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2. *Effect of the Intermediate Dissipative Medium on the Transmission of Elastic Waves through a Heterogeneous Medium having Periodic Structures.*

By Isao ONDA,

Earthquake Research Institute.

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Abstract

In order to make clear the role of the intermediate medium between heterogeneous media, it is assumed that the intermediate medium is dissipative. The case in which the effect of two heterogeneous media is subtractive is considerably limited; the special case appears only when both heterogeneous media are very thick and when the thickness of the intermediate medium is nearly equal to $(2m-1)/4$ times the wavelength of the wave propagated.

1. Introduction

One theory on the attenuation of seismic waves has been proposed by Prof. R. Yoshiyama in connection with the waves through a heterogeneous medium which has periodic parameters[1]. According to this theory, the amplitude of the transmitted wave with a certain period T diminishes as inversely proportional to $\cosh(\pi bx/aT)$, where a is the mean velocity, b the amplitude of velocity variation and x the distance through this medium. In the successive paper[2], the author having participated, a study was made of the transmission of waves through the structure in which a homogeneous medium with a finite thickness was inserted between two media with a periodic structure. They pointed out that the total effect of such a structure on the transmission of waves is dependent not only on addition but subtraction of their thickness, and that its dependence is governed by the phase difference of waves between two boundaries, ahead of and behind the intermediate medium.

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1) R. YOSHIYAMA, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction," *Bull. Earthq. Res. Inst.*, **38** (1960), 467-478.

2) R. YOSHIYAMA and I. ONDA, *loc. cit.*, Part 2, *Bull. Earthq. Res. Inst.*, **40** (1962), 391-398.

In this paper, the structure in which the intermediate medium is dissipative, instead of perfectly homogeneous, is taken into consideration in order to investigate the role of the intermediate medium.

2. Transmission through heterogeneous media inserting an intermediate dissipative medium

In Part II of "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction"[3], the problem of the waves propagated through the heterogeneous medium, which consists of two media with the periodic parameters and an intermediate medium being homogeneous, was studied. From this study it was pointed out that the effect of an intermediate medium cannot be neglected. In this paper, transmission of waves through the intermediate dissipative medium, instead of perfectly a homogeneous one, is considered in order to clarify the role of the intermediate medium.

The medium is divided into five parts as schematically shown in Fig. 1. The first and fifth media are both homogeneous, with the same velocity $a(1+b)$; both the second and the fourth media are heterogeneous, with the same velocity variation $c(x)=a(1+b\cos\gamma x')$, where x' is measured from x_1 for the second medium and from x_3 for the fourth medium; and the third medium is dissipative, different from the previous

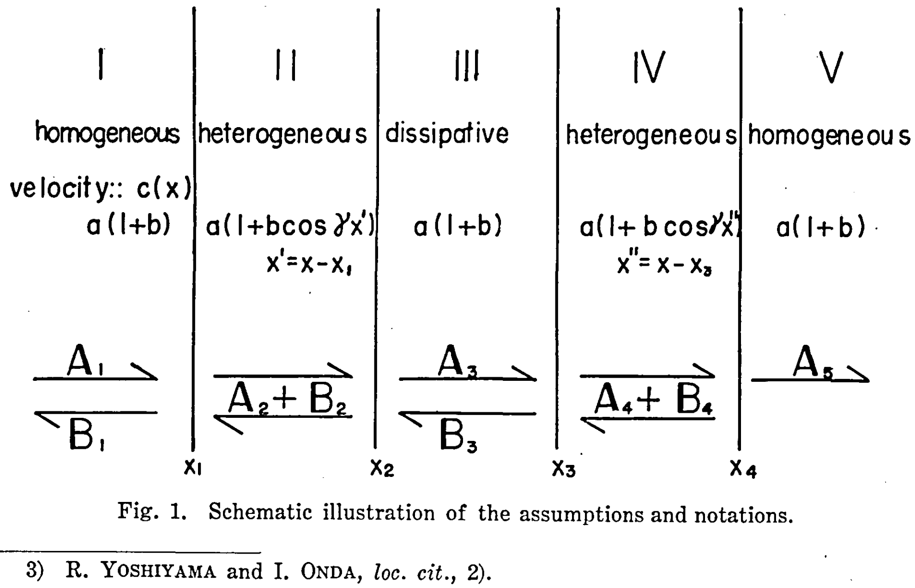

Fig. 1. Schematic illustration of the assumptions and notations.

