&+2\frac{x_1^3x_2}{r^6}\left(-43\frac{1}{S}+42\frac{p^2}{S^3}-33\frac{p^4}{S^5}+10\frac{p^6}{S^7}\right)+96\frac{x_1^3x_2}{r^8}(S-p)\\
&-20\frac{x_1x_2\xi_3^2}{S^7}-140\frac{x_1x_2x_3\xi_3}{S^7}+280\frac{x_1^3x_2x_3\xi_3}{S^9}\Bigg],
\end{aligned}
\right.
$$

$$\left\{\begin{aligned}
\eta_{22}^{11}=&\frac{\mu}{4\pi}\left[\frac{2}{3}\frac{1}{r^4}\left(-4S+18p+2\frac{p^2}{S}-67\frac{p^4}{S^3}+81\frac{p^6}{S^5}-30\frac{p^8}{S^7}\right)\right.\\
&+2\frac{x_1^2x_2^2}{r^8}\left(5S-48p+85\frac{p^2}{S}-75\frac{p^4}{S^3}+43\frac{p^6}{S^5}-10\frac{p^8}{S^7}\right)\\
&+2\frac{p^2}{S^5}+8\frac{x_3\xi_3}{S^5}-60\frac{x_1^2p\xi_3}{S^7}-20\frac{x_2^2p^2}{S^7}-20\frac{x_2^2p\xi_3}{S^7}+40\frac{x_1^2\xi_3^2}{S^7}\\
&\left.+40\frac{x_2^2\xi_3^2}{S^7}+280\frac{x_1^2x_2^2x_3\xi_3}{S^9}\right],\\
\eta_{22}^{22}=&\frac{\mu}{4\pi}\left[\frac{2}{3}\frac{1}{r^2}\left(4S-18p+13\frac{p^2}{S}+7\frac{p^4}{S^3}-6\frac{p^6}{S^5}\right)\right.\\
&+2\frac{x_2^2}{r^6}\left(-7S+48p-59\frac{p^2}{S}+9\frac{p^4}{S^3}+19\frac{p^6}{S^5}-10\frac{p^8}{S^7}\right)\\
&+2\frac{x_2^2}{r^8}\left(5S-48p+85\frac{p^2}{S}-75\frac{p^4}{S^3}+43\frac{p^6}{S^5}-10\frac{p^8}{S^7}\right)\\
&\left.+2\frac{p^2}{S^5}+24\frac{x_3\xi_3}{S^5}-2\frac{x_2^2}{S^5}-240\frac{x_2^2x_3\xi_3}{S^7}+280\frac{x_2^4x_3\xi_3}{S^9}\right],\\
\eta_{22}^{33}=&\frac{\mu}{4\pi}\left[-4\frac{p^2}{S^5}+8\frac{x_3\xi_3}{S^5}-4\frac{x_2^2}{S^5}+20\frac{p^4}{S^7}-60\frac{p^3\xi_3}{S^7}\right.\\
&\left.+40\frac{p^2\xi_3^2}{S^7}+40\frac{x_2^2\xi_3^2}{S^7}-20\frac{x_2^2x_3p}{S^7}+280\frac{x_2^2p^2x_3\xi_3}{S^9}\right],\\
\eta_{22}^{23}=&\frac{\mu}{4\pi}\left[8\frac{x_2x_3}{S^5}+20\frac{x_2x_3p^2}{S^7}-120\frac{x_2x_3p\xi_3}{S^7}-20\frac{x_2^3x_3}{S^7}+280\frac{x_2^3x_3p\xi_3}{S^9}\right],\\
\eta_{22}^{31}=&\frac{\mu}{4\pi}\left[20\frac{x_1x_3p^2}{S^7}-40\frac{x_1x_3p\xi_3}{S^7}-20\frac{x_1x_2^2x_3}{S^7}+280\frac{x_1x_2^2x_3p\xi_3}{S^9}\right],\\
\eta_{22}^{12}=&\frac{\mu}{4\pi}\left[2\frac{x_1x_2}{r^6}\left(-4S+24p-23\frac{p^2}{S}-12\frac{p^4}{S^3}+25\frac{p^6}{S^5}-10\frac{p^8}{S^7}\right)\right.\\
&+2\frac{x_1x_2^3}{r^8}\left(5S-48p+85\frac{p^2}{S}-75\frac{p^4}{S^3}+43\frac{p^6}{S^5}-10\frac{p^8}{S^7}\right)\\
&\left.-20\frac{x_1x_2\xi_3^2}{S^7}-140\frac{x_1x_2x_3\xi_3}{S^7}+280\frac{x_1x_2^3x_3\xi_3}{S^9}\right],
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\eta_{33}^{11}=&\frac{\mu}{4\pi}\left[\left\{-4\frac{p^2}{S^5}+\left(8\frac{p}{S^5}-20\frac{p^3}{S^7}\right)\xi_3+\left(-8\frac{1}{S^5}+40\frac{p^2}{S^7}\right)\xi_3^2\right\}\right.\\
&+x_1^2\left\{\left(-4\frac{1}{S^5}+40\frac{p^2}{S^7}\right)+\left(-100\frac{p}{S^7}+280\frac{p^3}{S^9}\right)\xi_3\right.\\
&\left.\left.+\left(40\frac{1}{S^7}-280\frac{p^2}{S^9}\right)\xi_3^2\right\}\right],
\end{aligned}\right.$$

$$\eta_{33}^{22}=\frac{\mu}{4\pi}\Big[\Big\{-4\frac{p^2}{S^5}+\Big(8\frac{p}{S^5}-20\frac{p^3}{S^7}\Big)\xi_3+\Big(-8\frac{1}{S^5}+40\frac{p^2}{S^7}\Big)\xi_3^2\Big\}$$

$$+x_2^2\Big\{\Big(-4\frac{1}{S^5}+40\frac{p^2}{S^7}\Big)+\Big(-100\frac{p}{S^7}+280\frac{p^3}{S^9}\Big)\xi_3$$

$$+\Big(40\frac{1}{S^7}-280\frac{p^2}{S^9}\Big)\xi_3^2\Big\}\Big],$$

$$\eta_{33}^{33}=\frac{\mu}{4\pi}\Big[\Big(-\frac{8}{3}\frac{1}{S^3}-16\frac{p^2}{S^5}+40\frac{p^4}{S^7}\Big)+\Big(24\frac{p}{S^5}-240\frac{p^3}{S^7}+280\frac{p^5}{S^9}\Big)\xi_3$$

$$+\Big(-24\frac{1}{S^5}+240\frac{p^2}{S^7}-280\frac{p^4}{S^9}\Big)\xi_3^2\Big],$$

$$\eta_{33}^{23}=\frac{\mu}{4\pi}\Big[x_2\Big\{\Big(-8\frac{p}{S^5}+40\frac{p^3}{S^7}\Big)+\Big(8\frac{1}{S^5}-160\frac{p^2}{S^7}+280\frac{p^4}{S^9}\Big)\xi_3$$

$$+\Big(120\frac{p}{S^7}-280\frac{p^3}{S^9}\Big)\xi_3^2\Big\}\Big],$$

$$\eta_{33}^{31}=\frac{\mu}{4\pi}\Big[x_1\Big\{\Big(-8\frac{p}{S^5}+40\frac{p^3}{S^7}\Big)+\Big(8\frac{1}{S^5}-160\frac{p^2}{S^7}+280\frac{p^4}{S^9}\Big)\xi_3$$

$$+\Big(120\frac{p}{S^7}-280\frac{p^3}{S^9}\Big)\xi_3^2\Big\}\Big],$$

$$\eta_{33}^{12}=\frac{\mu}{4\pi}\Big[x_1x_2\Big\{\Big(-4\frac{1}{S^5}+40\frac{p^2}{S^7}\Big)+\Big(-100\frac{p}{S^7}+280\frac{p^3}{S^9}\Big)\xi_3$$

$$+\Big(40\frac{1}{S^7}-280\frac{p^2}{S^9}\Big)\xi_3^2\Big\}\Big],$$

$$\eta_{23}^{11}=\frac{\mu}{4\pi}\Big[20\frac{x_1^2x_2\xi_3}{S^7}+20\frac{x_2p^2\xi_3}{S^7}-40\frac{x_2p\xi_3^2}{S^7}-280\frac{x_1^2x_2x_3p\xi_3}{S^9}\Big],$$

$$\eta_{23}^{22}=\frac{\mu}{4\pi}\Big[-8\frac{x_2\xi_3}{S^5}+20\frac{x_2^3\xi_3}{S^7}-20\frac{x_2p^2\xi_3}{S^7}+120\frac{x_2x_3p\xi_3}{S^7}-280\frac{x_2^3x_3p\xi_3}{S^9}\Big],$$

$$\eta_{23}^{33}=\frac{\mu}{4\pi}\Big[8\frac{x_2\xi_3}{S^5}-40\frac{x_2\xi_3^3}{S^7}+40\frac{x_2x_3\xi_3^2}{S^7}+80\frac{x_2x_3^2\xi_3}{S^7}-280\frac{x_2x_3p^3\xi_3}{S^9}\Big],$$

$$\eta_{23}^{23}=\frac{\mu}{4\pi}\Big[-8\frac{x_3\xi_3}{S^5}+40\frac{x_2^2x_3\xi_3}{S^7}+40\frac{x_3p^2\xi_3}{S^7}-280\frac{x_2^2x_3p^2\xi_3}{S^9}\Big],$$

$$\eta_{23}^{31}=\frac{\mu}{4\pi}\Big[40\frac{x_1x_2x_3\xi_3}{S^7}-280\frac{x_1x_2x_3p^2\xi_3}{S^9}\Big],$$

$$\eta_{23}^{12}=\frac{\mu}{4\pi}\Big[-4\frac{x_1\xi_3}{S^5}+20\frac{x_1x_2^2\xi_3}{S^7}+40\frac{x_1x_3p\xi_3}{S^7}-280\frac{x_1x_2^2x_3p\xi_3}{S^9}\Big],$$

$$\eta_{31}^{11}=\frac{\mu}{4\pi}\Big[-8\frac{x_1\xi_3}{S^5}+20\frac{x_1^3\xi_3}{S^7}-20\frac{x_1\xi_3^3}{S^7}+20\frac{x_1x_3^2\xi_3}{S^7}$$
$$+80\frac{x_1x_3p\xi_3}{S^7}-280\frac{x_1^3\xi_3p\xi_3}{S^9}\Big],$$

$$\eta_{31}^{22}=\frac{\mu}{4\pi}\Big[20\frac{x_1x_2^2\xi_3}{S^7}-20\frac{x_1\xi_3^3}{S^7}+20\frac{x_1x_3^2\xi_3}{S^7}-280\frac{x_1x_2^2x_3p\xi_3}{S^9}\Big],$$

$$\eta_{31}^{33}=\frac{\mu}{4\pi}\Big[8\frac{x_1\xi_3}{S^5}-40\frac{x_1\xi_3^3}{S^7}+40\frac{x_1x_3\xi_3^2}{S^7}+80\frac{x_1x_3^2\xi_3}{S^7}-280\frac{x_1x_3p^3\xi_3}{S^9}\Big],$$

$$\eta_{31}^{23}=\frac{\mu}{4\pi}\Big[40\frac{x_1x_2x_3\xi_3}{S^7}-280\frac{x_1x_2x_3p^2\xi_3}{S^9}\Big],$$

$$\eta_{31}^{31}=\frac{\mu}{4\pi}\Big[-8\frac{x_3\xi_3}{S^5}+40\frac{x_1^2x_3\xi_3}{S^7}+40\frac{x_3p^2\xi_3}{S^7}-280\frac{x_1^2x_3p^2\xi_3}{S^9}\Big],$$

$$\eta_{31}^{12}=\frac{\mu}{4\pi}\Big[-4\frac{x_2\xi_3}{S^5}+20\frac{x_1^2x_2\xi_3}{S^7}+40\frac{x_2x_3p\xi_3}{S^7}-280\frac{x_1^2x_2x_3p\xi_3}{S^9}\Big],$$

$$\eta_{12}^{11}=\frac{\mu}{4\pi}\Big[2\frac{x_1x_2}{r^4}\Big(20\frac{1}{S}-13\frac{p^2}{S^3}+5\frac{p^4}{S^5}\Big)-48\frac{x_1x_2}{r^6}(S-p)$$
$$-2\frac{x_1^3x_2}{r^6}\Big(43\frac{1}{S}-42\frac{p^2}{S^3}+33\frac{p^4}{S^5}-10\frac{p^6}{S^7}\Big)+96\frac{x_1^3x_2}{r^8}(S-p)$$
$$+20\frac{x_1x_2p\xi_3}{S^7}-120\frac{x_1x_2x_3\xi_3}{S^7}+280\frac{x_1^3x_2x_3\xi_3}{S^9}\Big],$$

$$\eta_{12}^{22}=\frac{\mu}{4\pi}\Big[2\frac{x_1x_2}{r^4}\Big(20\frac{1}{S}-13\frac{p^2}{S^3}+5\frac{p^4}{S^5}\Big)-48\frac{x_1x_2}{r^6}(S-p)$$
$$-2\frac{x_1x_2^3}{r^6}\Big(43\frac{1}{S}-42\frac{p^2}{S^3}+33\frac{p^4}{S^5}-10\frac{p^6}{S^7}\Big)+96\frac{x_1x_2^3}{r^8}(S-p)$$
$$+20\frac{x_1x_2p\xi_3}{S^7}-120\frac{x_1x_2x_3\xi_3}{S^7}+280\frac{x_1x_2^3x_3\xi_3}{S^9}\Big],$$

$$\eta_{12}^{33}=\frac{\mu}{S^5}\Big[-4\frac{x_1x_2}{S^5}+40\frac{x_1x_2\xi_3^2}{S^7}-20\frac{x_1x_2x_3p}{S^7}+280\frac{x_1x_2x_3p^2\xi_3}{S^9}\Big],$$

$$\eta_{12}^{23}=\frac{\mu}{4\pi}\Big[4\frac{x_1x_3}{S^5}-20\frac{x_1x_2^2x_3}{S^7}-40\frac{x_1x_3p\xi_3}{S^7}+280\frac{x_1x_2^2x_3p\xi_3}{S^9}\Big],$$

$$\eta_{12}^{31}=\frac{\mu}{4\pi}\Big[4\frac{x_2x_3}{S^5}-20\frac{x_1^2x_2x_3}{S^7}-40\frac{x_2x_3p\xi_3}{S^7}+280\frac{x_1^2x_2x_3p\xi_3}{S^9}\Big],$$

$$\eta_{12}^{12}=\frac{\mu}{4\pi}\Big[\frac{2}{3}\frac{1}{r^2}\Big(16\frac{1}{S}-13\frac{p^2}{S^3}+6\frac{p^4}{S^5}\Big)-12\frac{1}{r^4}(S-p)$$
$$-2\frac{x_1^2x_2^2}{r^6}\Big(43\frac{1}{S}-42\frac{p^2}{S^3}+33\frac{p^4}{S^5}-10\frac{p^6}{S^7}\Big)+96\frac{x_1^2x_2^2}{r^8}(S-p)$$
$$-32\frac{x_3\xi_3}{S^5}+40\frac{x_3p^2\xi_3}{S^7}+280\frac{x_1^2x_2^2x_3\xi_3}{S^9}\Big].$$

After replacing x_1 and x_2 by $x_1-\xi_1$ and $x_2-\xi_2$ respectively in the foregoing expressions, using W_{kl}^m in (2.28) and H_{kl}^{mn} in (2.29), we have the displacement components and stress components at $Q(x_1, x_2, x_3)$ as

$$u_m(Q)=\iint \Delta u_k(P)\,W_{kl}^m\nu_l(P)d\Sigma\,, \tag{2.30}$$

$$\tau_{mn}(Q)=\iint \Delta u_k(P)H_{kl}^{mn}\nu_l(P)d\Sigma\,, \tag{2.31}$$

where the integrations are taken over the coordinates of variable point $P(\xi_1, \xi_2, \xi_3)$ on the dislocation surface.

