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7. *On the Bottom Friction in an Oscillatory Current.*

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Abstract

The frictional coefficient C of an oscillating turbulent flow is estimated on the assumption that the eddy viscosity is proportional to the amplitude of the bottom friction velocity and the height above the bottom. The dependence of C on parameters such as the amplitude of the vertically averaged horizontal velocity, the period of the oscillation, the depth of water, and the roughness length z_0 (for the case of a rough boundary) or the molecular viscosity ν (for the case of a smooth boundary) is shown graphically by choosing suitable non-dimensional parameters. The frictional coefficient for the case of a laminar oscillatory flow is also discussed.

1. Introduction

In many practical applications of the dynamical equations of motion, such as the case of a numerical experiment on long waves in shallow water, the law of bottom stress is assumed as something like $C|\bar{u}|\bar{u}$ where $\bar{u}$ is the depth-mean velocity in a water column and C is the frictional coefficient. There are several estimates of C in tidal currents by means of the dynamical method (Taylor, 1918; Grace, 1936, 1937; Bowden and Fairbairn, 1952). On the other hand, the bottom friction is intimately related to the mean velocity profile or the turbulent velocity fluctuations near the bottom and there are some observations which show the logarithmic law of velocity profile near the bottom (Lesser, 1951; Charnock, 1959). Furthermore, the turbulent velocity fluctuations in a tidal current are measured by means of an electromagnetic flow meter (Bowden and Fairbairn, 1956; Bowden, 1961) and the Reynolds stresses near the bottom are computed. From these results, it appears that the frictional coefficient (for depth-mean velocity) of a tidal current is about $1.5\sim2.5\times10^{-3}$. However, it is not yet certain whether the same frictional law holds for a long wave of shorter periods, because it is plausible that the structure of the bottom frictional layer in such an oscillatory current may be different from that of the tidal current.

On the other hand, the frictional law of the steady turbulent flow in pipes and open channels is extensively studied by hydraulic engineers but the frictional law of the oscillatory flow seems to be hardly investigated. Recently, the decay of waves in shallow water has become a subject of coastal engineers and some experimental results are discussed in conjunction with the theoretical estimate based on the condition of a laminar flow (Biesel, 1949; Eagleson, 1962; Grosch, 1962), and the transition from the laminar to turbulent boundary layer in an oscillating flow over smooth and rough bottoms is also studied experimentally (Li, 1954; Vincent, 1957; Collins, 1963).

The present paper is a theoretical attempt, though admittedly not so complete, to find the frictional coefficient C of the oscillatory flow in a fully turbulent state by assuming a suitable relation between velocity shear and stress in water and to examine the dependence of the frictional coefficient on the amplitude and period of oscillating currents as well as to the depth of water.

2. Basic consideration

We consider that the motion is predominantly horizontal in one-direction and water is homogeneous with constant depth. Then, the linearized equation of horizontal motion may be written as

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x} + \frac{\partial \tau}{\partial z}, \tag{2-1}$$

where t is time, x and z are the horizontal and vertical co-ordinates with the origin at the bottom of the water and the z-axis positive upwards, u is the horizontal velocity, ζ is the elevation of the water surface from the undisturbed free surface, z_h is the depth of water, and g is the acceleration due to gravity. The tangential stress in the x-direction is given by $\rho\tau$ with ρ the density of water.

Now putting formally

$$-g\frac{\partial \zeta}{\partial x} = \frac{\partial U}{\partial t}, \tag{2-2}$$

we may rewrite (2-1) in the form

$$\frac{\partial}{\partial t}(u - U) = \frac{\partial \tau}{\partial z}. \tag{2-3}$$

In terms of the vertically averaged velocity, (2–3) becomes

$$\frac{\partial}{\partial t}(\bar{u}-U)=\frac{1}{z_h}(\tau_S-\tau_B)\,, \tag{2–4}$$

where τ_S and τ_B are the surface and the bottom stresses respectively, ($\tau_S=0$ for the present discussion) and $\bar{u}$ is the mean velocity defined by

$$\bar{u}=\frac{1}{z_h}\int_0^{z_h} u\,dz\,. \tag{2–5}$$

The equation of continuity may be given by

$$\frac{\partial\bar{u}}{\partial x}=-\frac{1}{z_h}\frac{\partial\zeta}{\partial t}\,, \tag{2–6}$$

and from (2–4) and (2–6) together with (2–2), it follows

$$\frac{\partial^2\bar{u}}{\partial t^2}-gz_h\frac{\partial^2\bar{u}}{\partial x^2}=\frac{1}{z_h}\frac{\partial}{\partial t}(\tau_S-\tau_B)\,, \tag{2–7}$$

or in terms of U,

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial U}{\partial t}\right)-gz_h\frac{\partial^2}{\partial x^2}\left(\frac{\partial U}{\partial t}\right)=g\frac{\partial^2}{\partial x^2}(\tau_S-\tau_B)\,. \tag{2–8}$$

The equation (2–7) is the common representation of long waves of small amplitude and if the bottom stress is given in terms of $\bar{u}$, the solution can be found under given initial and boundary conditions.