3) R. YOSHIYAMA and I. ONDA, *loc. cit.*, 2).

paper, with the velocity $a(1+b)$. The density ρ is assumed as constant throughout the whole media, that is to say ρ_0 and velocities $c(x)$ are assumed as continuous at each boundary, to minimize the effect of the boundary reflection and to emphasize the effect of the periodic structures. The displacements in each part are expressed, omitting the common time factor $\exp(ipt)$, and assuming $|b| \ll 1$, as follows:

$$\mathrm{I}:\quad \mathbf{A}_1=\frac{A_1}{\sqrt{\rho_0 c_0}}\exp\{-ik_0(x-x_1)\}\,,\qquad \mathbf{B}_1=\frac{B_1}{\sqrt{\rho_0 c_0}}\exp\{ik_0(x-x_1)\}\,,$$

$$k_0=p/c_0\,.$$

$$\mathrm{II}:\quad \mathbf{A}_2+\mathbf{B}_2=\frac{1}{\sqrt{\rho_0 c_0}}\{A_2\psi(-z_2)\exp(-\mu z_2)+B_2\psi(z_2)\exp(\mu z_2)\}\,,$$

$$z_2=\frac{\gamma a\sqrt{1-b^2}}{2}\int_{x_1}^{x}\frac{dx}{c_2(x)}\,,\qquad c_2(x)=a\{1+b\cos\gamma(x-x_1)\}\,,$$

$$\mathrm{III}:\quad \mathbf{A}_3=\frac{A_3}{\sqrt{\rho_0 c_0}}\exp\{-ik_0(1-i\alpha)(x-x_2)\}\,,$$

$$\mathbf{B}_2=\frac{B_3}{\sqrt{\rho_0 c_0}}\exp\{ik_0(1-i\alpha)(x-x_2)\}\,.$$

$$\mathrm{IV}:\quad \mathbf{A}_4+\mathbf{B}_4=\frac{1}{\sqrt{\rho_0 c_4}}\{A_4\psi(-z_4)\exp(-\mu z_4)+B_4\psi(z_4)\exp(\mu z_4)\}\,,$$

$$z_4=\frac{\gamma a\sqrt{1-b^2}}{2}\int_{x_3}^{x}\frac{dx}{c_4(x)}\,,\qquad c_4(x)=a\{1+b\cos\gamma(x-x_3)\}\,.$$

$$\mathrm{V}:\quad \mathbf{A}_5=\frac{A_5}{\sqrt{\rho_0 c_0}}\exp\{-ik_0(x-x_4)\}\,.$$

where the functions $\psi(\pm z)$ are periodic functions of z, which enter into the general solutions of Mathieu's equation, approximated respectively by $\sin(z\mp\pi/4)$ for the largest μ of the expressions in the second and fourth media, which is nearly equal to half an amplitude of the velocity variations, $\mu=b/2$. If the factor of the attenuation in the third medium is written as $\exp(-\kappa x)$, α is the ratio of the attenuation coefficient κ $(=\pi/QcT)$ to the wave number, k_0, so that α is regarded as a half of $1/Q$, and therefore for the sake of argument the order of magnitude of α may be neglected.

Now the wave alone is considered, the wavelength of which propagated through the homogeneous medium is doubled in the periodic structures, because such a wave has the largest attenuation in propagation[4].

The coefficients A_1, B_1, etc. are connected with each other by putting the displacement and stress to be continuous at each boundary. In the result, the transmission coefficient is obtained by the form

$$\frac{A_5}{A_1}=\frac{\exp(-iz_4)}{\cos kx\cosh\mu(z_2+z_4)+i\sin kx\cosh\mu(z_2-z_4)+0(\alpha)}\,,$$

where

$$kx=k_0(1-i\alpha)x_0+z_2=k_0(x_3-x_1)-i\kappa x_0\,,\quad\text{and}\quad x_0=x_3-x_2\,.$$

In such a limit as α tends to zero, the expression obtained above agrees with the result of the previous paper[5], remembering that both z_2 and z_4 are integer time of π, that is, if $z_2=l\pi$ and $z_4=m\pi$, $\cos kx=(-1)^l\cdot\cos k_0x_0$, $\sin kx=(-1)^l\sin k_0x_0$ and $\exp(-iz_4)=(-1)^m$.