The displacement field on the surface

If we put $x_3=0$ in the expressions of w_{kl}^m and ω_{kl}^m, we get a system of W_{kl}^m's in equation (2.28) for the displacement field on the surface, which are surprisingly simplified and in refined forms and may be written for a point $P(0, 0, \xi_3)$ and a point $Q(x_1, x_2, 0)$ as follows.

$$\begin{cases} W_{11}^1=\dfrac{1}{4\pi}\dfrac{x_1}{r^4}\rho\Big\{C+\dfrac{x_1^2}{r^2}F\Big\}\,, \\[2ex] W_{11}^2=\dfrac{1}{4\pi}\dfrac{x_2}{r^2}\rho\Big\{D+\dfrac{x_1^2}{r^2}F\Big\}\,, \\[2ex] W_{11}^3=\dfrac{1}{4\pi}\dfrac{1}{r^2}\Big\{B+\dfrac{x_1^2}{r^2}E\Big\}\,, \end{cases} \qquad \begin{cases} W_{23}^1=\dfrac{1}{4\pi}\Big\{-6\dfrac{x_1x_2\xi_3}{\rho^5}\Big\}\,, \\[2ex] W_{23}^2=\dfrac{1}{4\pi}\Big\{-6\dfrac{x_2^2\xi_3}{\rho^5}\Big\}\,, \\[2ex] W_{23}^3=\dfrac{1}{4\pi}\Big\{6\dfrac{x_2\xi_3^2}{\rho^5}\Big\}\,, \end{cases}$$

$$\begin{cases} W_{22}^1=\dfrac{1}{4\pi}\dfrac{x_1}{r^4}\rho\Big\{D+\dfrac{x_2^2}{r^2}F\Big\}\,, \\[2ex] W_{22}^2=\dfrac{1}{4\pi}\dfrac{x_2}{r^4}\rho\Big\{C+\dfrac{x_2^2}{r^2}F\Big\}\,, \\[2ex] W_{22}^3=\dfrac{1}{4\pi}\dfrac{1}{r^2}\Big\{B+\dfrac{x_2^2}{r^2}E\Big\}\,, \end{cases} \qquad \begin{cases} W_{31}^1=\dfrac{1}{4\pi}\Big\{-6\dfrac{x_1^2\xi_3}{\rho^5}\Big\}\,, \\[2ex] W_{31}^2=\dfrac{1}{4\pi}\Big\{-6\dfrac{x_1x_2\xi_3}{\rho^5}\Big\}\,, \\[2ex] W_{31}^3=\dfrac{1}{4\pi}\Big\{6\dfrac{x_1\xi_3^2}{\rho^5}\Big\}\,, \end{cases}$$

$$\begin{cases} W_{33}^1 = \dfrac{1}{4\pi}\left\{6\dfrac{x_1\xi_3^2}{\rho^5}\right\}, \\ W_{33}^2 = \dfrac{1}{4\pi}\left\{6\dfrac{x_2\xi_3^2}{\rho^5}\right\}, \\ W_{33}^3 = \dfrac{1}{4\pi}\left\{-6\dfrac{\xi_3^3}{\rho^5}\right\}, \end{cases} \qquad \begin{cases} W_{12}^1 = \dfrac{1}{4\pi}\dfrac{x_2}{r^4}\rho\left\{A+\dfrac{x_1^2}{r^2}F\right\}, \\ W_{12}^2 = \dfrac{1}{4\pi}\dfrac{x_1}{r^4}\rho\left\{A+\dfrac{x_2^2}{r^2}F\right\}, \\ W_{12}^3 = \dfrac{1}{4\pi}\dfrac{x_1x_2}{r^4}E, \end{cases}$$

where

$$r=\sqrt{x_1^2+x_2^2},$$
$$\rho=\sqrt{x_1^2+x_2^2+\xi_3^2},$$

and A, B, C, D, E and F are polynomials in $\zeta=\xi_3/\rho$ as

$$\begin{aligned} &A(\zeta)=1-2\zeta+\zeta^2, \\ &B(\zeta)=-1+2\zeta-\zeta^3, \\ &C(\zeta)=2-6\zeta+5\zeta^2-\zeta^4, \\ &D(\zeta)=-2\zeta+3\zeta^2-\zeta^4=C(\zeta)-2A(\zeta), \\ &E(\zeta)=2-9\zeta+13\zeta^3-6\zeta^5, \\ &F(\zeta)=3+8\zeta-24\zeta^2+19\zeta^4-6\zeta^6. \end{aligned}$$

Here we investigate further the field due to a single nucleus. As to the nuclei, the cases when $(kl)=11$, 33, 23 and 12 are sufficient to be taken up because of the property of symmetry. We substitute polar coordinates $x_1=r\cos\varphi$, $x_2=r\sin\varphi$ and obtain the radial and tangential displacement components W_{kl}^r and W_{kl}^φ in the plane $x_3=0$. We find

$$\begin{cases} W_{11}^r = \dfrac{1}{8\pi}\dfrac{\rho}{r^3}\left\{(C+D+F)+(C-D+F)\cos 2\varphi\right\}, \\ W_{11}^\varphi = \dfrac{1}{8\pi}\dfrac{\rho}{r^3}(-C+D)\sin 2\varphi, \\ W_{11}^3 = \dfrac{1}{8\pi}\dfrac{1}{r^2}\{(2B+E)+E\cos 2\varphi\}, \end{cases}$$

$$\begin{cases} W_{33}^r = \dfrac{3}{2\pi}\dfrac{\xi_3^2 r}{\rho^5}, \\ W_{33}^\varphi=0, \\ W_{33}^3 = -\dfrac{3}{2\pi}\dfrac{\xi_3^3}{\rho^5}, \end{cases}$$

$$\left\{\begin{aligned} W_{23}^{r} &= -\frac{3}{2\pi}\frac{\xi_3 r^2}{\rho^5}\sin\varphi\ , \\ W_{23}^{\varphi} &= 0\ , \\ W_{23}^{3} &= \frac{3}{2\pi}\frac{\xi_3^2 r}{\rho^5}\sin\varphi\ , \end{aligned}\right.$$

$$\left\{\begin{aligned} W_{12}^{r} &= \frac{1}{8\pi}\frac{\rho}{r^3}(2A+F)\sin 2\varphi\ , \\ W_{12}^{\varphi} &= \frac{1}{8\pi}\frac{\rho}{r^3}(2A)\cos 2\varphi\ , \\ W_{12}^{3} &= \frac{1}{8\pi}\frac{1}{r^2}E\sin 2\varphi\ . \end{aligned}\right.$$

The displacement fields W_{11}, W_{33}, W_{23} and W_{12} are shown in Fig. 6～17, where the depth of nuclei is taken to be unit ($\xi_3=1$) and the

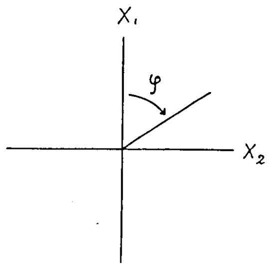

Fig. 5. Direction of horizontal displacements is measured clockwise from x_1-axis toward x_2-axis.

interval of the lattice is equal to unit. The values in the figures of resultant horizontal or downward displacements must be multiplied by 10^{-3} if true values are required. The direction of horizontal displacements are measured clockwise from the direction of x_1-axis toward the direction of x_2-axis, as shown in Fig. 5.

The displacement field of W_{11} is somewhat complicated in the neighborhood of the origin and enlargements of that portion are shown in Fig. 18～20, where the depth of the nucleus is 10 times the lattice interval.

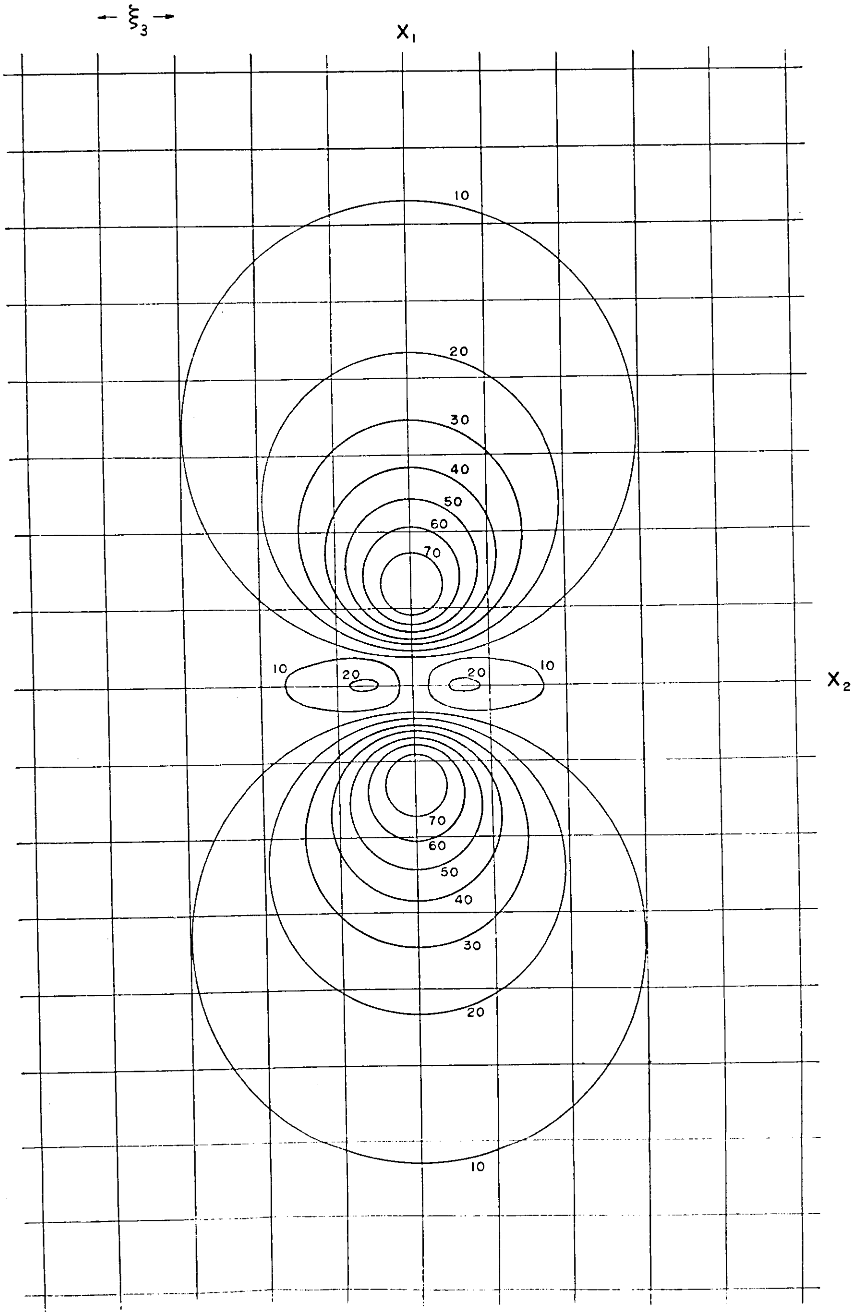

Fig. 6. Resultant horizontal displacements of the free surface for W_{11}.

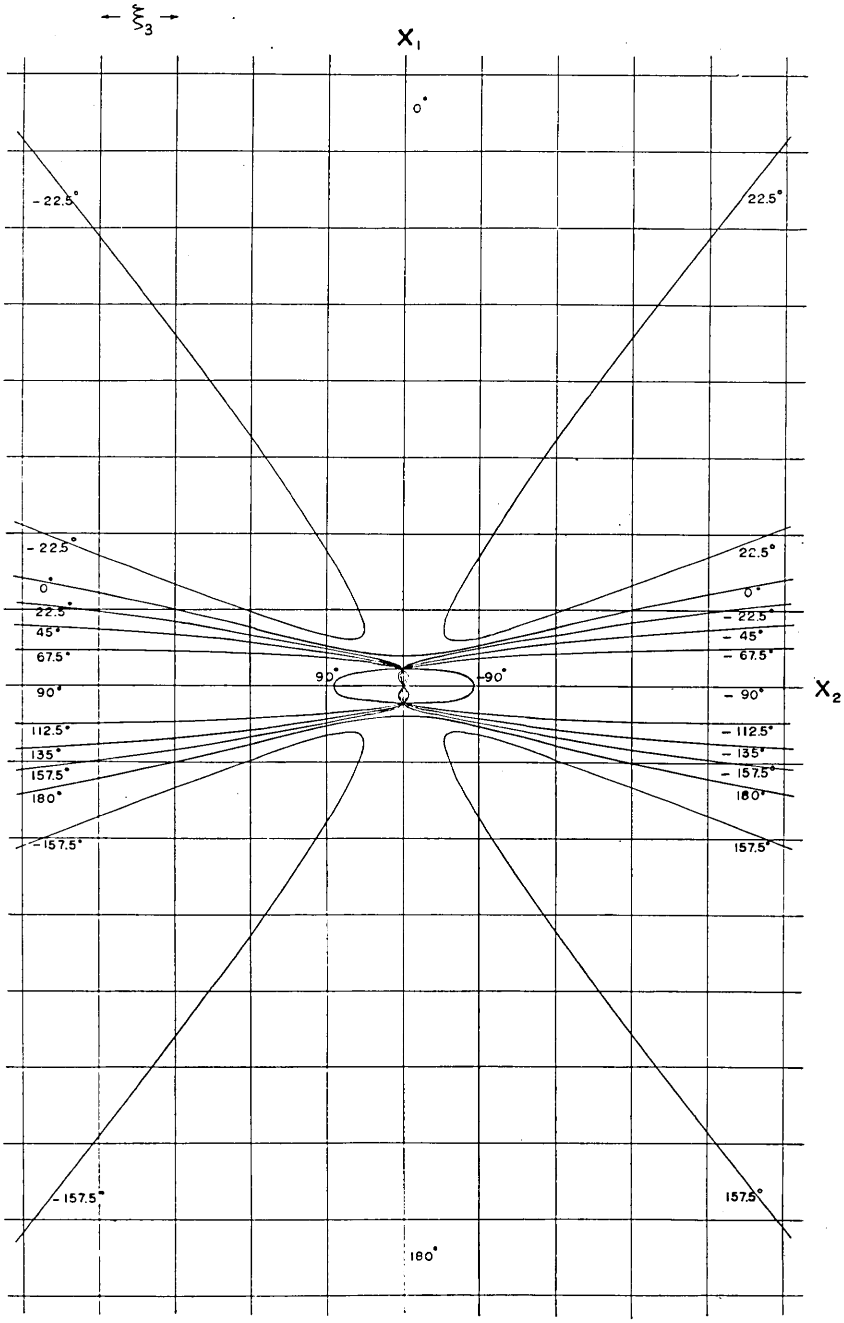

Fig. 7. Direction of horizotal displacements of the free surface for W_{11}.

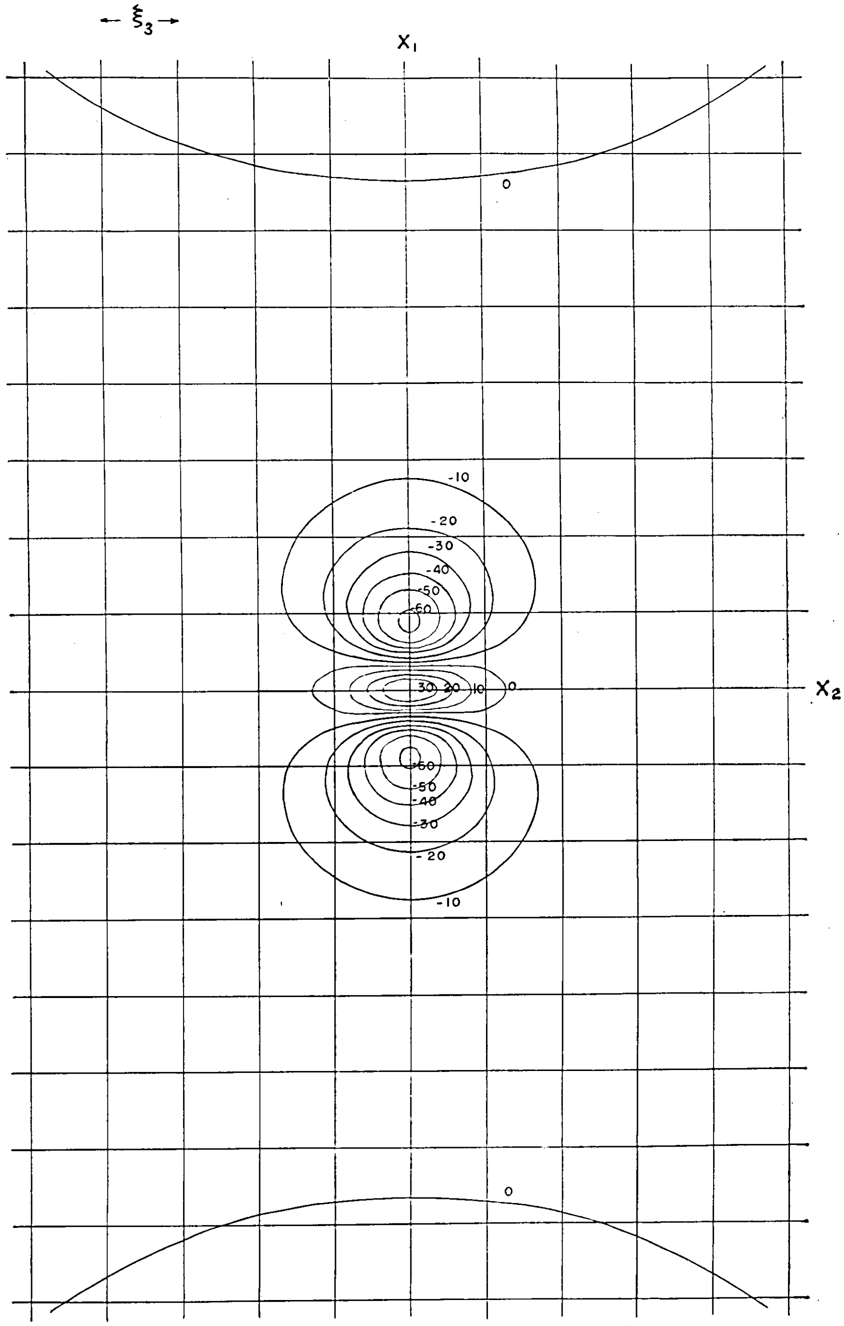

Fig. 8. Downward displacements of the free surface for W_{11}.