The bottom friction velocity u_B^* defined by

$$\tau_B=|u_B^*|u_B^*\,, \tag{2–9}$$

may be approximated for a periodic motion in time ($u_B^*=\hat{u}_B^*\cos(\sigma t+\varepsilon)$) by

$$\tau_B=\tilde{u}_B^*\,u_B^*\,, \tag{2–10}$$

where

$$\tilde{u}_B^*=\frac{8}{3\pi}\hat{u}_B^*\,, \tag{2–11}$$

and $\hat{u}_B^*$ is the amplitude of u_B^* (See Proudman ; Dynamical Oceanography, § 151, 1953).

Now, for the relation between the velocity shear and tangential stress, we resort to the usual formulation of a turbulent flow. On the ground of dimensional analysis, the turbulent eddy viscosity K_z in neutral stability may be put formally proportional to the characteristic velocity of turbulence $\langle v^2\rangle^{1/2}$ and the effective size of the turbulent eddy Λ such that

$$K_z \propto \langle v^2\rangle^{1/2}\Lambda\,, \tag{2-12}$$

and

$$K_z\frac{\partial u}{\partial z}=\tau\,. \tag{2-13}$$

Here, we assume that the effective size of the turbulent eddy is proportional to the height above the bottom and the characteristic turbulent velocity $\langle v^2\rangle^{1/2}$ is proportional to $\tilde{u}_B^*$[1]. More specifically, we assume

$$K_z=k\tilde{u}_B^*(z+z_0)\,, \tag{2-14}$$

where k is von Kármán's constant $(=0.4)$ and z_0 is the roughness length of a rough surface. Similar assumption for the eddy viscosity is already used in the discussion of the atmospheric frictional layer near the ground (e. g., Ellison, 1956). For the case of a smooth surface, z_0 should be understood as the thickness of a laminar sub-layer. Approximating the frictional velocity in general[2] by

$$u^*=\tau/\tilde{u}_B^*\,, \tag{2-15}$$

(2-12), (2-13), and (2-15) give

$$\frac{\partial u}{\partial z}=\frac{u^*}{k(z+z_0)}\,. \tag{2-16}$$

1) The size of turbulent eddies may decrease near the free surface because of the presence of the free surface $z=z_h$, and the characteristic turbulent velocity may not be constant throughout the vertical column of water but a function of the velocity shear and the height above the bottom as assumed in the Prandtl's mixing-length theory. However, a refined distribution of K_z does not seem to be worth while to try in the present crude discussion unless more definite knowledge about the turbulent structure of an oscillatory flow is obtained.

2) Essentially, u^* is not the friction velocity in an ordinary sense, but is a quantity defined by (2-15) and coincides with the bottom friction velocity u_B^* at the bottom. For convenience, we call u^* the friction velocity in the present paper.

3. The frictional coefficient for the case of a turbulent flow over a rough bottom

If the motion is assumed to be periodic in time, we may put

$$u^* = R_e[u^{*\prime} e^{i\sigma t}], \tag{3-1}$$

where R_e means the real part of the complex quantity. In the following discussion, we use the complex amplitude $u^{*\prime}$ with the prime dropped for simplicity unless otherwise stated. The same rule applies for other dependent variables too.