Next, using the relations $\exp(-2iz_2)=1$ and $z_2+z_4+k_0x_0=k_0(x_4-x_1)$, the above expression can be rewritten as follow;

$$\frac{A_5}{A_1}=\frac{\exp\{-ik_0(x_4-x_1)\}}{\exp(\kappa x_0)\cosh\mu(z_2+z_4)}\left[\frac{1+\exp(-2\mu\zeta)}{1+\exp(-2\mu\zeta-2\kappa x_0)\exp(-2ik_0x_0)}\right],$$

where

$$\tanh\mu\zeta=\frac{\cosh\mu(z_2-z_4)}{\cosh\mu(z_2+z_4)}\quad\text{or}\quad\exp(-2\mu\zeta)=\tanh\mu z_2\cdot\tanh\mu z_4\,.$$

The numerator is interpreted as the wave to transmit from $x=x_1$ through to x_4, whereas the denominator explains for the wave to attenuate due to transmission through the periodic structures and the intermediate dissipative medium. The term in square brackets denotes the magnitude of dependence on subtraction of thicknesses of periodic structures and on thickness of an intermediate medium.

3. Effect of subtraction of thicknesses of two periodic structures

Next, the behaviour of the term in the square brackets is examined: this term is designated by the symbol S, and is dependent on three independent variables $2\mu\zeta$, $2\kappa x_0$, and $2k_0x_0$. $2\mu\zeta$ is determined by the

4) R. YOSHIYAMA, *loc. cit.*, 1).

5) R. YOSHIYAMA and I. ONDA, *loc. cit.*, 2). In that paper, a few expressions must be corrected: the numerator in the formulas on the top line and in the abstract (in Japanese) of p. 398 should read $(-1)^{l+m}$ for 1.

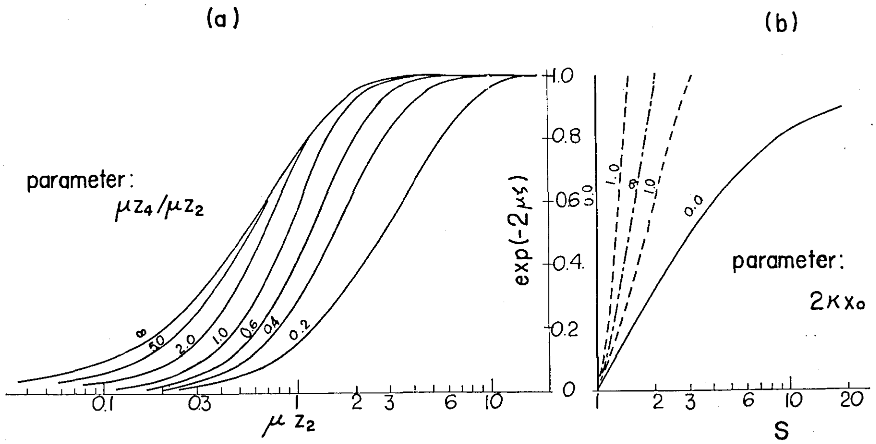

Fig. 2 (a). Relations among μz_2, μz_4 and $\exp(-2\mu\zeta)$.
(b). Extreme values of S for $\exp(-2\mu\zeta)$.

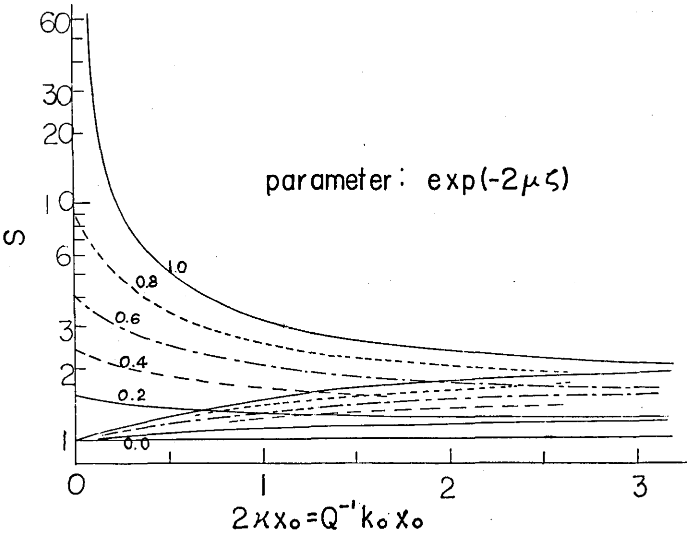

Fig. 3. Extreme values of S for $Q^{-1}k_0x_0$.