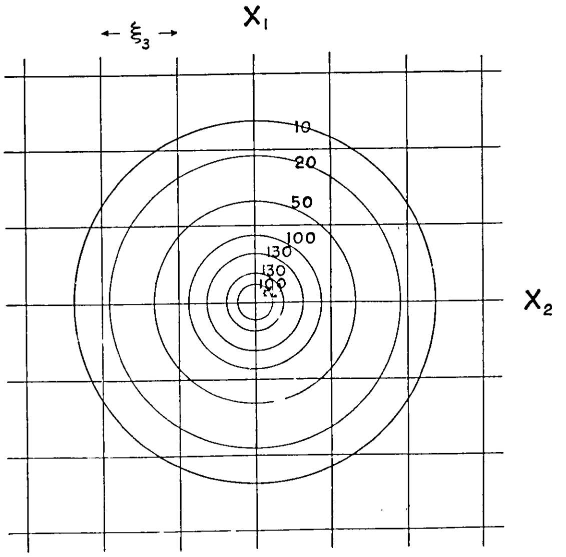

Fig. 9. Resultant horizontal displacements of the free surface for W_{33}.

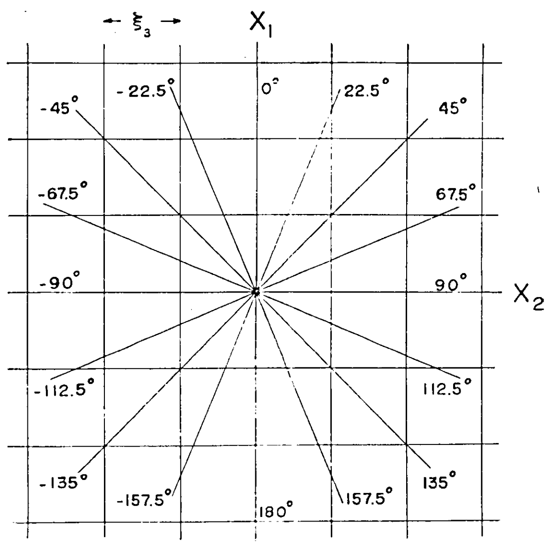

Fig. 10. Direction of horizontal displacements of the free surface for W_{33}.

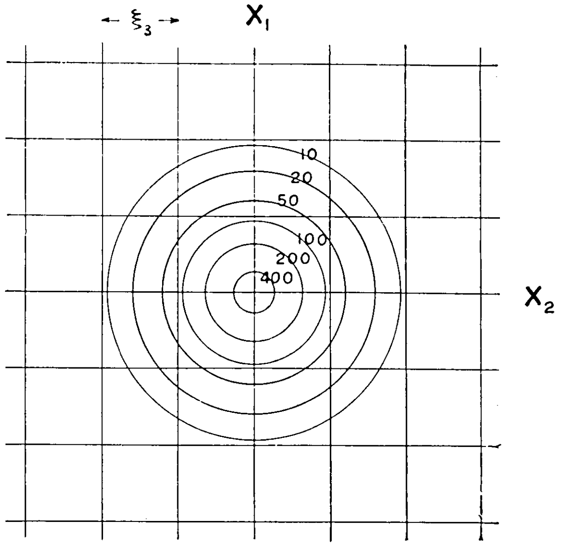

Fig. 11. Downward displacements of the free surface for W_{33}.

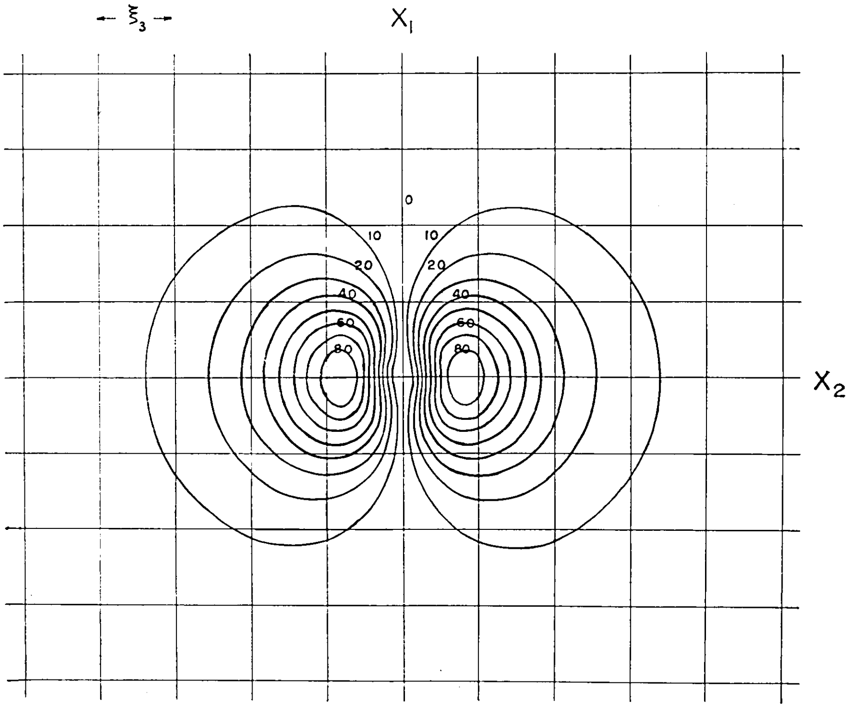

Fig. 12. Resultant horizontal displacements of the free surface for W_{23}.

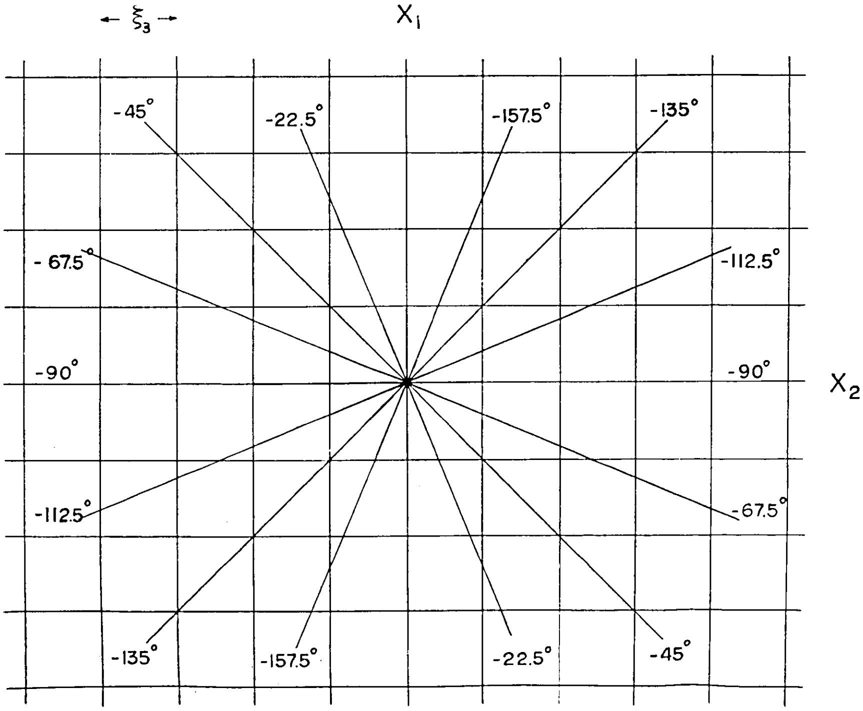

Fig. 13. Direction of horizontal displacements of the free surface for W_{23}.

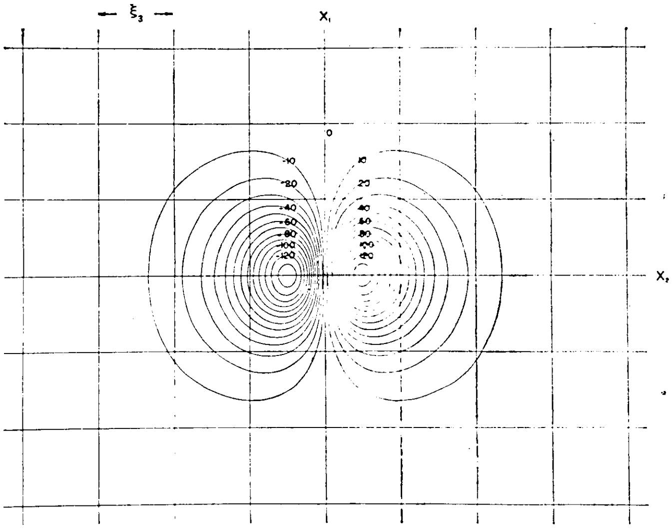

Fig. 14. Downward displacements of the free surface for W_{23}.

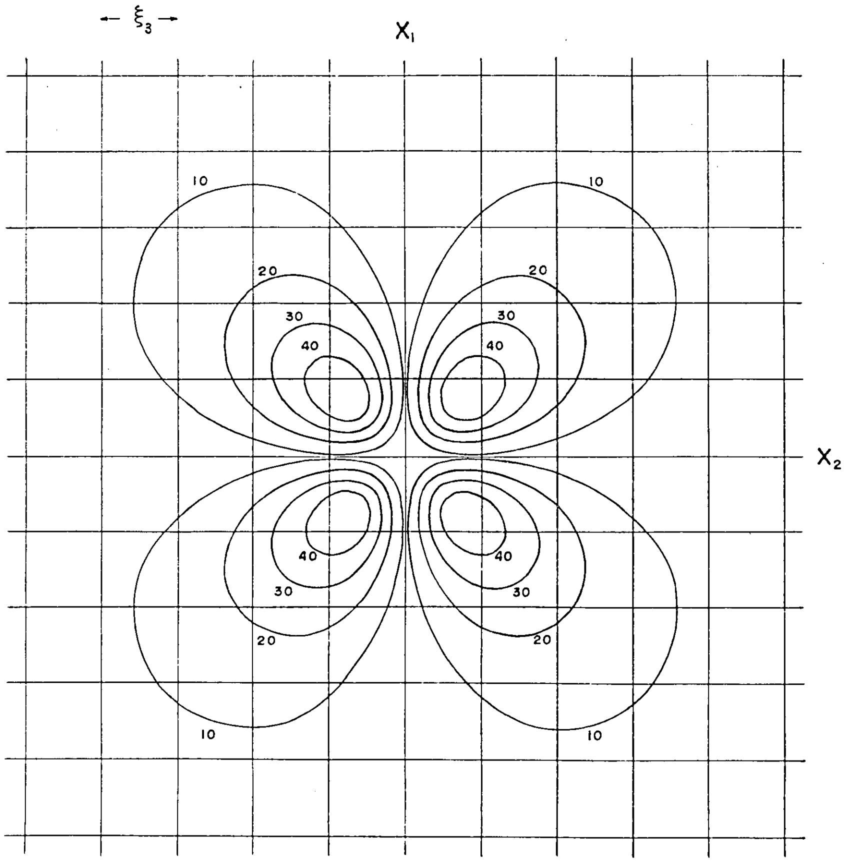

Fig. 15. Resultant horizontal displacements of the free surface for W_{12}.

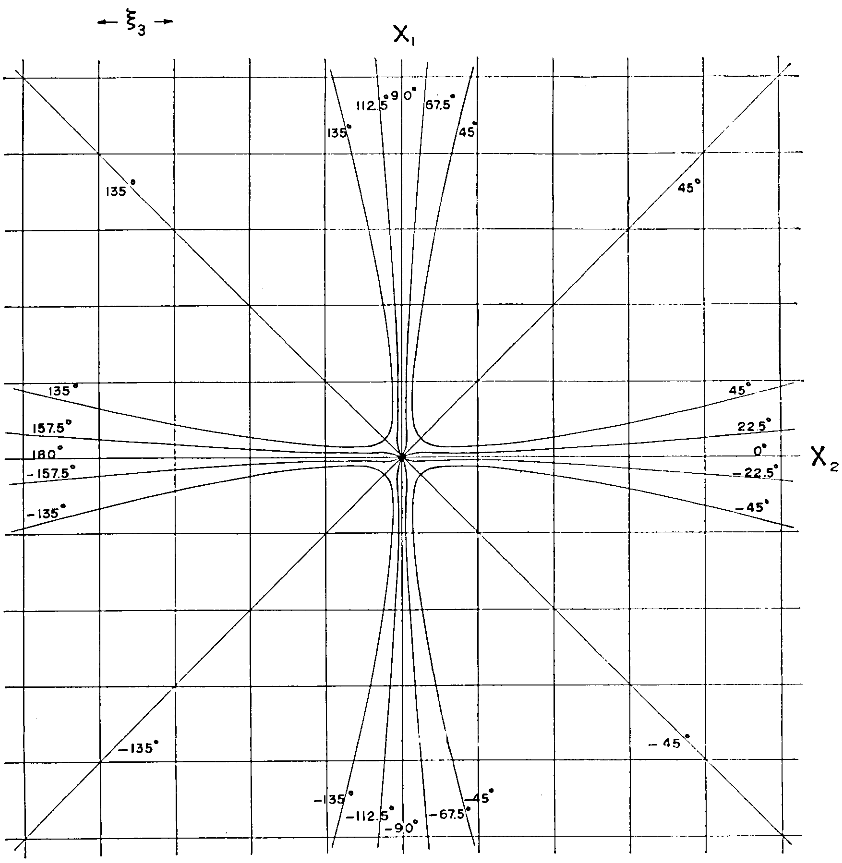

Fig. 16. Direction of horizotal displacements of the free surface for W_{12}.

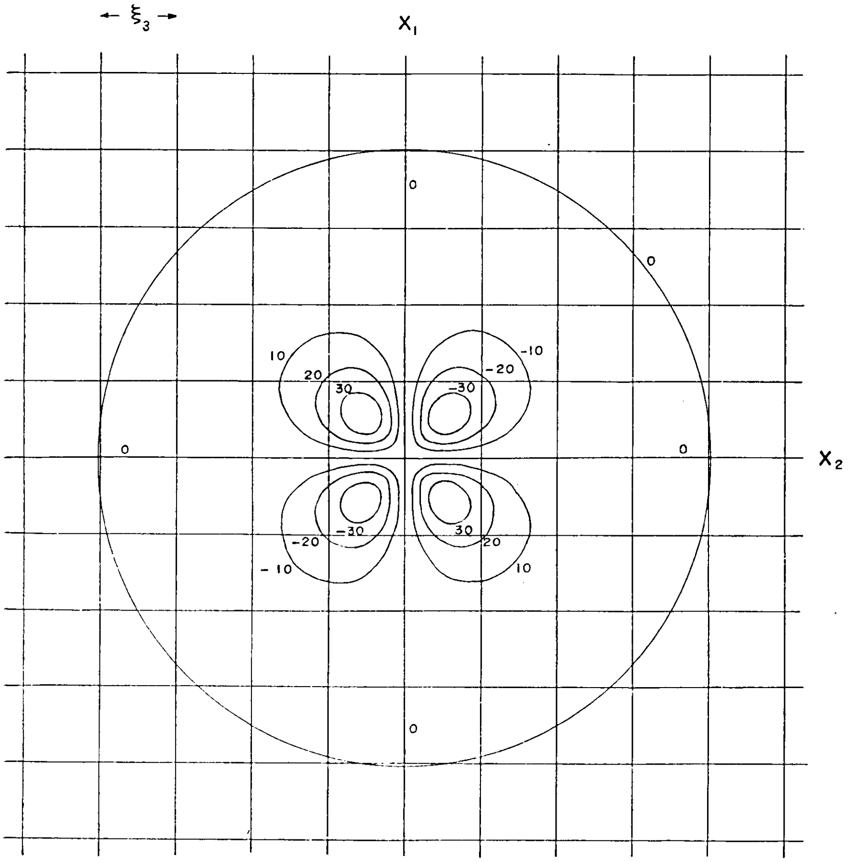

Fig. 17. Downward displacements of the free surface for W_{12}.