From (2-3) and (2-16) together with (2-15), the equation for the friction velocity u^* is derived:

$$\frac{\partial^2 u^*}{\partial z'^2} - i\frac{K^2}{z'} u^* = 0, \tag{3-2}$$

where

$$K^2 = \sigma/(k\tilde{u}_B^*), \quad \text{and} \quad z' = z + z_0. \tag{3-3}$$

Under the stress free condition at the surface:

$$u^* = 0 \quad \text{at} \quad z = z_h, \tag{3-4}$$

the solution of (3-2) becomes

$$\frac{u^*}{u_B^*} = \left(\frac{y}{y_0}\right) \frac{Z_1(ye^{-i\pi/4}, c_h)}{Z_1(y_0 e^{-i\pi/4}, c_h)}, \tag{3-5}$$

where

$$y = 2Kz'^{1/2}, \quad y_0 = 2Kz_0^{1/2}, \quad y_h = 2K(z_h + z_0)^{1/2},$$

and c_h is determined by

$$Z_1(y_h e^{-i\pi/4}, c_h) = 0. \tag{3-6}$$

Here,

$$Z_n(Y, c_h) = (a+ib)J_n(Y) + iN_n(Y), \tag{3-7}$$

and $J_n(Y)$ and $N_n(Y)$ are Bessel and Neumann functions of the order n and $c_h = a+ib$. For small values of Y, $Z_n(Y, c_h)$ can be expressed in series form as follows:

$$Z_0(ye^{-i\pi/4}, c_h) = \left(a+\frac{1}{2}\right) - \left(\frac{y}{2}\right)^2\left\{b+\frac{2}{\pi}\left(\gamma+\log\frac{y}{2}-1\right)\right\}$$
$$+i\left\{b+\frac{2}{\pi}\left(\gamma+\log\frac{y}{2}\right)+\left(\frac{y}{2}\right)^2\left(a+\frac{1}{2}\right)\right\}, \qquad (3\text{–}8)$$

and

$$Z_1(ye^{-i\pi/4}, c_h)$$
$$=\left(\frac{y}{2}\right)e^{-i\pi/4}\left[\left(a+\frac{1}{2}\right)+\frac{1}{\pi}\left(\frac{2}{y}\right)^2-\frac{1}{2}\left(\frac{y}{2}\right)^2\left\{b+\frac{2}{\pi}\left(\gamma+\log\frac{y}{2}-\frac{5}{4}\right)\right\}\right.$$
$$\left.+i\left\{b+\frac{2}{\pi}\left(\gamma+\log\frac{y}{2}-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{y}{2}\right)^2\left(a+\frac{1}{2}\right)\right\}\right], \qquad (3\text{–}9)$$

where $\gamma=0.5772....$ (Euler's constant).

The values of a and b computed numerically from (3–6) are shown in Fig. 1 as a function of y_h.

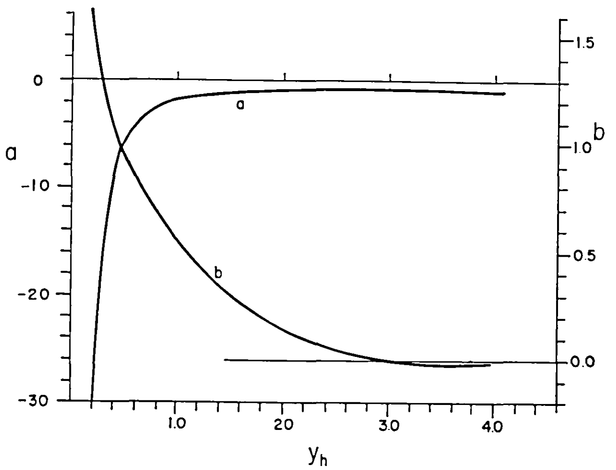

Fig. 1. a and b ($c_h=a+ib$) as a function of y_h.

For small values of y_h, say $y_h<1$,

$$a+\frac{1}{2}=-\frac{1}{\pi}\left(\frac{2}{y_h}\right)^2, \qquad (3\text{–}10)$$

and

$$b=-\frac{2}{\pi}\left(\gamma+\log\frac{y_h}{2}-\frac{3}{4}\right). \qquad (3\text{–}11)$$

For large values of y_h, say $y_h>3$,

$$a=-1\,, \quad \text{and} \quad b=0\,. \qquad (3\text{–}12)$$

Therefore, it follows:

$$\frac{u^*}{u_B^*}=\frac{1-(y/y_h)^2}{1-(y_0/y_h)^2}, \quad \text{for small } y_h\,, \tag{3-13}$$

and

$$\frac{u^*}{u_B^*}=-i\pi\left(\frac{y}{2}\right)e^{-i\pi/4}H_1^{(2)}(ye^{-i\pi/4})\,, \quad \text{for large } y_h\,, \tag{3-14}$$

where $H_1^{(2)}(Y)$ is the Hankel function of the second kind. (3-13) shows that for small y_h the stress τ decreases linearly with respect to depth from the bottom to the surface, and (3-14) shows that the decrease of stress is exponential for large values of y.