values of μz_2 and μz_4, so that the relation between μz_2 and $\exp(-2\mu\zeta)$ is given for some parameters, the ratio of μz_4 to μz_2, as graphically shown in Fig. 2a. It is seen from this figure that, if either μz_4 or μz_2 is small, the magnitude of $\exp(-2\mu\zeta)$ is small although another is large. However, $\exp(-2\mu\zeta)$ cannot be assumed as small for moderate values of μz_2 and μz_4.

If $2\mu\zeta$ and $2\kappa x_0$ are given, the expression S has the values between respective curves of S corresponding to $2k_0x_0=\pm\pi$, thence it is possible to denote the existence region of S for some parameters $2\kappa x_0$ as functions of $\exp(2\mu\zeta)$. Fig. 2b illustrates the extreme values of S for parameters $2\kappa x_0=0$, 1 and infinity. On the other hand, Fig. 3 illustrates the extreme values of S for some parameters $\exp(-2\mu\zeta)$. From these figures, it is seen that S can become very large only in the case of both $2\kappa x_0\rightarrow 0$ and $\exp(-2\mu\zeta)\rightarrow 1$, that is to say, in this special case, subtraction of thicknesses of two periodic structures plays an important part in the transmission of waves through such a medium; while the existence region of S becomes narrow rapidly as $2\kappa x_0$ increases apart from zero. It may be interpreted that the effect of subtraction of them becomes small, as the result of considering the intermediate medium with dissipation. It is remarked that the greater the value of $\exp(-2\mu\zeta)$ is, the greater both the width of existence region and the lowest value of S become.

As z_2 and z_4 are multiples of π respectively, the wavelength or the period of waves propagated is fixed, according to periodic structures under consideration. $1/Q$ to the maximum amplitude observed in the shallow earthquakes will perhaps be given by part of the surface waves. The absolute value of S is illustrated in Fig. 4, where $\exp(-2\mu\zeta)$ is assumed as 0.8 and $1/Q$ in the intermediated medium is taken as 0.01 in order to facilitate the understanding. It is understood from this figure that the effect of subtraction of them appears only for the intermediate medium with thickness nearly equal to $(2m-1)/4$ times the

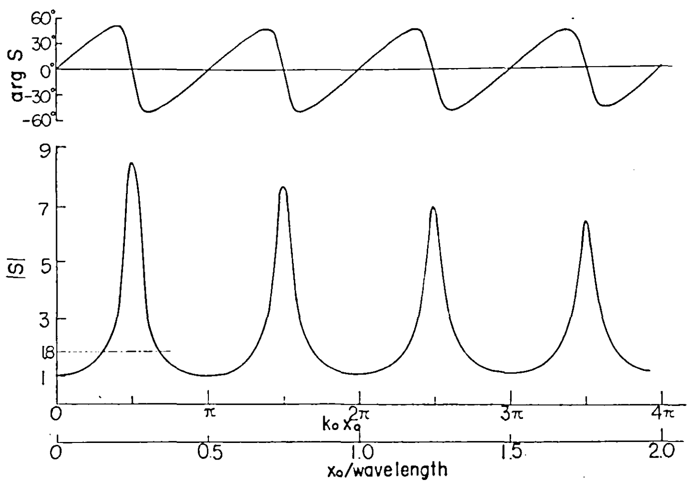

Fig. 4. Variation of S for $\exp(-2\mu\zeta)=0.8$, $Q^{-1}=0.01$.

wavelength of waves propagated.

The average value of S is calculated as follows:

$$\overline{S}=\frac{1}{\xi}\int_0^{\xi} Sd(k_0x_0)=\{1+\exp(-2\mu\zeta)\}\left[1+\frac{1}{\xi}\ln\frac{1+\exp\{-2\mu\zeta-(Q^{-1}+2i)\xi\}}{1+\exp(-2\mu\zeta)}\right],$$

and, if ξ is a multiple of π, it is expressed by the form

$$\begin{aligned}\overline{S}&=\{1+\exp(-2\mu\zeta)\}\left[1+\frac{1}{m\pi}\ln\frac{1+\exp(-2\mu\zeta-Q^{-1}m\pi)}{1+\exp(-2\mu\zeta)}\right]\\&=\{1+\exp(-2\mu\zeta)\}[1-O\{Q^{-1}\exp(-2\mu\zeta)\}]\,,\end{aligned}$$

which is the average from the first minimum of S to m-th (or from $x_0=0$ to half an integer times the wavelength of waves propagated) and is nearly equal to $1+\exp(-2\mu\zeta)$. It is certain that the effect of subtraction of them is small, except for a few special cases.