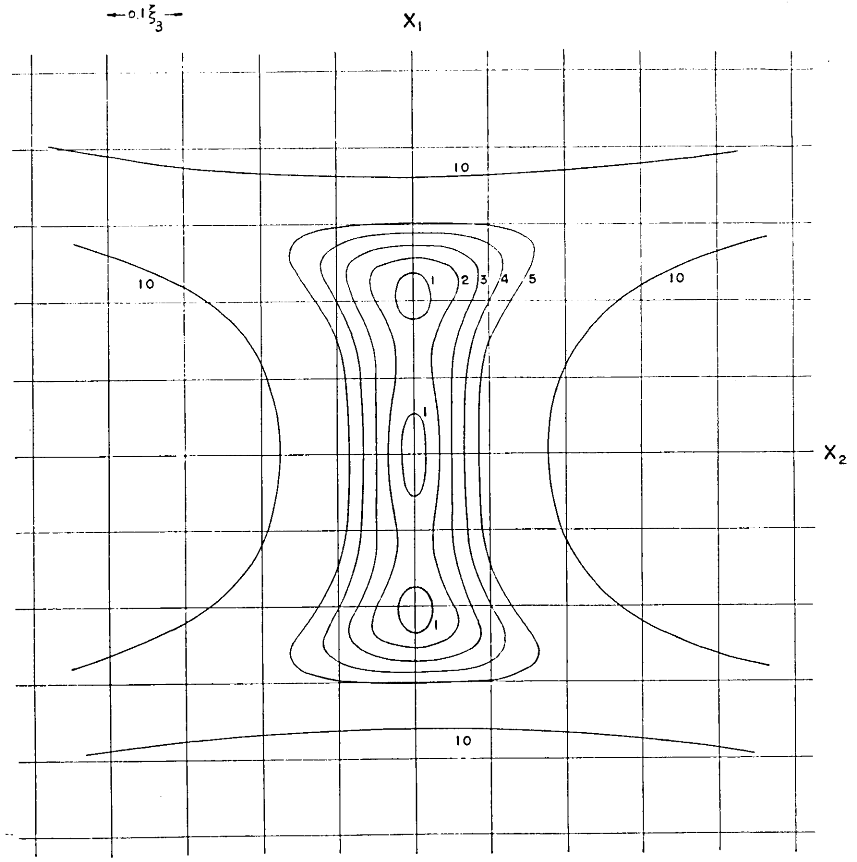

Fig. 18. Resultant horizontal displacements of the free surface for W_{11}.

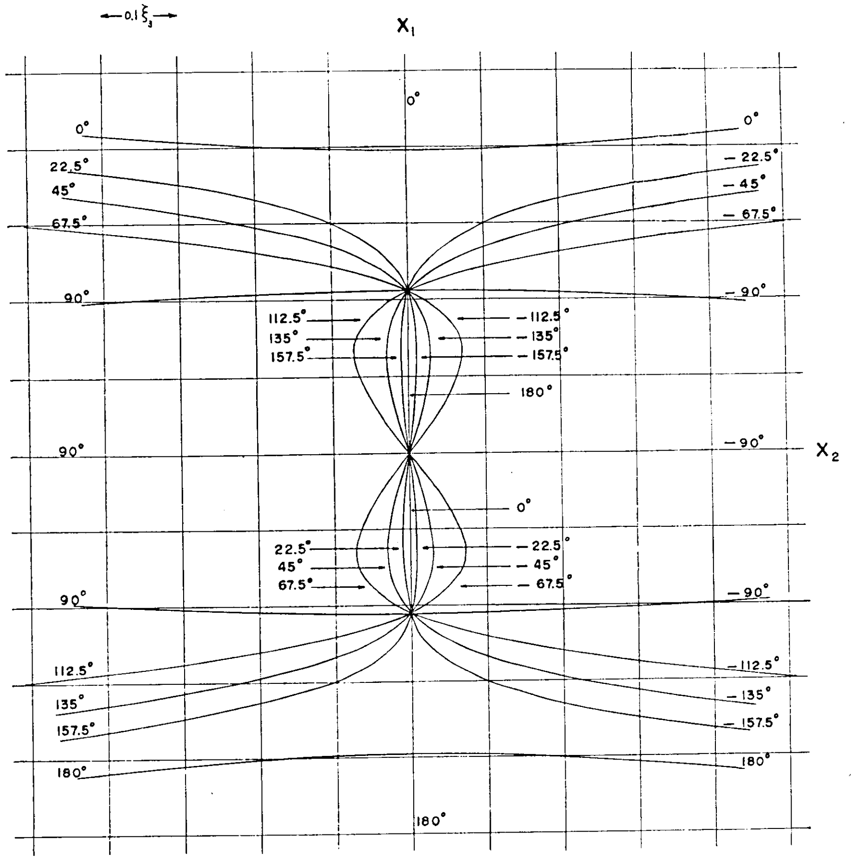

Fig. 19. Direction of horizonal displacements of the free surface for W_{11}.

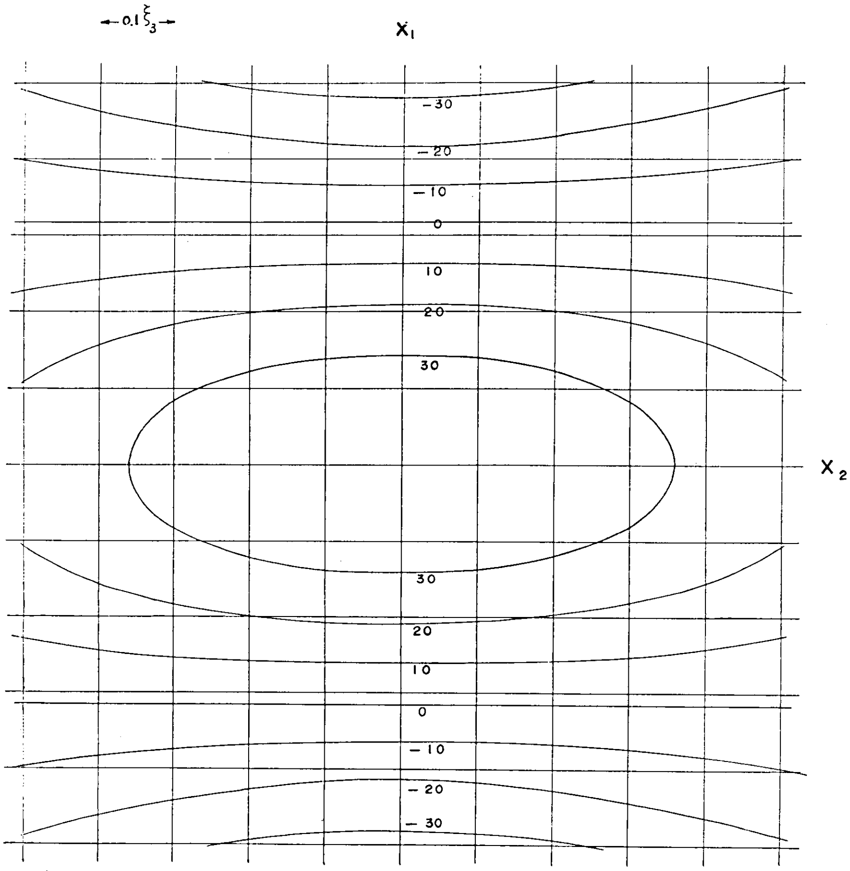

Fig. 20. Downward displacements of the free surface for W_{11}.

2-2. Examples of Displacement Field on the Free Surface due to Finite Dislocations of Simple Forms.

In integral representations of the displacement field due to a dislocation in the medium, if the surface Σ is perpendicular or parallel to the free surface and its form is rectangular and if the displacement discontinuity Δu is constant on the dislocation surface Σ, the integration can be represented by elementary functions.

(a) When Σ is perpendicular to the free surface, we consider the following cases in the order shown.

(i) When only u_1 is discontinuous on Σ, Case (1-1),

(ii) When only u_2 is discontinuous on Σ, Case (2-1),

(iii) When only u_3 is discontinuous on Σ, Case (3-1).

If we put

$$X_1 = x_1 - \xi_1\,,$$

$$X_2 = x_2 - \xi_2\,,$$

$$\rho = \sqrt{X_1^2 + X_2^2 + \xi_2^3}\,,$$

and we use the notation $\|$ as

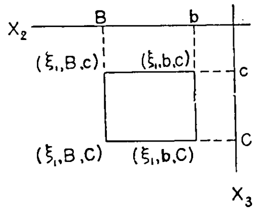

Fig. 21. Rectangular dislocation surface Σ perpendicular to the free surface.

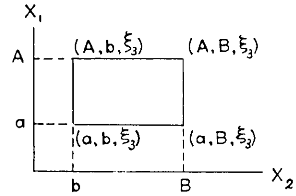

Fig. 22. Rectangular dislocation surface Σ parallel to the free surface.

$$f(\xi_2, \xi_3)\| = f(b, c) - f(b, C) - f(B, c) + f(B, C)\,,$$

following Chinnery, where (b, c), (B, c), (B, C) and (b, C) are apices of Σ, then the expressions are as follows.

(i)

$$4\pi\frac{u_1}{\Delta u_1} = \left\{ -\frac{X_1X_2\xi_3(\rho+2\xi_3)}{\rho(\rho+\xi_3)^2(\rho-\xi_3)} - 2\frac{X_1X_2\xi_3}{(X_1^2+\xi_3^2)\rho} - 2\arctan\left(\frac{X_2\xi_3}{X_1\rho}\right)\right\}\Bigg\|\,,$$

$$4\pi\frac{u_2}{\Delta u_1} = \left\{ \frac{1}{2}\left(\frac{\xi_3}{\rho+\xi_3}\right) + \frac{X_1^2\xi_3(\rho+2\xi_3)}{\rho(\rho+\xi_3)^2(\rho-\xi_3)} + \frac{1}{4}\log\left(\frac{\rho-\xi_3}{\rho+\xi_3}\right)\right\}\Bigg\|\,,$$

$$4\pi\frac{u_3}{\Delta u_1} = \left\{ \frac{X_2}{\rho+\xi_3} - 2\frac{X_1^2X_2}{(X_1^2+\xi_3^2)\rho}\right\}\Bigg\|\,,$$

(ii)

$$4\pi\frac{u_1}{\Delta u_2} = \left\{ \frac{1}{2}\left(\frac{\xi_3}{\rho+\xi_3}\right) + \frac{X_1^2\xi_3(\rho+2\xi_3)}{\rho(\rho+\xi_3)^2(\rho-\xi_3)} - \frac{1}{4}\log\left(\frac{\rho-\xi_3}{\rho+\xi_3}\right)\right\}\Bigg\|\,,$$

$$4\pi\frac{u_2}{\Delta u_2} = \left\{ \frac{X_1X_2\xi_3(\rho+2\xi_3)}{\rho(\rho+\xi_3)^2(\rho-\xi_3)} - 2\arctan\left(\frac{X_2\xi_3}{X_1\rho}\right)\right\}\Bigg\|\,,$$

$$4\pi\frac{u_3}{\Delta u_2} = \left\{ \frac{X_1(\rho+2\xi_3)}{\rho(\rho+\xi_3)}\right\}\Bigg\|\,,$$

(iii)

$$4\pi\frac{u_1}{\Delta u_3} = \left\{ -2\frac{X_1^2X_2}{(X_1^2+\xi_3^2)\rho}\right\}\Bigg\|\,,$$

$$4\pi\frac{u_2}{\Delta u_3} = \left\{ 2\frac{X_1}{\rho}\right\}\Bigg\|\,,$$

$$4\pi\frac{u_3}{\Delta u_3} = \left\{ 2\frac{X_1X_2\xi_3}{(X_1^2+\xi_3^2)\rho} - 2\arctan\left(\frac{X_2\xi_3}{X_1\rho}\right)\right\}\Bigg\|\,.$$

(b) When Σ is parallel to the surface, the cases we are to consider are:

(iv) When only u_1 is discontinuous on Σ, Case (1-3),
(v) When only u_2 is discontinuous on Σ, Case (2-3),
(vi) When only u_3 is discontinuous on Σ, Case (3-3).

If we use the notation $\|$ as

$$f(\xi_1, \xi_2)\| = f(a, b) - f(a, B) - f(A, b) + f(A, B) ,$$

where (a, b), (a, B), (A, B) and (A, b) are apices of the rectangular Σ, then the expressions are as follows.

(iv)

$$4\pi\frac{u_1}{\Delta u_1} = \left\{ 2\frac{X_1X_2\xi_3}{(X_1^2+\xi_3^2)\rho} - 2\arctan\left(\frac{X_1X_2}{\xi_3\rho}\right)\right\}\Bigg\| ,$$

$$4\pi\frac{u_2}{\Delta u_1} = \left\{-2\frac{\xi_3}{\rho}\right\}\Bigg\| ,$$

$$4\pi\frac{u_3}{\Delta u_1} = \left\{-2\frac{X_2\xi_3^2}{(X_1^2+\xi_3^2)\rho}\right\}\Bigg\| ,$$

(v)

$$4\pi\frac{u_1}{\Delta u_2} = \left\{-2\frac{\xi_3}{\rho}\right\}\Bigg\| ,$$

$$4\pi\frac{u_2}{\Delta u_2} = \left\{ 2\frac{X_1X_2\xi_3}{(X_2^2+\xi_3^2)\rho} - 2\arctan\left(\frac{X_1X_2}{\xi_3\rho}\right)\right\}\Bigg\| ,$$

$$4\pi\frac{u_3}{\Delta u_2} = \left\{-2\frac{X_1\xi_3^2}{(X_2^2+\xi_3^2)\rho}\right\}\Bigg\| ,$$

(vi)

$$4\pi\frac{u_1}{\Delta u_3} = \left\{-2\frac{X_2\xi_3^2}{(X_1^2+\xi_3^2)\rho}\right\}\Bigg\| ,$$

$$4\pi\frac{u_2}{\Delta u_3} = \left\{-2\frac{X_1\xi_3^2}{(X_2^2+\xi_3^2)\rho}\right\}\Bigg\| ,$$

$$4\pi\frac{u_3}{\Delta u_3} = \left\{-2\frac{X_1X_2\xi_3}{(X_1^2+\xi_3^2)\rho} - 2\frac{X_1X_2\xi_3}{(X_2^2+\xi_3^2)\rho} - 2\arctan\left(\frac{X_1X_2}{\xi_3\rho}\right)\right\}\Bigg\| .$$

For cases (a), when Σ intersects the surface of the medium, the results of calculations for various points on the free surface are shown in Fig. 23~34, where the depths C of the dislocation surface Σ are

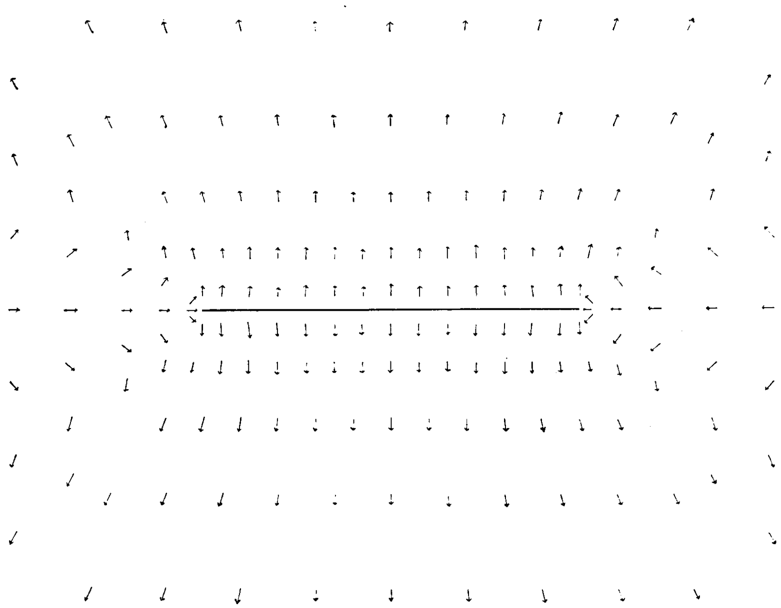

Fig. 23. Direction of horizontal displacements of the free surface for Case (1-1), $C=0.1$.