Now, in terms of y, (2-16) may be transformed into

$$\frac{\partial u}{\partial y}=\frac{2u^*}{ky}\,, \tag{3-15}$$

and the substitution of (3-5) into (3-15) yields

$$\frac{u}{u_B^*}=\frac{1}{k}\frac{Z_0(y_0e^{-i\pi/4},\,c_h)-Z_0(ye^{-i\pi/4},\,c_h)}{(y_0/2)e^{-i\pi/4}Z_1(y_0e^{-i\pi/4},\,c_h)}\,, \tag{3-16}$$

where the boundary condition at the bottom such that $u=0$ at $y=y_0$ is taken into consideration. It can be easily shown that for small values of y ($y<1$), the velocity profile in terms of z is given by

$$\frac{u}{u_B^*}=\frac{1}{k}\log\left(\frac{z+z_0}{z_0}\right). \tag{3-17}$$

Therefore, if the depth of water is small, the logarithmic formula of the velocity distribution holds for the entire domain.

In terms of y, (2-5) becomes

$$\bar{u}=\left(\frac{2}{y_h^2}\right)\int_{y_0}^{y_h}uy\,dy\,, \tag{3-18}$$

and the substitution of (3-16) yields

$$\frac{\bar{u}}{u_B^*}=\frac{1}{ky_h^2}\left\{4i+\frac{(y_h^2-y_0^2)Z_0(y_0e^{-i\pi/4},\,c_h)}{(y_0/2)e^{-i\pi/4}Z_1(y_0e^{-i\pi/4},\,c_h)}\right\}. \tag{3-19}$$

Since in general y_0 is very small and $(y_0/y_h)^2 \ll 1$, (3-19) is reduced to

$$\frac{\bar{u}}{u_B^*}=\alpha_1+i\alpha_2=Ae^{i\theta}\,, \tag{3-20}$$

where

$$\alpha_1=-\left(\frac{\pi}{k}\right)\left\{b+\frac{2}{\pi}\left(\gamma+\log\frac{y_0}{2}\right)\right\}>0\,,$$

$$\alpha_2=\left(\frac{\pi}{k}\right)\left\{a+\frac{1}{2}+\frac{1}{\pi}\left(\frac{2}{y_h}\right)^2\right\}\leqslant 0\,,$$

$$A=\sqrt{\alpha_1^2+\alpha_2^2}\,,$$

and

$$\theta=Tan^{-1}(\alpha_2/\alpha_1)\leqslant 0\,.$$

For small values of y_h, it follows,

$$A=\frac{1}{k}\left\{2\log\left(\frac{y_h}{y_0}\right)-\frac{3}{2}\right\}\,,\quad \text{and}\quad \theta=0\,, \tag{3-21}$$

and for large values of y_h,

$$\left.\begin{aligned} A&=\left(\frac{\pi}{k}\right)\left\{\left(\frac{2}{\pi}\right)^2\left(\gamma+\log\frac{y_0}{2}\right)^2+\frac{1}{4}\right\}^{1/2}\,,\\ \theta&=Tan^{-1}\left\{(\pi/4)\Big/\left(\gamma+\log\frac{y_0}{2}\right)\right\}\,. \end{aligned}\right\} \tag{3-22}$$

The frictional coefficient C is now defined by

$$\tau_B=\tilde{u}_B^*u_B^*=C\left(\frac{8}{3\pi}\right)\hat{\bar{u}}\bar{u}\,, \tag{3-23}$$

where $\hat{\bar{u}}$ is the amplitude of $\bar{u}$.[3] Therefore, the substitution of (3–20) yields

$$C=A^{-2}e^{-i\theta}\,. \tag{3-24}$$

Taking (3–21) and (3–22) into consideration, it is found that for $y_h<1$, the frictional coefficient is almost independent of wave period and amplitude and only a function of z_h/z_0, or in other words, the steady flow condition is applicable to the oscillatory flow in this range. On the other hand, for $y_h>3$ the frictional coefficient is a function of y_0

3) Strictly speaking, the instantaneous friction coefficient C should be defined by

$$R_e(\tau_B e^{i\sigma t})=C|R_e(\bar{u}e^{i\sigma t})|R_e(\bar{u}e^{i\sigma t})\,,$$

but for the periodic motion, the following approximation may be introduced;

$$|R_e(\bar{u}e^{i\sigma t})|R_e(\bar{u}e^{i\sigma t})\simeq\left(\frac{8}{3\pi}\right)\hat{\bar{u}}R_e(\bar{u}e^{i\sigma t})\,.$$

only so that C is independent of the depth but dependent on wave period and amplitude.