In addition, the phase lag of S is illustrated in the upper part of Fig. 4, and tends to zero at the thickness of the intermediate medium at which the absolute value of S denotes no variation with respect to its thickness. When S is small, the phase lag varies linearly and slowly, while, for large S, the variation of the phase lag becomes very large. That is to say, properties of the transmission coefficient are very complicated for the special thickness above-stated and these special cases will seldom appear.

4. Concluding remarks

In the previous paper[6], the wave propagation through the medium in which a homogeneous medium was inserted between two heterogeneous media was studied, and it was pointed out that the transmitted waves was affected by thickness of the intermediate medium. So, in order to make clear the role of the intermediate medium, it is assumed that it is dissipative instead of perfectly homogeneous, and the amplitude of the transmitted wave with the most attenuate character is calculated. By defining S for the ratio of the contribution toward addition of thicknesses of two heterogeneous media to that toward subtraction of them, the transmission coefficient is expressed by the form

$$\frac{A_5}{A_1}=\frac{\exp\{-i(x_4-x_1)\}}{\exp(\kappa x_0)\cosh\mu(z_2+z_4)}\cdot S\,,$$

in which the numerator represents the phase transmission through the

6) R. YOSHIYAMA and I. ONDA, *loc. cit.*, 2).

whole medium, while the denominator represents the attenuation factor, the first of which is the contribution toward the intermediate dissipative medium, the second of which is that toward addition of thicknesses of the heterogeneous media, and S is interpreted as the correction for subtraction of them.

S behaves itself in the finite range for certain values of μz_2, μz_4 and κx_0, and is nearly equal to unity except for the case in which not only the product of $\tanh \mu z_2$ and $\tanh \mu z_4$ is nearly equal to unity but also thickness of the intermediate medium is nearly equal to $(2m-1)/4$ times the wavelength of the wave propagated. The former exception must satisfy that both heterogeneous media are very thick. If either condition is not satisfied, the effect of subtraction of them need not be taken into consideration.

In conclusion, it is little expected that such special cases appear.

In addition, if the seismic waves are propagated through the heterogeneous medium approximated as the alternately stratified structures consisting of homogeneous medium and heterogeneous one, it is suggested that the total effect of apparent attenuation on the transmitted waves is expressed by addition of effects of each medium.

Acknowledgement

The author wishes to express his sincere thanks to Professor Ryoichi Yoshiyama for his helpful suggestions and encouragement throughout the course of this study.

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2. 不均質媒質を透過する弾性波に及ぼす中間層の吸収性の影響

地震研究所　音　田　功

“不均質な媒質内を伝わる弾性波の安定性について”の第2報で均質な中間層を挿入してその透過波が均質層の厚さによつて影響を受けることを示した．この論文ではこの中間層の役割を説明するために，中間層が吸収性を有すると仮定して透過波を計算した．その結果，二つの不均質層の影響がそれぞれの厚さの引き算になる場合は可成り限られていることがわかつた．すなわち，それらの引き算の寄与する部分と寄せ算の寄与する部分の商を S と定義すれば，Sにはある存在範囲があることが示され，そして一例として internal friction $1/Q = 0.01$ の場合の S の変化を第4図に示した．結局，それらの引き算の寄与が大きく現われるのは $\tanh \mu z_2$ と $\tanh \mu z_4$ の積がほぼ1に等しく，かつ中間層の厚さが波の波長の $(2m-1)/4$ 倍にほぼ等しいときに限られ，前者の条件は両方の不均質媒質の厚さが充分に厚いことを必要としている．これらの二つの条件が共に満されていないときには，それらの寄せ算の寄与だけで透過波の減衰の振舞いが近似される．換言すれば，それらの引き算が透過波の減衰に対して重要な役割を演ずる場合はほとんど期待されないといえよう．