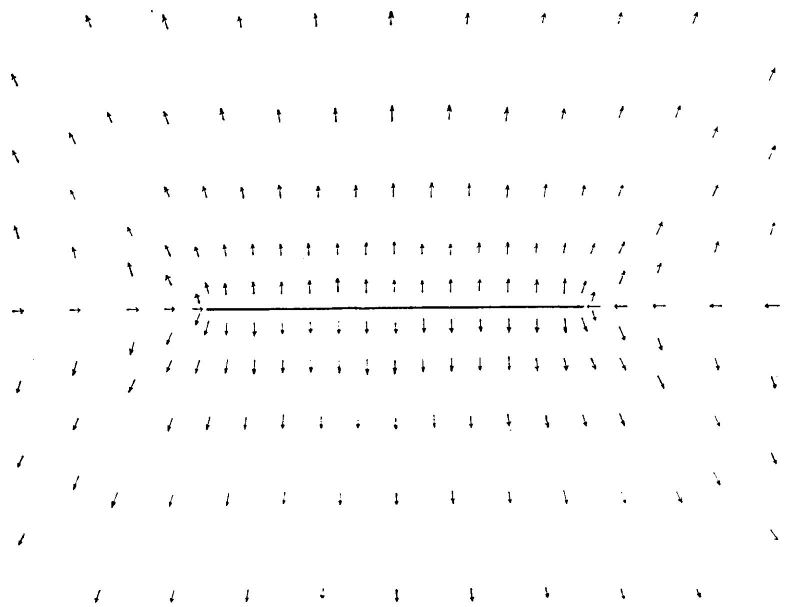

Fig. 24. Direction of horizontal displacements of the free surface for Case (1-1), $C=2.0$.

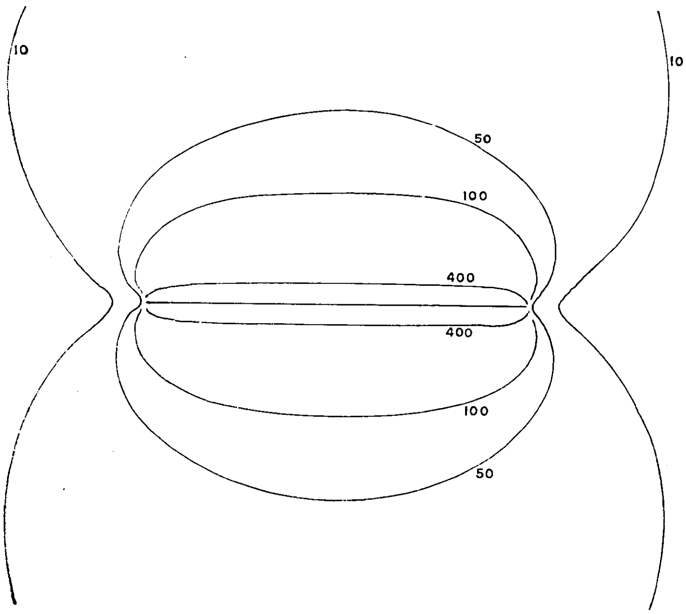

Fig. 25. Resultant horizontal displacements for Case (1-1), $C=0.1$. Contour values in units of $10^{-3}\Delta u_1$.

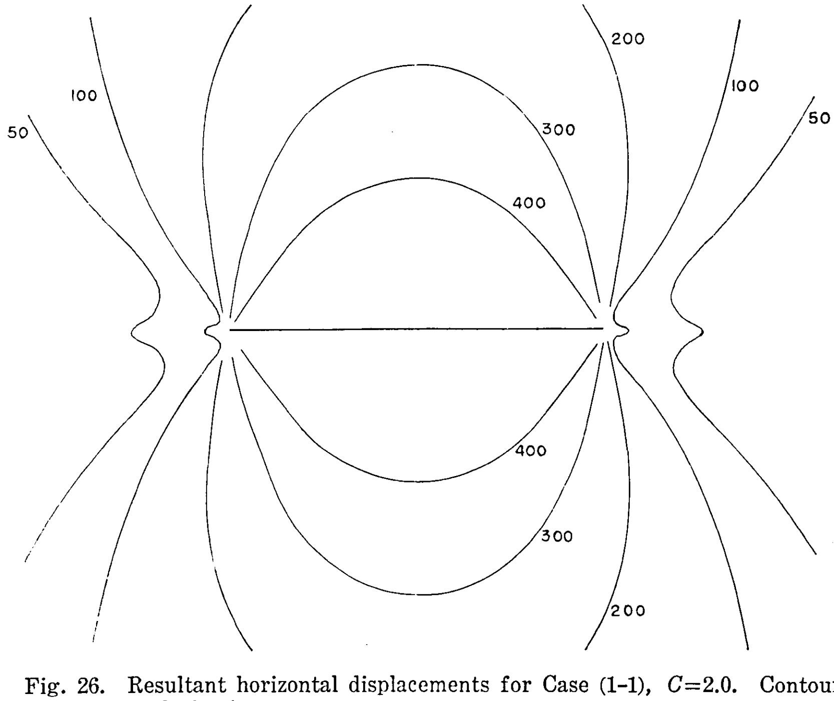

Fig. 26. Resultant horizontal displacements for Case (1-1), $C=2.0$. Contour values in units of $10^{-3}\Delta u_1$.

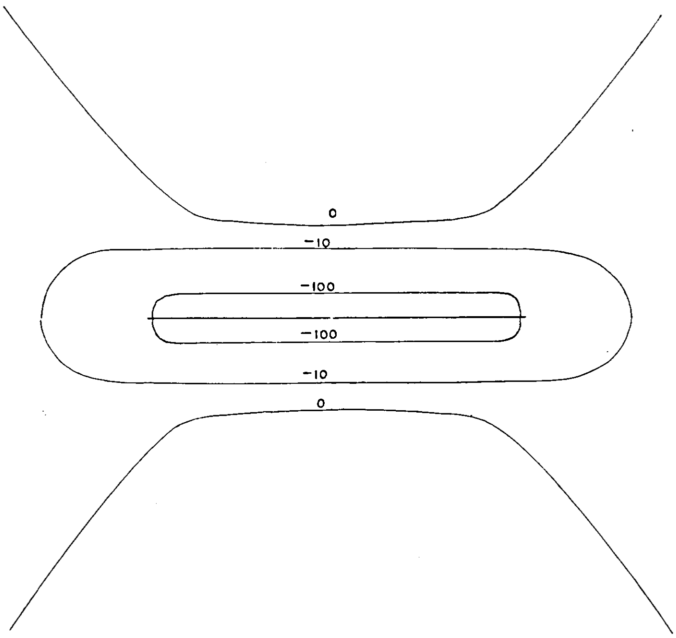

Fig. 27. Downward displacements u_3 for Case (1-1), C=0.1. Contour values in units of $10^{-3}\Delta u_1$.

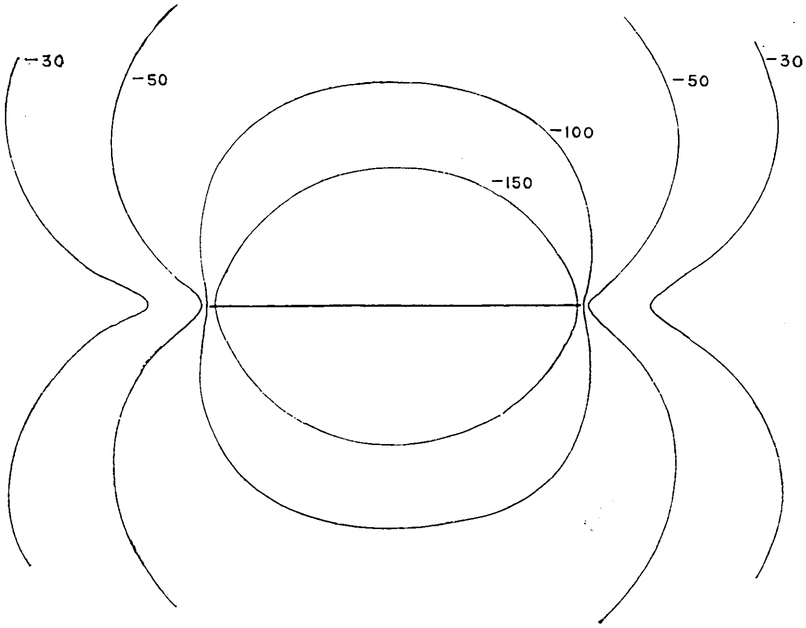

Fig. 28. Downward displacements u_3 for Case (1-1), C=2.0. Contour values in units of $10^{-3}\Delta u_1$.

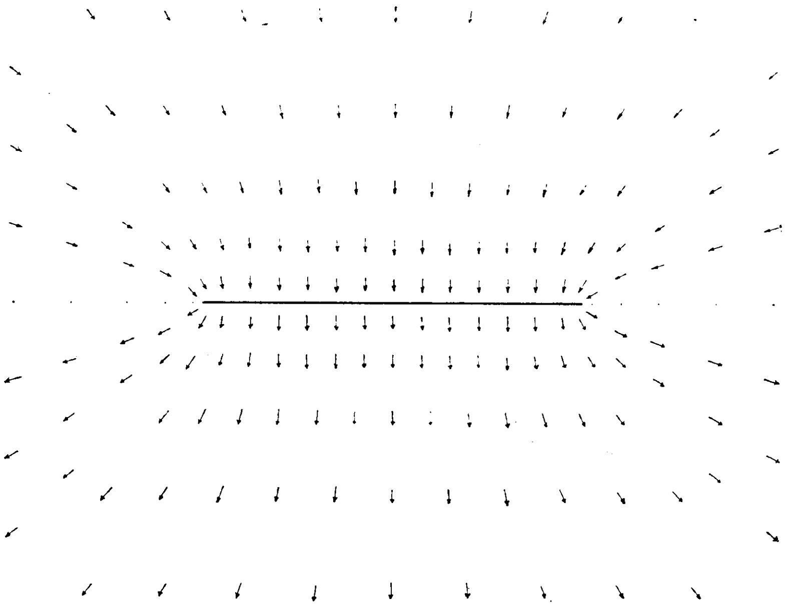

Fig. 29. Direction of horizontal displacements of the free surface for Case (3-1), $C=0.1$.

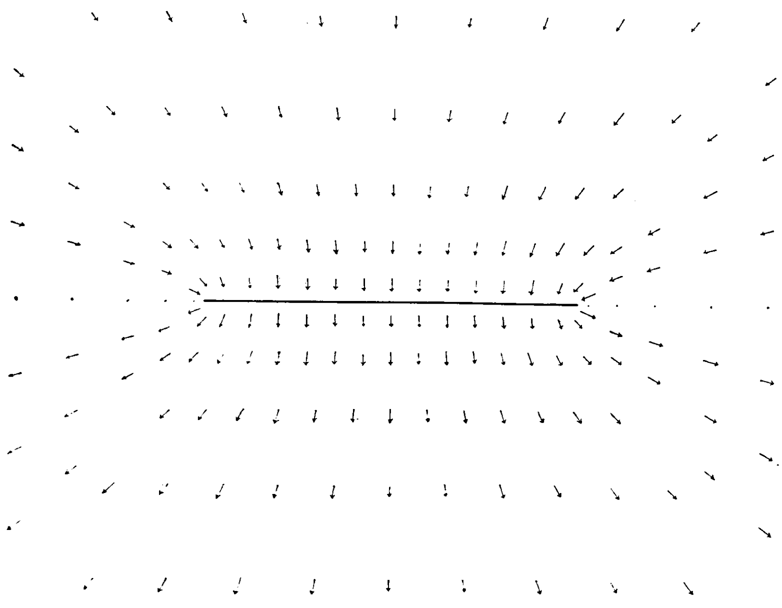

Fig. 30. Direction of horizontal displacements of the free surface for Case (3-1), $C=2.0$.

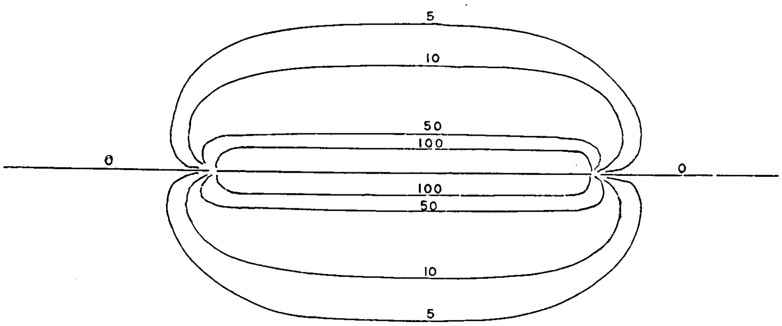

Fig. 31. Resultant horizontal displacements for Case (3-1), $C=0.1$. Contour values in units of $10^{-3}\Delta u_3$.

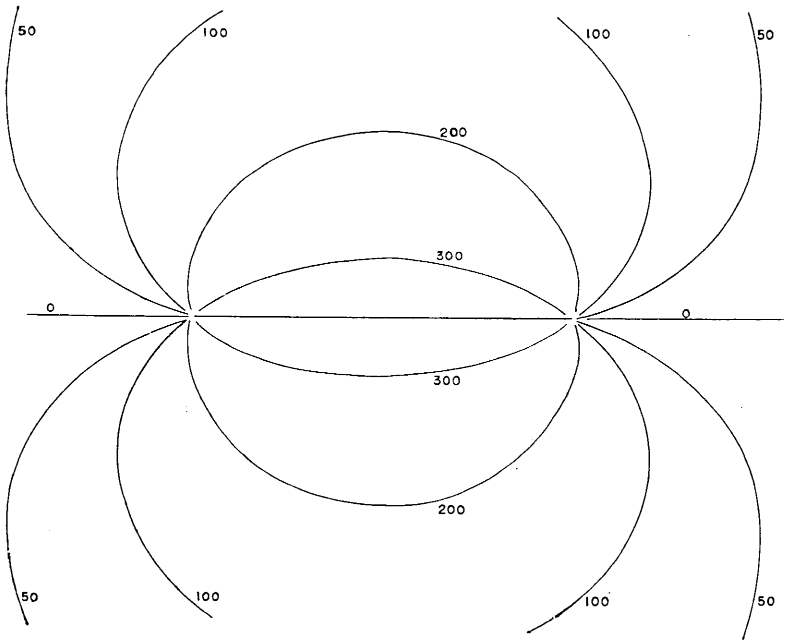

Fig. 32. Resultant horizontal displacements for Case (3-1), $C=2.0$. Contour values in units of $10^{-3}\Delta u_3$.

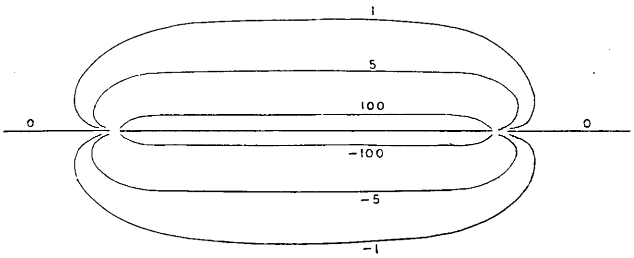

Fig. 33. Downward displacements u_3 for Case (3-1), $C=0.1$. Contour values in units of $10^{-3}\Delta u_3$.