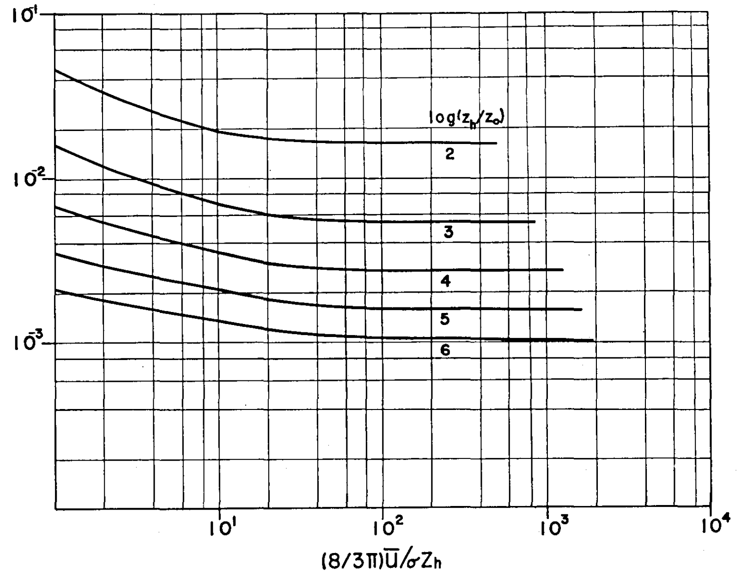

Fig. 2a. Amplitude $\hat{C}$ of the frictional coefficient C as a function of $\bar{u}/(\sigma z_h)$ for the oscillating turbulent flow over a rough surface.

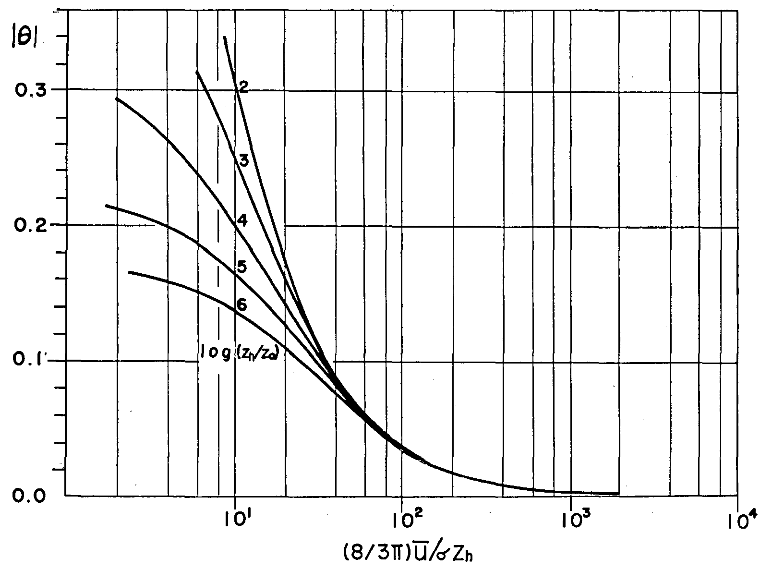

Fig. 2b. Phase θ of the frictional coefficient C as a function of $\bar{u}/(\sigma z_h)$ for the oscillating turbulent flow over a rough surface.

Combining the two relations, (3–20) and (3–24), $\hat{C}$ (amplitude of C) can be given as a function of $\bar{u}/(\sigma z_h)$ with z_h/z_0 as a parameter as shown in Fig. 2a. For small values of $\bar{u}/(\sigma z_h)$, $\hat{C}$ increases with decreasing

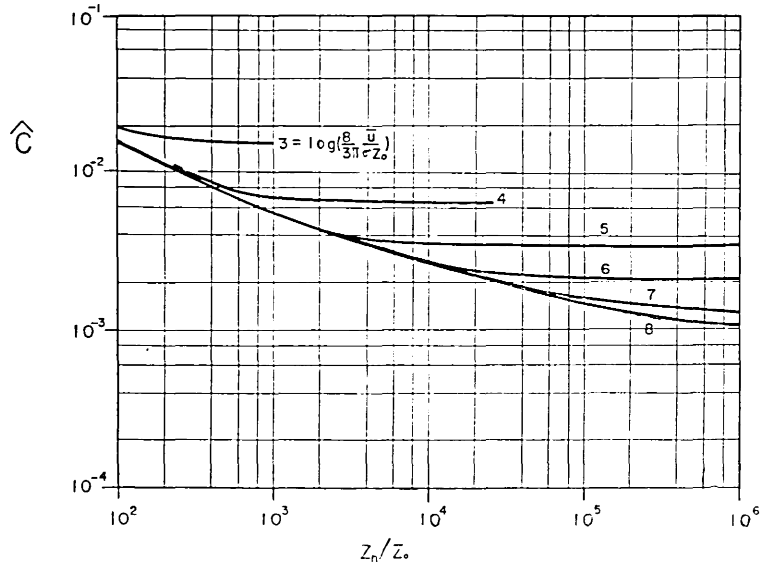

Fig. 3a. Amplitude $\hat{C}$ of the frictional C as a function of z_h/z_0 for the oscillating turbulent flow over a rough surface.

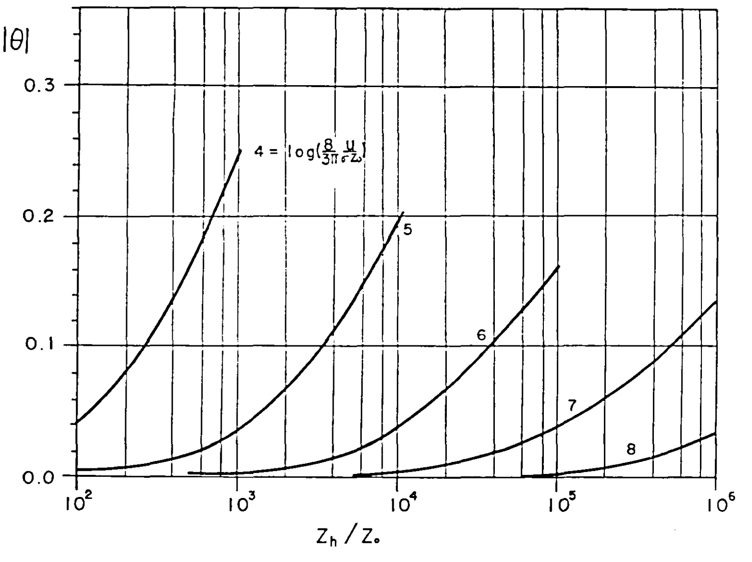

Fig. 3b. Phase θ of the frictional coefficient C as a function of z_h/z_0 for the oscillating turbulent flow over a rough surface.

values of $\bar{u}/(\sigma z_h)$ for a fixed value of z_h/z_0, and for $\bar{u}/(\sigma z_h)$ larger than, say 10^2, $\hat{C}$ stays almost constant. Fig. 2 b shows the corresponding phase angle $|\theta|$ as a function of $\bar{u}/(\sigma z_h)$ with z_h/z_0 as a parameter, and it is seen that the phase angle is noticable only for small values of $\bar{u}/(\sigma z_h)$ where the increase of $\hat{C}$ is observed.

For a fixed values of $\bar{u}/(\sigma z_0)$, the increase of z_h/z_0 is followed by the decrease of $\hat{C}$ and the increase of $|\theta|$ as shown Fig. 3 a and Fig. 3 b. However, for large values of z_h/z_0, $\hat{C}$ becomes constant for a fixed value of $\bar{u}/(\sigma z_0)$, showing the independence of $\hat{C}$ on z_h. In this range, C can be given as a function of $\bar{u}/(\sigma z_0)$ as shown in Fig. 3 c. Thus, the frictional coefficient C increases with decreasing $\bar{u}$, decreasing period of oscillation in the range of $\bar{u}/(\sigma z_h) \leq 10^2$, and with decreasing depth of water for large values of $\bar{u}/(\sigma z_h)$.