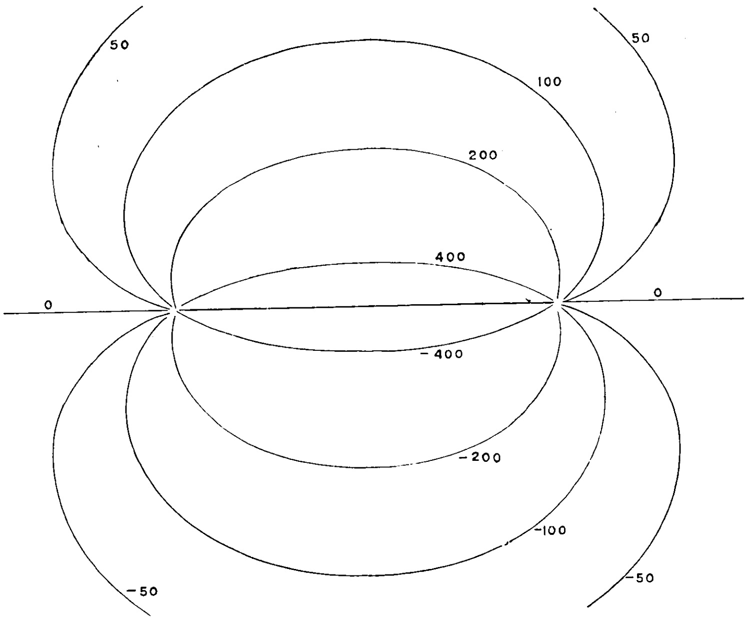

Fig. 34. Downward displacements u_3 for Case (3-1), $C=2.0$. Contour values in units of $10^{-3}\Delta u_3$.

measured in units of the semi-length of Σ. For Case (2-1), the strike-slip model, detailed figures are shown in Chinnery (1961).

For cases (b), when Σ is a square, the results of calculations are shown in Fig. 35~38, where distances and depths are measured in units of the semi-length of the side of square.

For cases (a), examples of the fall off of displacement components along x_1-axis, perpendicular to the fault surface Σ, passing through the

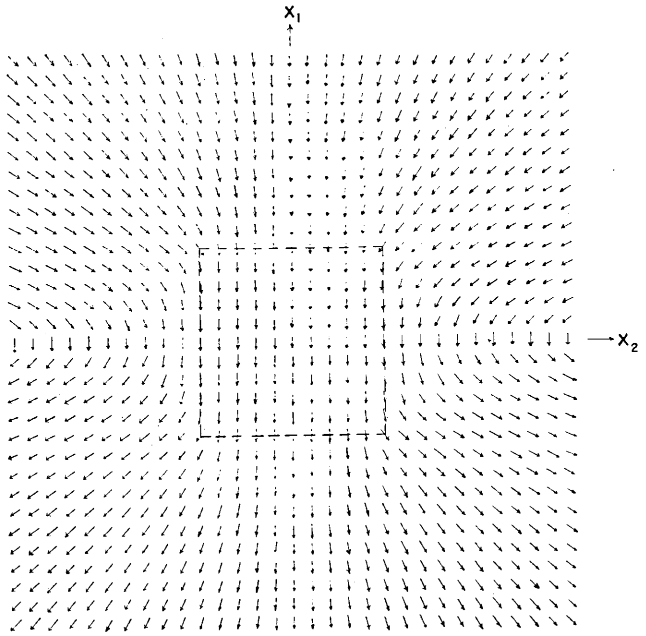

Fig. 35. Direction of horizontal displacements of the free surface for Case (1-3), C=0.1.

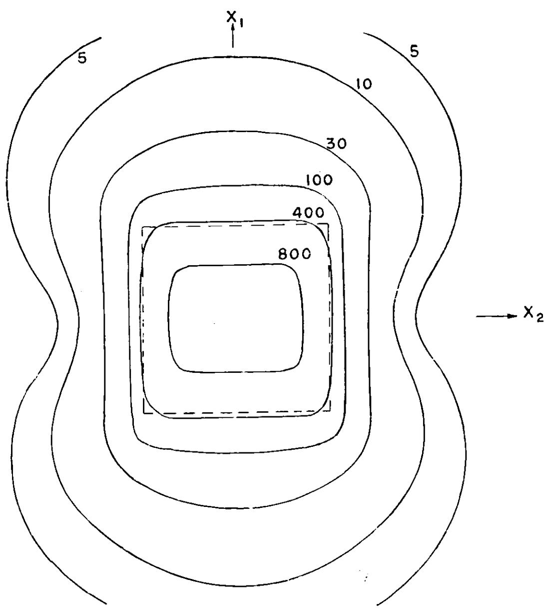

Fig. 36. Resultant horizontal displacements for Case (1-3), C=0.1. Contour values in units of $10^{-3}\Delta u_1$.

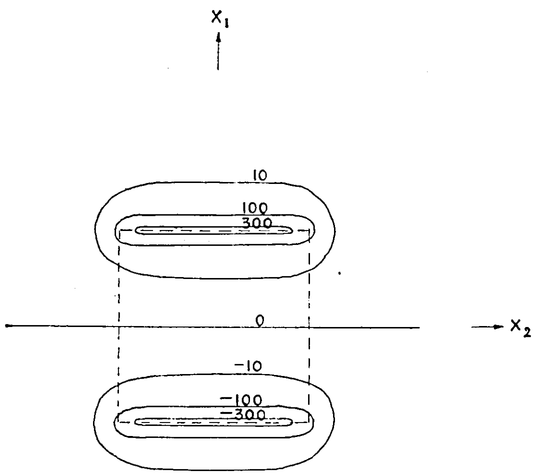

Fig. 37. Downward displacements u_3 for Case (1-3), C=0.1. Contour values in units of $10^{-3}\Delta u_1$.

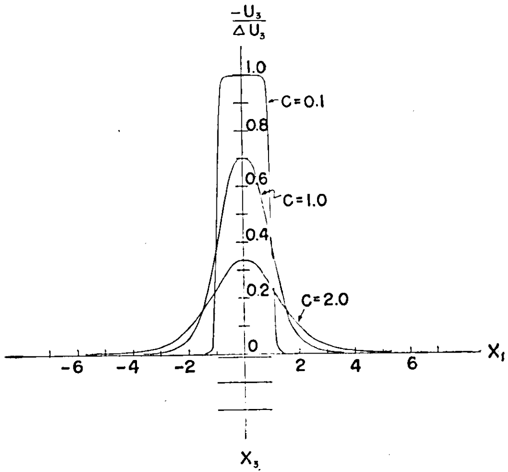

Fig. 38. Downward displacements along x_1-axis for Case (3-3), C=0.1, 1.0 and 2.0.

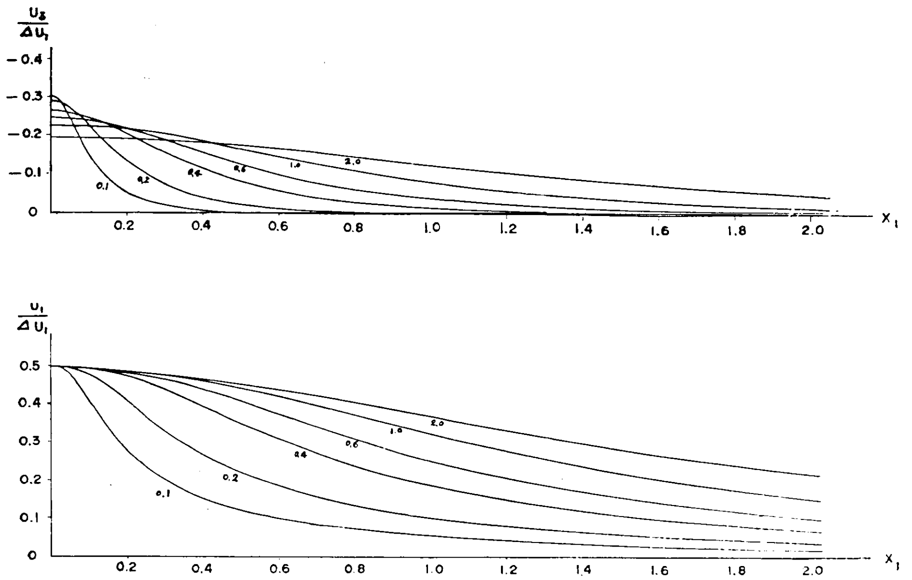

Fig. 39. Downward displacements (above) and horizontal displacement (below) along x_1-axis for rectangular Σ of length 2 and of various depths, for Case (1-1).

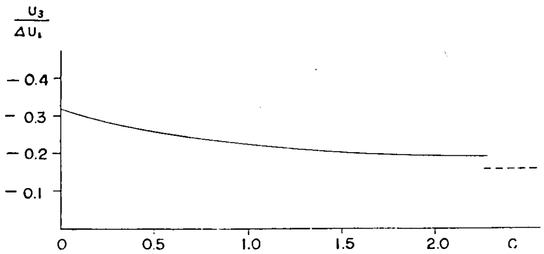

Fig. 40. Downward displacements at the origin for various depths C of Σ for Case (1-1).

origin of the free surface are shown in Fig. 39~41 (for Case (2-1) see Chinnery (1961)).

For a rectangular dislocation surface S which is inclined form the vertical plane, if we choose the normal to S as shown in Fig. 42 and we have $\nu_1=\cos\theta$, $\nu_2=0$, $\nu_3=-\sin\theta$, and if only Δu_2 does not vanish on S, we have

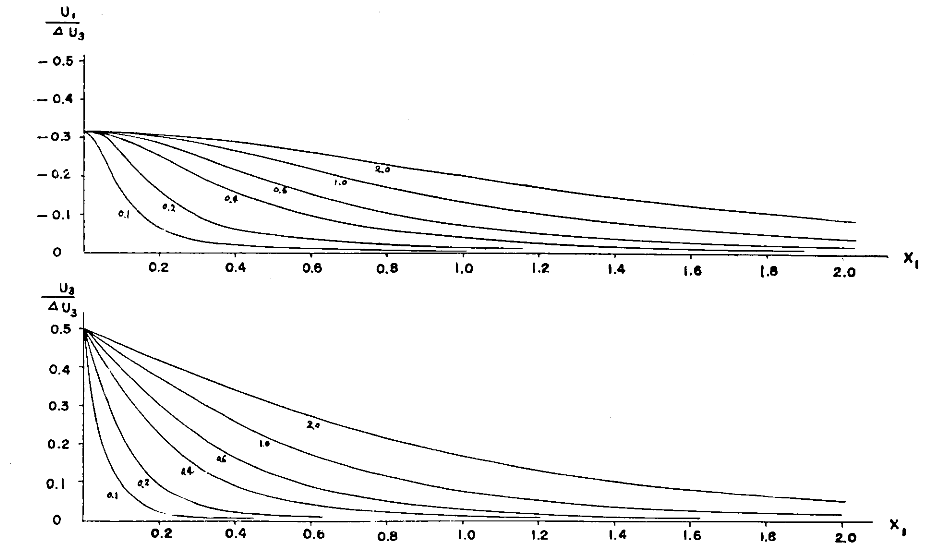

Fig. 41. Horizontal displacements (above) and downward displacements (below) along x_1-axis for rectangular Σ of length 2 and of various depths, for Case (3-1).

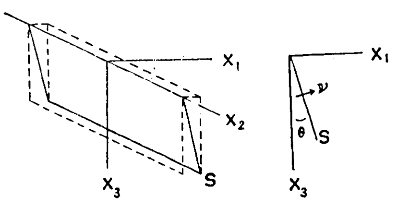

Fig. 42. Rectangular dislocation surface S meeting the free surface obliquely.

$$\frac{u_m}{\Delta u_2}=\iint W^m_{2l}\nu_l dS$$

$$=\iint W^m_{21}d\xi_2 d\xi_3-\iint W^m_{23}d\xi_1 d\xi_2$$

$$=\iint (W^m_{23}-\tan\theta\, W^m_{21})d\xi_2 d\xi_3\ .$$

$$(d\xi_1=\tan\theta\ d\xi_3)$$

If we want to evaluate these formulae by means of numerical integration, we are led to the following calculations for a point $(x_1, x_2, 0)$:

$$\frac{u_1}{\Delta u_2}=\frac{1}{4\pi}\sum\left[\frac{X_2\rho}{r^4}\left\{A\left(\frac{\xi_3}{\rho}\right)+\frac{X_1^2}{r^2}F\left(\frac{\xi_3}{\rho}\right)\right\}+\tan\theta\cdot 6\ \frac{X_1X_2\xi_3}{\rho^5}\right]\Delta S_1\ ,$$

$$\frac{u_2}{\Delta u_2}=\frac{1}{4\pi}\sum\left[\frac{X_1\rho}{r^4}\left\{A\left(\frac{\xi_3}{\rho}\right)+\frac{X_2^2}{r^2}F\left(\frac{\xi_3}{\rho}\right)\right\}+\tan\theta\cdot 6\ \frac{X_2^2\xi_3}{\rho^5}\right]\Delta S_1\ ,$$

$$\frac{u_3}{\Delta u_2}=\frac{1}{4\pi}\sum\left[\frac{X_1X_2}{r^4}E\left(\frac{\xi_3}{\rho}\right)-\tan\theta\cdot 6\ \frac{X_2\xi_3^2}{\rho^5}\right]\Delta S_1\ ,$$

where ΔS_1 is a surface element of the projection of S on the $\xi_2\xi_3$-plane and the summation is taken over the dislocation surface S.

Appendixes

Appendix 1. Volterra's formula. The m-component of displacement vector at an arbitrary point $Q(x_1, x_2, x_3)$ due to a dislocation in an infinite elastic medium, $u_m(Q)$, is determined by the formula

$$u_m(Q)=\iint_{\Sigma} \Delta u_k T^m_{kl}(P, Q)\nu_l(P)d\Sigma \ , \tag{a-1}$$

where P is a point on the dislocation surface Σ over which the integral is taken and $T^m_{kl}(P, Q)$ is the (kl)-component of the stress tensor at P due to a unit body force acting at Q in the m-direction. Here we introduce equation (a-1) following Steketee (1958 b) for static cases, though it can also be obtained from more general dynamic relations (Maruyama 1963).

The reciprocal theorem of Betti in static cases states:

$$\iiint u^{(1)}_k F^{(2)}_k \rho dV+ \iint u^{(1)}_k \tau^{(2)}_{kl}\nu_l dS=\iiint u^{(2)}_k F^{(1)}_k \rho dV+\iint u^{(2)}_k \tau^{(1)}_{kl}\nu_l dS \ , \tag{a-2}$$

where the superscripts inside parentheses refer to the two possible but different sets of displacements, stresses and body forces for a particular elastic body which occupies a region $D+S$ with S as its boundary. In equation (a-2) ρ denotes density of the elastic body, $F^{(1)}_k$ and $F^{(2)}_k$ the body forces. We imagine a dislocation surface Σ in the body and apply the equation (a-2) to the body with the boundaries S, Σ^+ and Σ^-. If we apply proper tractions over S, Σ^+ and Σ^-, the body will be deformed as if it were a portion of an infinite elastic medium. When a force F^m_k in the positive m-direction is applied at Q, and when such surface tractions as will be generated by the force in an infinite medium are applied on S, Σ^+ and Σ^-, the displacement field and the stress field in D will be the same as in an infinite medium and they may be expressed in the well-known formula (Love p. 185). Then as the first set in equation (a-2) we take a unit body force acting at Q in the m-direction, the displacement field $U^m_k(P, Q)$ and the stress field $T^m_{kl}(P, Q)$ which will be generated at P in an infinite elastic medium by the force. As the second set we may take an arbitrary possible displacement field $u_k(P)$ and stress field $\tau_{kl}(P)$ when there are no body forces. The reciprocal theorem may then be written in the form

$$\iiint_D F^m_k u_k \rho dV+ \iint_{S+\Sigma^++\Sigma^-} u_k T^m_{kl}\nu_l dS= \iint_{S+\Sigma^++\Sigma^-} U^m_k \tau_{kl}\nu_l dS \ .$$

Since $U_k^m(P, Q)$ and $T_{kl}^m(P, Q)$ are continuous across Σ, using equation (1.5) we have

$$\iiint_D F_k^m u_k \rho dV = \iint_\Sigma (u_k^+ - u_k^-) T_{kl}^m \nu_l d\Sigma - \iint_\Sigma U_k^m (\tau_{kl}^+ - \tau_{kl}^-) \nu_l d\Sigma$$
$$+ \iint_S u_k T_{kl}^m \nu_l dS - \iint_S U_k^m \tau_{kl} \nu_l dS \,. \tag{a-3}$$

We take the magnitude of the body force F_k^m operative at Q in the m-direction to be unit, as

$$\iiint_{V_0} F_k^m \rho dV = \begin{cases} 1: & \text{if } k=m \text{ when } V_0 \text{ includes } Q \\ 0: & \text{otherwise}\,, \end{cases}$$

hence the left-hand side of equation (a-3) gives us $u_m(Q)$. In the case of Somigliana dislocation, by definition the second term on the right-hand side vanishes. If the outer surface S is left free from forces the fourth term vanishes. If we assume that S recedes to infinity requiring at the same time that Q is not at infinity, the third term vanishes in so far as u_k's vanish at infinity. Thus from equation (1.2) we get equation (a-1) for an infinite elastic body.

Appendix 2. Line integral representation of the displacement field due to a Volterra dislocation. A general Volterra dislocation is specified by the discontinuity $\Delta \boldsymbol{u}$ across Σ which should be of a rigid body displacement type as

$$\Delta \boldsymbol{u} = \boldsymbol{b} + \boldsymbol{\omega} \times \boldsymbol{\xi} \,. \tag{a-4}$$

We deal with two cases separately: One is the case when $\Delta \boldsymbol{u} = \boldsymbol{b}$, and the other when $\Delta \boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{\xi}$.

(i) When $\Delta \boldsymbol{u} = \boldsymbol{b}$. If a unit body force is acting in the m-direction at $Q(\boldsymbol{x})$, the displacement $\boldsymbol{u}^m$ at $P(\boldsymbol{\xi})$ is expressed as

$$8\pi\rho \boldsymbol{u}^m = \frac{1}{a^2}(\boldsymbol{e}_m \cdot \text{grad})\text{grad}\, r + \frac{1}{b^2}\text{rot}(\boldsymbol{e}_m \times \text{grad}\, r) \,, \tag{a-5}$$

where a, b, are the P- and S-wave velocities, grad and rot (also div and ∇^2 later) are taken with respect to the coordinates of P,

$$\text{grad}\, f = \text{grad}_P f \,, \quad \text{rot}\, \boldsymbol{v} = \text{rot}_P \boldsymbol{v} \,,$$

and

$$r = \sqrt{(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2} \,.$$

The stress component T^m_{kl} at $P(\xi)$ derived from the stress field $\boldsymbol{u}^m$ is computed from the relation

$$T^m_{kl}=\lambda\delta_{kl}u^m_{n,n}+\mu(u^m_{k,l}+u^m_{l,k}) \ , \tag{a-6}$$

where partial differentiations are perfomed with respect to the coordinates of $Q(\xi)$ and $\partial u/\partial\xi_k$ is abbreviated by $u_{,k}$.

The product of the constant vector b_k and the stress tensor T^m_{kl} is a vector $\boldsymbol{A}^m$ which may be written as

$$\boldsymbol{A}^m=\lambda\boldsymbol{b}\ \mathrm{div}\ \boldsymbol{u}+\mu\ \mathrm{grad}(\boldsymbol{b}\cdot\boldsymbol{u})+\mu(\boldsymbol{b}\cdot\mathrm{grad})\boldsymbol{u} \ . \tag{a-7}$$

Substituting (a-5) in equation (a-7) and using the relations,

$$\boldsymbol{b}\ \mathrm{div}(\mathrm{grad}\ r)=\mathrm{rot}(\boldsymbol{b}\times\mathrm{grad}\ r)+(\boldsymbol{b}\cdot\mathrm{grad})\mathrm{grad}\ r \ ,$$

$$\mathrm{rot}(\boldsymbol{e}_m\times\mathrm{grad}\ r)=\boldsymbol{e}_m\ \mathrm{div}\ \mathrm{grad}\ r-(\boldsymbol{e}_m\cdot\mathrm{grad})\mathrm{grad}\ r \ ,$$

$$\mathrm{grad}\ \nabla^2 r\times(\boldsymbol{e}_m\times\boldsymbol{b})=(\boldsymbol{b}\cdot\mathrm{grad})\mathrm{rot}(\boldsymbol{e}_m\times\mathrm{grad}\ r)-(\boldsymbol{e}_m\cdot\mathrm{grad})\mathrm{rot}(\boldsymbol{b}\times\mathrm{grad}\ r) \ ,$$

we obtain

$$4\pi\boldsymbol{A}^m=(\boldsymbol{e}_m\cdot\boldsymbol{b})\mathrm{grad}\Big(\frac{1}{r}\Big)+\ \mathrm{grad}\Big(\frac{1}{r}\Big)\times(\boldsymbol{e}_m\times\boldsymbol{b})+\alpha(\boldsymbol{e}_m\cdot\mathrm{grad})\mathrm{rot}(\boldsymbol{b}\times\mathrm{grad}\ r) \ .$$

By relations

$$\Big[\Big(\mathrm{grad}\frac{1}{r}\Big)\times(\boldsymbol{e}_m\times\boldsymbol{b})\Big]\cdot\boldsymbol{\nu}=\boldsymbol{e}_m\cdot\Big[\boldsymbol{b}\times\Big(\boldsymbol{\nu}\times\Big(\mathrm{grad}\frac{1}{r}\Big)\Big)\Big] \ ,$$

$$(\boldsymbol{e}_m\cdot\mathrm{grad}_P)\mathrm{rot}(\boldsymbol{b}\times\mathrm{grad}\ r)=-(\boldsymbol{e}_m\cdot\mathrm{grad}_Q)\mathrm{rot}(\boldsymbol{b}\times\mathrm{grad}\ r) \ ,$$

where suffix Q means the derivative taken with respect to the coordinates of Q, we have

$$\begin{aligned}\boldsymbol{A}^m\cdot\boldsymbol{\nu}=&\frac{1}{4\pi}(\boldsymbol{e}_m\cdot\boldsymbol{b})\ \mathrm{grad}_P\Big(\frac{1}{r}\Big)\cdot\boldsymbol{\nu}\\&+\frac{1}{4\pi}\boldsymbol{e}_m\cdot\Big[\boldsymbol{b}\times\Big(\boldsymbol{\nu}\times\mathrm{grad}\frac{1}{r}\Big)\Big]\\&-\frac{1}{4\pi}\alpha(\boldsymbol{e}_m\cdot\mathrm{grad}_Q)[\mathrm{rot}(\boldsymbol{b}\times\mathrm{grad}\ r)\cdot\boldsymbol{\nu}] \ .\end{aligned}$$

By means of the formulae

$$\iint\mathrm{grad}_P\Big(\frac{1}{r}\Big)\cdot\boldsymbol{\nu}d\Sigma=\iint\frac{\partial}{\partial\nu}\Big(\frac{1}{r}\Big)d\Sigma=\Omega \ ,$$

$$\iint \nu \times \text{grad}\, \varphi\, d\Sigma = \int \varphi\, d\boldsymbol{\xi}\,,$$

$$\iint \text{rot}\, \boldsymbol{A} \cdot \nu\, d\Sigma = \int \boldsymbol{A} \cdot d\boldsymbol{\xi}\,,$$

where $d\boldsymbol{\xi}$ is an element of the line σ and the line integrals are taken once round the dislocation line in the positive sense and Ω is the solid angle subtended by the loop σ at the point Q, we have

$$\begin{aligned} u_m(Q) = & \frac{1}{4\pi}\boldsymbol{e}_m \cdot \boldsymbol{b}\Omega \\ & +\frac{1}{4\pi}\boldsymbol{e}_m \cdot \left[\boldsymbol{b} \times \int \frac{1}{r} d\boldsymbol{\xi}\right] \\ & -\frac{\alpha}{4\pi}(\boldsymbol{e}_m \cdot \text{gard}_Q)\int (\boldsymbol{b} \times \text{grad}\, r) \cdot d\boldsymbol{\xi}\,. \end{aligned}$$

This may be written in vector form as

$$\boldsymbol{u}(Q) = \frac{1}{4\pi}\left\{\boldsymbol{b}\Omega + \boldsymbol{b} \times \int \frac{1}{r} d\boldsymbol{\xi} + \alpha\, \text{grad}_Q \int (\boldsymbol{b} \times \boldsymbol{r}) \cdot \frac{d\boldsymbol{\xi}}{r}\right\}. \tag{a-8}$$

(ii) when $\Delta \boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{\xi}$ (or $\Delta u_k = \Omega_{kh}\xi_k$, $\Omega_{kh} = -\Omega_{hk}$) the product of the vector $(\boldsymbol{\omega} \times \boldsymbol{\xi})$ and the stress tensor T^m_{kl} in equation (a-6) may be written as

$$\Delta u_k T^m_{kl} = \lambda \Omega_{lh}\xi_h u^m_{n,n} + \mu(\Omega_{kh}\xi_h u^m_k)_{,l} + \mu\Omega_{lk}u^m_k + \mu\Omega_{kh}\xi_h u^m_{k,l}\,,$$

or in vector form as

$$\begin{aligned} \boldsymbol{A}^m = & \lambda(\boldsymbol{\omega} \times \boldsymbol{\xi})\, \text{div}\, \boldsymbol{u}^m + \mu\, \text{grad}\, [(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \boldsymbol{u}^m] \\ & + \mu[(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \text{grad}]\boldsymbol{u}^m + \mu(\boldsymbol{\omega} \times \boldsymbol{u}^m)\,. \end{aligned} \tag{a-9}$$

By substituting (a-4) to (a-9) we have

$$\begin{aligned} 8\pi\rho \boldsymbol{A}^m = & \frac{\lambda}{a^2}(\boldsymbol{\omega} \times \boldsymbol{\xi})(\boldsymbol{e}_m \cdot \text{grad})\nabla^2 r \\ & + \frac{\mu}{a^2}\, \text{grad}\, [(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot (\boldsymbol{e}_m \cdot \text{grad})\, \text{grad}\, r] \\ & + \frac{\mu}{b^2}\, \text{grad}\, [(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \text{rot}\, (\boldsymbol{e}_m \times \text{grad}\, r)] \\ & + \frac{\mu}{a^2}[(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \text{grad}](\boldsymbol{e}_m \cdot \text{grad})\, \text{grad}\, r \end{aligned}$$

$$+\frac{\mu}{b^2}[(\omega\times\xi)\cdot\mathrm{grad}]\,\mathrm{rot}\,(e_m\times\mathrm{grad}\,r)$$

$$+\frac{\mu}{a^2}(e_m\cdot\mathrm{grad})(\omega\times\mathrm{grad}\,r)$$

$$+\frac{\mu}{b^2}\omega\times\mathrm{rot}\,(e_m\times\mathrm{grad}\,r)\,. \tag{a-10}$$

We can obtain the following relations

$$\begin{aligned}&\mathrm{grad}\,[(\omega\times\xi)\cdot(e_m\cdot\mathrm{grad})\,\mathrm{grad}\,r]+(e_m\cdot\mathrm{grad})(\omega\times\mathrm{grad}\,r)\\&\quad=[(\omega\times\xi)\cdot\mathrm{grad}](e_m\cdot\mathrm{grad})\,\mathrm{grad}\,r\,,\end{aligned} \tag{a-11}$$

and

$$\begin{aligned}&\mathrm{grad}\,[(\omega\times\xi)\cdot\mathrm{rot}\,(e_m\times\mathrm{grad}\,r)]\\&\quad=-[(\omega\times x)\cdot\mathrm{grad}\,r](e_m\cdot\mathrm{grad})\,\mathrm{grad}\,r\\&\qquad+[(\omega\times x)\cdot e_m]\,\mathrm{grad}\,\nabla^2 r+[(e_m\times\omega)\cdot\mathrm{grad}]\,\mathrm{grad}\,r\,,\end{aligned} \tag{a-12}$$

where we considered the definition $\xi=x-r$. Further we can easily obtain the relation

$$\begin{aligned}&\omega\times\mathrm{rot}\,(e_m\times\mathrm{grad}\,r)+[(e_m\times\omega)\cdot\mathrm{grad}]\,\mathrm{grad}\,r\\&\quad=\mathrm{rot}\,[(\omega\times e_m)\times\mathrm{grad}\,r]-(e_m\cdot\mathrm{grad})(\omega\times\mathrm{grad}\,r)\,.\end{aligned} \tag{a-13}$$

By substituting (a-11), (a-12) and (a-13) into (a-10), we have

$$\begin{aligned}8\pi\rho A^m=&\frac{\lambda}{a^2}(\omega\times\xi)(e_m\cdot\mathrm{grad})\,\nabla^2 r\\&+\frac{\mu}{a^2}[(\omega\times\xi)\cdot\mathrm{grad}](e_m\cdot\mathrm{grad})\,\mathrm{grad}\,r\\&+\frac{\mu}{b^2}\Big\{-[(\omega\times x)\cdot\mathrm{grad}\,r](e_m\cdot\mathrm{grad})\,\mathrm{grad}\,r\\&+[(\omega\times x)\cdot e_m]\,\mathrm{grad}\,\nabla^2 r\Big\}\\&+\frac{\mu}{a^2}[(\omega\times\xi)\cdot\mathrm{grad}](e_m\cdot\mathrm{grad})\,\mathrm{grad}\,r\\&+\frac{\mu}{b^2}[(\omega\times\xi)\cdot\mathrm{grad}]\,\mathrm{rot}\,(e_m\times\mathrm{grad}\,r)\\&+\frac{\mu}{b^2}\,\mathrm{rot}\,[(\omega\times e_m)\times\mathrm{grad}\,r]\\&-\frac{\mu}{b^2}(e_m\cdot\mathrm{grad})(\omega\times\mathrm{grad}\,r)\,.\end{aligned} \tag{a-14}$$

By the relation

$$\mathrm{rot}\ (\boldsymbol{e}_m \times \mathrm{grad}\ r) = \boldsymbol{e}_m \nabla^2 r - (\boldsymbol{e}_m \cdot \mathrm{grad})\ \mathrm{grad}\ r\ ,$$

we find

$$\begin{aligned}
&[(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \mathrm{grad}]\ \mathrm{rot}\ (\boldsymbol{e}_m \times \mathrm{grad}\ r) \\
&\quad = [(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \mathrm{grad}]\{\boldsymbol{e}_m \nabla^2 r - (\boldsymbol{e}_m \cdot \mathrm{grad})\ \mathrm{grad}\ r\} \\
&\quad = \boldsymbol{e}_m [(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \mathrm{grad}\ \nabla^2 r] - [(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \mathrm{grad}](\boldsymbol{e}_m \cdot \mathrm{grad})\ \mathrm{grad}\ r\ .
\end{aligned}$$

If we substitute this into (a-14) we get

$$\begin{aligned}
8\pi\rho \boldsymbol{A}^m = &-\frac{\lambda}{a^2}[(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \mathrm{grad}](\boldsymbol{e}_m \cdot \mathrm{grad})\ \mathrm{grad}\ r \\
&-\frac{\mu}{b^2}[(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \mathrm{grad}](\boldsymbol{e}_m \cdot \mathrm{grad})\ \mathrm{grad}\ r \\
&+\frac{\lambda}{a^2}(\boldsymbol{\omega} \times \boldsymbol{\xi})(\boldsymbol{e}_m \cdot \mathrm{grad})\ \nabla^2 r \\
&+\frac{\mu}{b^2}\boldsymbol{e}_m[(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \mathrm{grad}\ \nabla^2 r] \\
&+\frac{\mu}{b^2}[(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \boldsymbol{e}_m]\ \mathrm{grad}\ \nabla^2 r \\
&+\frac{\mu}{b^2}\ \mathrm{rot}\ [(\boldsymbol{\omega} \times \boldsymbol{e}_m) \times \mathrm{grad}\ r] \\
&-\frac{\mu}{b^2}(\boldsymbol{e}_m \cdot \mathrm{grad})(\boldsymbol{\omega} \times \mathrm{grad}\ r)\ ,
\end{aligned} \qquad \text{(a-15)}$$

where we considered the relation

$$\frac{\mu}{b^2} = \frac{\lambda + 2\mu}{a^2}\ .$$

The first term of the right-hand side of (a-15) may be written

$$\begin{aligned}
&[(\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot \mathrm{grad}[(\boldsymbol{e}_m \cdot \mathrm{grad})\ \mathrm{grad}\ r \\
&\quad = [(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \mathrm{grad}](\boldsymbol{e}_m \cdot \mathrm{grad})\ \mathrm{grad}\ r \\
&\qquad - (\boldsymbol{\omega} \times \boldsymbol{r})(\boldsymbol{e}_m \cdot \mathrm{grad}) \nabla^2 r \\
&\qquad - (\boldsymbol{\omega} \cdot \mathrm{grad})(\boldsymbol{e}_m \times \mathrm{grad}\ r)\ .
\end{aligned} \qquad \text{(a-16)}$$