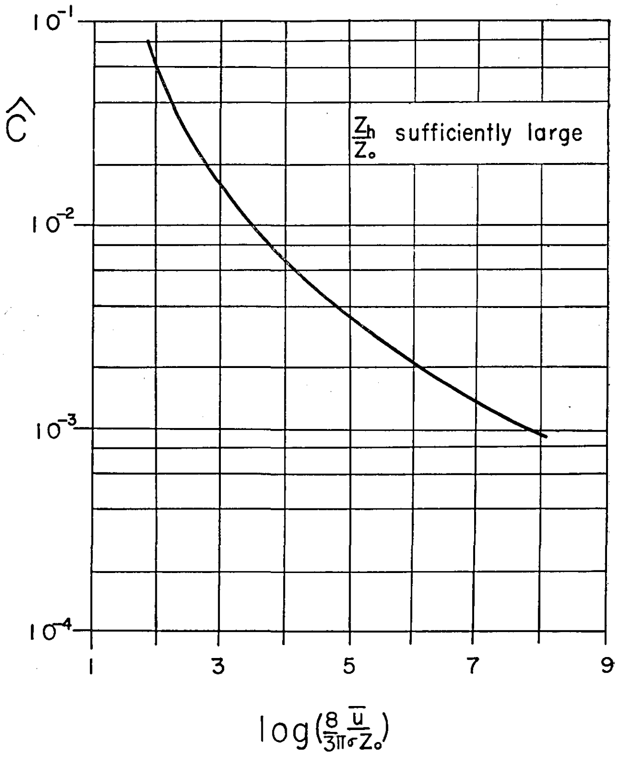

Fig. 3 c. Amplitude $\hat{C}$ of the frictional coefficient C as a function of $\bar{u}/(\sigma z_0)$ for sufficiently large values of z_h/z_0.

4. The dependence of τ_B and $\bar{u}$ on U (the pressure gradient term)

The substitution of (2–10) and (3–20) into (2–4) yields

$$u_B^* = Ue^{i\delta}/[\alpha_1^2+\{\alpha_2-\tilde{u}_B^*/(\sigma z_h)\}^2]^{1/2}\,, \tag{4–1}$$

where

$$\tan\delta=\{\tilde{u}_B^*/(\sigma z_h)-\alpha_2\}/\alpha_1\,. \tag{4–2}$$

Since $\alpha_1>0$, and $\alpha_2\leqslant 0$, it follows that $\delta\geqslant 0$.

Putting $\tilde{u}_B^*/(\sigma z_h)=\eta$, we have $u_B^*=(3\pi/8)\eta\sigma z_h e^{i\delta}$, if U is assumed real, and it follows from (4–1),

$$\eta^2\{\alpha_1^2+(\alpha_2-\eta)^2\}=\left(\frac{8}{3\pi}\cdot\frac{U}{\sigma z_h}\right)^2. \tag{4–3}$$

Remembering that α_1 and α_2 are functions of y_h and y_0, and $\eta=10/y_h^2$ by definition, we can solve $U/(\sigma z_h)$ in terms of η with y_0 or z_h/z_0 as a parameter.

In particular, for small values of η, we have

$$\eta^2=\left(\frac{8}{3\pi}\right)^2\frac{U^2}{(\sigma z_h)^2(\alpha_1^2+\alpha_2^2)}\,,\quad\text{and}\quad\delta=-\theta\,, \tag{4–4}$$

so that

$$\tau_B=\left(\frac{8}{3\pi}\right)\frac{e^{-i\theta}}{A^2}U^2\,. \tag{4–5}$$

On the other hand, for large values of η, we have

$$\eta^2=\left(\frac{8}{3\pi}\right)\frac{U}{\sigma z_h}\,,\quad\text{and}\quad\delta=\frac{\pi}{2}\,, \tag{4–6}$$

so that

$$\tau_B=(\sigma z_h)e^{i\pi/2}U\,. \tag{4–7}$$

Furthermore, from (3–20) and (4–1), it follows,

$$\frac{\hat{\bar{u}}}{U}=\left\{\frac{\alpha_1^2+\alpha_2^2}{\alpha_1^2+(\alpha_2-\eta)^2}\right\}^{1/2},$$

so that $\hat{\bar{u}}/U\simeq 1$ for small values of η and $\hat{\bar{u}}/U\simeq\eta^{-1}$ for large values of η. The phase difference between U and u is given by $\delta+\theta$.

As shown in Fig. 4a, the ratio $\hat{\bar{u}}/U$ of the mean velocity to the velocity with no friction stays almost unity for small values of $U/(\sigma z_h)$ but decreases for increasing values of $U/(\sigma z_h)$ as is expected from the