Substituting (a-11) in (a-15) we obtain

$$\begin{aligned} 8\pi\rho \boldsymbol{A}^m = &-\frac{2(\lambda+\mu)}{a^2}[(\omega\times\boldsymbol{x})\cdot\text{grad}](\boldsymbol{e}_m\cdot\text{grad})\,\text{grad}\, r \\ &+\frac{\lambda}{a^2}(\omega\times\boldsymbol{x})(\boldsymbol{e}_m\cdot\text{grad})\nabla^2 r \\ &+\frac{\mu}{b^2}\boldsymbol{e}_m[(\omega\times\boldsymbol{x})\cdot\text{grad}\,\nabla^2 r] \\ &+\frac{\lambda}{a^2}(\omega\cdot\text{grad})(\boldsymbol{e}_m\times\text{grad}\, r) \\ &-\frac{\mu}{b^2}(\boldsymbol{e}_m\cdot\text{grad})(\omega\times\text{grad}\, r) \\ &+\frac{\mu}{b^2}[(\omega\times\boldsymbol{x})\cdot\boldsymbol{e}_m]\,\text{grad}\,\nabla^2 r \\ &+\frac{\mu}{b^2}\,\text{rot}\,[(\omega\times\boldsymbol{e}_m)\times\text{grad}\, r]\,. \end{aligned} \tag{a-17}$$

Using the relations

$$\begin{aligned} &(\boldsymbol{e}_m\cdot\text{grad})\,\text{rot}\,[(\omega\times\boldsymbol{x})\times\text{grad}\, r] \\ &\quad=(\omega\times\boldsymbol{x})(\boldsymbol{e}_m\cdot\text{grad})\,\nabla^2 r-[(\omega\times\boldsymbol{x})\cdot\text{grad}](\boldsymbol{e}_m\cdot\text{grad})\,\text{grad}\, r\,, \\ &\text{grad}\,\nabla^2 r\times[\boldsymbol{e}_m\times(\omega\times\boldsymbol{x})] \\ &\quad=\boldsymbol{e}_m[(\omega\times\boldsymbol{x})\cdot\text{grad}\,\nabla^2 r]-(\omega\times\boldsymbol{x})(\boldsymbol{e}_m\cdot\text{grad}\,\nabla^2 r)\,, \\ &\text{rot}\,[(\omega\times\boldsymbol{e}_m)\times\text{grad}\, r] \\ &\quad=-(\omega\cdot\text{grad})(\boldsymbol{e}_m\times\text{grad}\, r)+(\boldsymbol{e}_m\cdot\text{grad})(\omega\times\text{grad}\, r)\,, \end{aligned}$$

and

$$\text{rot}\,[\boldsymbol{e}_m\times(\omega\times\text{grad}\, r)]=-(\boldsymbol{e}_m\cdot\text{grad})(\omega\times\text{grad}\, r)\,,$$

equation (a-17) becomes

$$\begin{aligned} 8\pi\rho \boldsymbol{A}^m = &\frac{2(\lambda+\mu)}{a^2}(\boldsymbol{e}_m\cdot\text{grad})\,\text{rot}\,[(\omega\times\boldsymbol{x})\times\text{grad}\, r] \\ &+\frac{\mu}{b^2}\,\text{grad}\,\nabla^2 r\times[\boldsymbol{e}_m\times(\omega\times\boldsymbol{x})] \\ &+\frac{2\mu}{a^2}\,\text{rot}\,[(\omega\times\boldsymbol{e}_m)\times\text{grad}\, r] \\ &+\frac{2\mu}{a^2}\,\text{rot}\,[\boldsymbol{e}_m\times(\omega\times\text{grad}\, r)] \\ &+\frac{\mu}{b^2}[(\omega\times\boldsymbol{x})\cdot\boldsymbol{e}_m]\,\text{grad}\,\nabla^2 r\,. \end{aligned} \tag{a-18}$$

The first term in (a-18) becomes

$$(e_m\cdot\text{grad})\,\text{rot}\,[(\omega\times x)\times\text{grad}\,r]$$
$$=(e_m\cdot\text{grad})\,\text{rot}\,[(\omega\times\xi)\times\text{grad}\,r]+2\,\text{rot}\,[e_m\times(\omega\times\text{grad}\,r)]\,. \quad \text{(a-19)}$$

If we denote the differentiation operator with respect to the coordinates of Q by adding suffix Q as

$$\frac{\partial}{\partial x_m}f=(e_m\cdot\text{grad})_Q f\,,$$

we can easily arrive at the relation

$$(e_m\cdot\text{grad})_P[(\omega\times\xi)\times\text{grad}_P\,r]$$
$$=-(e_m\cdot\text{grad})_Q[(\omega\times\xi)\times\text{grad}_P\,r]+(\omega\times e_m)\times\text{grad}_P\,r\,. \quad \text{(a-20)}$$

By substituting (a-19) and (a-20) into (a-18) we have

$$8\pi\rho A^m=-\frac{2(\lambda+\mu)}{a^2}(e_m\cdot\text{grad})_Q\,\text{rot}\,[(\omega\times\xi)\times\text{grad}\,r]$$
$$+\frac{2(2\lambda+3\mu)}{a^2}\,\text{rot}\,[e_m\times(\omega\times\text{grad}\,r)]$$
$$+\frac{2\mu}{b^2}\,\text{rot}\,[(\omega\times e_m)\times\text{grad}\,r]$$
$$+\frac{2\mu}{b^2}\,\text{grad}\,\nabla^2 r\times[e_m\times(\omega\times x)]$$
$$+\frac{\mu}{b^2}[(\omega\times x)\cdot e_m]\,\text{grad}\,\nabla^2 r\,. \quad \text{(a-21)}$$

The surface integral from the first term of the right-hand side of (a-20) becomes

$$\iint\text{rot}\,[(\omega\times\xi)\times\text{grad}\,r]\cdot\nu d\Sigma=-\int[(\omega\times\xi)\times r]\cdot\frac{d\xi}{r}\,.$$

From the second term we have

$$\iint\text{rot}\,[e_m\times(\omega\times\text{grad}\,r)]\cdot\nu d\Sigma$$
$$=\int[e_m\times(\omega\times\text{grad}\,r)]\cdot d\xi$$
$$=e_m\cdot\left(\int(\omega\times\text{grad}\,r)\times d\xi\right)$$
$$=e_m\cdot\int\left(\frac{r}{r}\times\omega\right)\times d\xi\,.$$

From the third term we have

$$\iint \operatorname{rot}[(\boldsymbol{\omega}\times\boldsymbol{e}_m)\times\operatorname{grad} r]\cdot\nu d\Sigma$$
$$=\int[(\boldsymbol{\omega}\times\boldsymbol{e}_m)\times\operatorname{grad} r]\cdot d\boldsymbol{\xi}$$
$$=\int(\boldsymbol{\omega}\times\boldsymbol{e}_m)\cdot(\operatorname{grad} r\times d\boldsymbol{\xi})$$
$$=\boldsymbol{e}_m\cdot\int\boldsymbol{\omega}\times\left(\frac{\boldsymbol{r}}{r}\times d\boldsymbol{\xi}\right).$$

From the fourth term we have

$$\iint\{\operatorname{grad}\nabla^2 r\times[\boldsymbol{e}_m\times(\boldsymbol{\omega}\times\boldsymbol{x})]\}\cdot\nu d\Sigma$$
$$=\iint[\boldsymbol{e}_m\times(\boldsymbol{\omega}\times\boldsymbol{x})]\cdot(\boldsymbol{\nu}\times\operatorname{grad}\nabla^2 r)d\Sigma$$
$$=[\boldsymbol{e}_m\times(\boldsymbol{\omega}\times\boldsymbol{x})]\cdot\int\frac{2}{r}d\boldsymbol{\xi}$$
$$=2\boldsymbol{e}_m\cdot\left[(\boldsymbol{\omega}\times\boldsymbol{x})\times\int\frac{1}{r}d\boldsymbol{\xi}\right].$$

From the fifth term we have

$$(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\boldsymbol{e}_m\iint\operatorname{grad}\nabla^2 r\cdot\nu d\Sigma$$
$$=2(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\boldsymbol{e}_m\iint\frac{\partial}{\partial\nu}\left(\frac{1}{r}\right)d\Sigma$$
$$=2\boldsymbol{e}_m\cdot[(\boldsymbol{\omega}\times\boldsymbol{x})\Omega].$$

Thus

$$\begin{aligned}4\pi u_m(Q)=&\ \alpha(\boldsymbol{e}_m\operatorname{grad})_Q\int\left[(\boldsymbol{\omega}\times\boldsymbol{\xi})\times\frac{\boldsymbol{r}}{r}\right]\cdot d\boldsymbol{\xi}\\ &-(1+\alpha)\boldsymbol{e}_m\cdot\int\left(\boldsymbol{\omega}\times\frac{\boldsymbol{r}}{r}\right)\times d\boldsymbol{\xi}\\ &+\boldsymbol{e}_m\cdot\int\boldsymbol{\omega}\times\left(\frac{\boldsymbol{r}}{r}\times d\boldsymbol{\xi}\right)\\ &+\boldsymbol{e}_m\cdot\int(\boldsymbol{\omega}\times\boldsymbol{x})\times\frac{1}{r}d\boldsymbol{\xi}\\ &+\boldsymbol{e}_m\cdot[(\boldsymbol{\omega}\times\boldsymbol{x})\Omega],\end{aligned}\tag{a-22}$$

where

$$\alpha = \frac{\lambda + \mu}{\lambda + 2\mu} .$$

Employing the relation $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{r}$, we can write in the vector form

$$\begin{aligned} 4\pi \boldsymbol{u}_Q = & (\boldsymbol{\omega} \times \boldsymbol{x})\Omega \\ & + \int (\boldsymbol{\omega} \times \boldsymbol{\xi}) \times \frac{1}{r} d\boldsymbol{\xi} \\ & + \alpha\, \mathrm{grad}_Q \int \left[(\boldsymbol{\omega} \times \boldsymbol{\xi}) \times \frac{\boldsymbol{r}}{r} \right] \cdot d\boldsymbol{\xi} \\ & + \int \boldsymbol{\omega} \times \left(\frac{\boldsymbol{r}}{r} \times d\boldsymbol{\xi} \right) \\ & - \alpha \int \left(\boldsymbol{\omega} \times \frac{\boldsymbol{r}}{r} \right) \times d\boldsymbol{\xi} . \end{aligned} \qquad \text{(a-23)}$$

For a general Volterra dislocation (a-5), as the result of the computations in (i) and (ii) we have

$$\begin{aligned} \boldsymbol{u}(Q) = \frac{1}{4\pi} \Big\{ & (\boldsymbol{b} + \boldsymbol{\omega} \times \boldsymbol{x})\Omega + \int \Delta \boldsymbol{u} \times \frac{1}{r} d\boldsymbol{\xi} \\ & + \alpha\, \mathrm{grad}_Q \int (\Delta \boldsymbol{u} \times \boldsymbol{r}) \cdot \frac{1}{r} d\boldsymbol{\xi} \\ & + \boldsymbol{\omega} \times \int \boldsymbol{r} \times \frac{1}{r} d\boldsymbol{\xi} \\ & - \alpha \int (\boldsymbol{\omega} \times \boldsymbol{r}) \times \frac{1}{r} d\boldsymbol{\xi} \Big\} . \end{aligned} \qquad \text{(a-24)}$$

Acknowledgments

The writer wishes to express his sincere thanks to Dr. K. Aki and Dr. S. Miyamura of the Earthquake Research Institute for encouragement and support in the course of this study. Thanks are also due to Prof. H. Takeuchi who encouraged the writer to prepare the manuscript.

This work was partly supported by a Grant from the National Science Foundation.

References

1) BURGERS, J. M., "Some consdiderations on the fields of stress connected with dislocations in a regular crystal lattice," *Proc. Acad. Sci. Amst.*, **42** (1939), 293.

2) CHINNERY, M. A., "The deformation of the ground around surface faults," *Bull. Seism. Soc. Amer.*, **51** (1961), 355-372.

3) CHINNERY, M. A., "The stress changes that accompany strike-slip faulting," *Bull. Seism. Soc. Amer.*, **53** (1963), 921-932.

4) HONDA, H. and T. MIURA, "On the strain produced in a semi-infinite elastic solid by statical surface force, with some applications to seismology," *Geophys. Mag.*, **9** (1935), 61-81.

5) KASAHARA, K., "The nature of seismic origins as inferred from seismological and geodetic observations (1)," *Bull. Earthq. Res. Inst.*, **35** (1957), 473-532.

6) KASAHARA, K., "Physical conditions of earthquake faults II," *Bull. Earthq. Res. Inst.*, **37** (1959), 251-273.

7) KNOPOFF, L., "Energy release in earthquakes," *Geophys. J., R. A. S.*, **1** (1958), 44-52.

8) LOVE, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, 4th Ed. (1927).

9) MARUYAMA, T., "On the force equivalents of dynamical elastic dislocations with reference to the earthquake mechanism," *Bull. Earthq. Res. Inst.*, **41** (1963), 467-486.

10) NABARRO, F. R. N., "The synthesis of elastic dislocation fields," *Phil. Mag.*, **42** (1951), No. 334, 1224.

11) NABARRO, F. R. N., "The mathematical theory of stationary dislocations," *Advances in Phys.*, **1** (1952), No. 3.

12) PRANDTL, L., "Ein Gedankenmodell zur kinetischen Theorie der festen Koerper," *Z. angew. Math. Mech.*, **8** (1928), 85.

13) SEZAWA. K., "The tilting of the surface of a semi-infinite solid due to internal nuclei of strain," *Bull. Earthq. Res. Inst.*, **7** (1929), 1-14.

14) SOEDA, K., "On the deformation produced in a semi-infinite elastic solid by an internal source of stress," *Kenshin Ziho*, **13** (1942), 263-291, (In Japanese).

15) STEKETEE, J. A., "On Volterra's dislocations in a semi-infinite elastic medium," *Can. J. Phys.*, **36** (1958 a), 192-205.

16) STEKETEE, J. A., "Some geophysical applications of the elasticity theory of dislocations," *Can. J. Phys.*, **36** (1958 b), 1168-1198.

17) TAYLOR, G. I., "The mechanism of plastic deformation of crystals," *Proc. Roy. Soc.*, A, **145** (1935), 362.

18) VOLTERRA, V., "Sur l'èquilibre des corps èlastiques multiplement connexes," *Ann. sci. école norm. supérieure, Paris*, **24** (1907), 401-517.

19) VVEDENSKAYA, A. V., "The determination of displacement fields by means of dislocation theory," *Akad. Nauk SSSR Izv. ser. Geofiz.*, (1956), No. 3, 227.

20) VVEDENSKAYA, A. V., "The displacement field associated with discontinuities in an elastic medium," *Akad. Nauk Izv. ser. Geofiz.*, (1959), No. 4, 516.

21) WATSON, G. N., *Treatise on the Theory of Bessel Functions, Cambridge* (1922).

22) WHIPPLE, F. J. W., "On the theory of the strain in an elastic solid bounded by a plane where there is a nucleus of strain at an internal point, and on the relation of the theory to seismology," *Mon. Not. Roy. Astron. Soc. Geophys. Supple.*, **3** (1936), 380-388.

23) YAMAKAWA, N., "On the strain produced in a semi-infinite elastic solid by an interior

source of stress," *Zisin*, **8** (1955), 84-98, (In Japanese).
24) YOFFE, E. H., "A dislocation at a free surface," *Phil. Mag.*, **6** (1961), No. 69, 1147.

16. 無限および半無限媒質における静的弾性転位

東京大学大学院地球物理課程 丸 山 卓 男

地震現象にともなう地殻変動は古くから知られているけれども，その変動の様相の詳細を説明することには十分成功してはいない．地下の状態と地震発生の機構について知識を得るためには，現在の段階では，地震にともなう地殻変動を説明しうる様々のモデルを用意していることが有用であろう．

半無限弾性体の変形という立場から，従来考えられているモデルは，点状の歪核や小さな球面上に与えられた応力分布にもとずく弾性変形である．その後 Kasahara, Knopoff などによつて strike slip fault を簡単な二次元のモデルとしてとらえる方法が行なわれた．

転位の弾性論は，有限の広がりと，任意の形をもつ一つの曲面を境としてその両側が相対的に移動する，三次元の fault を扱うことを可能にする．Steketee は鉛直の strike slip を表現するための式を導いた．筆者はここで dip slip や fissure type の変形や，傾きのある strike slip などを扱う上に必要な，変位および応力のための表現を求めた．

特に地表上の変位を，簡単な型の fracture の場合について，いくつかの例について計算して結果を図示した．これらは具体的な地表の変動の観測値を調べる際に役立つであろう．

弾性転位という概念を Steketee の流儀にしたがつて地殻変動や地震の分野に適用するときは，結晶転位論における場合とはやや趣きが異なる．これらの関連の一端にふれて，Steketee の叙述の一部分を補つた．