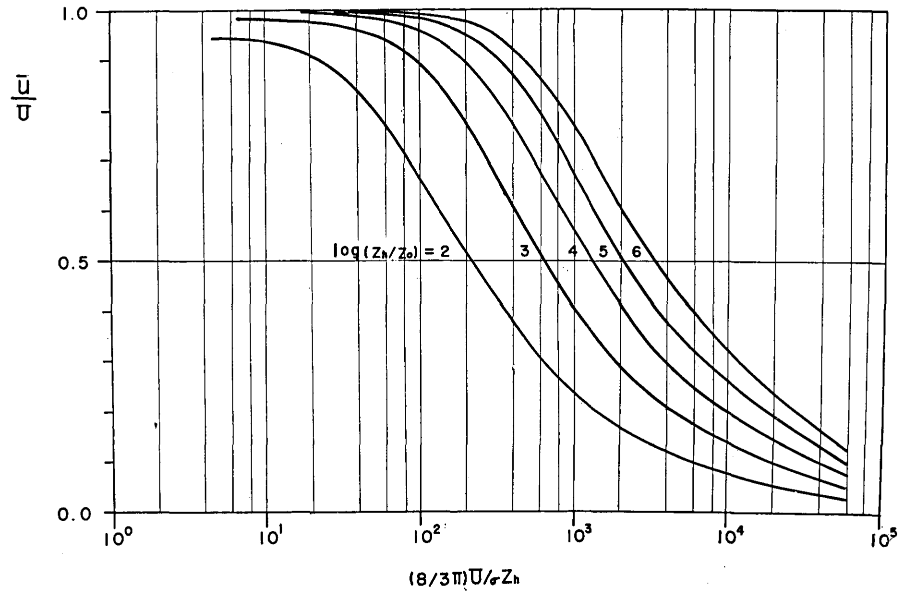

Fig. 4a. The ratio of the depth-mean velocity $\bar{u}$ to the velocity U for the case of no friction as a function of $U/(\sigma z_h)$ with z_h/z_0 as a parameter.

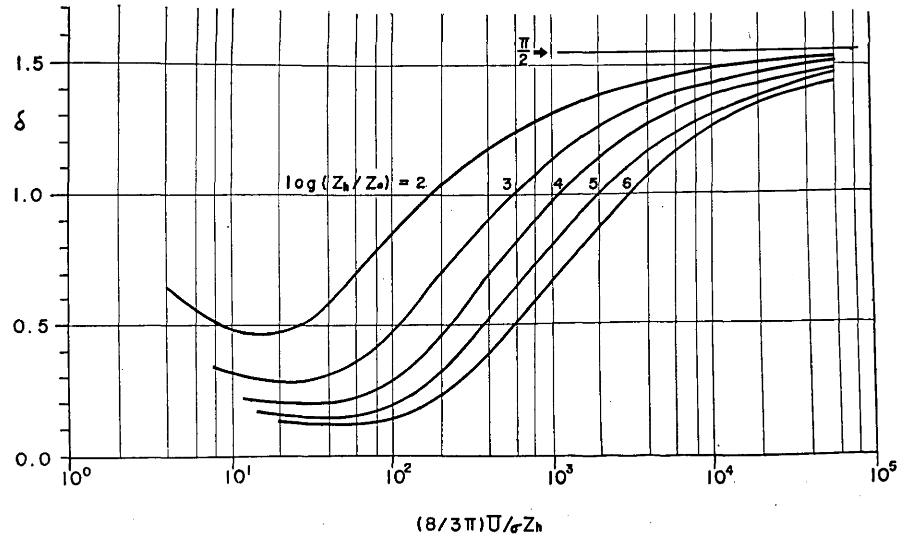

Fig. 4b. Phase difference δ between $v_B{}^*$ and U.

simple consideration of the frictional effect. In other words, in the dynamical equation of motion the acceleration of water dominates over friction for, say, $U/(\sigma z_h)<10^2$ as in the case of ordinary gravity waves without friction, but, with the increase of $U/(\sigma z_h)$, the frictional effect becomes increasingly important and in the limiting case of $U/(\sigma z_h)>10^5$ the wave is in a state of balance between the pressure gradient force and the frictional force. Fig. 4 b shows the phase difference δ, between U and u_B^*, which varies form $|\theta|$ to $\pi/2$ with increasing $U/(\sigma z_h)$. Taking Fig. 2 b into consideration, the phase difference between U and $\bar{u}$ increases from zero to $\pi/2$ with the increase of $U/(\sigma z_h)$.

It is understood that the dynamics of the unsteady river flow such as the flood wave can be treated as a quasi-steady state if $U/(\sigma z_h)$ is very large and tidal waves in rivers may be considered as half inertial and half frictional to balance the pressure gradient.

5. The frictional coefficient for the case of a turbulent flow over a smooth bottom

For a smooth boundary, some modification of the results in the previous section is necessary, because the bottom surface is located at $z=-z_0$ and the roughness parameter z_0 should be replaced by the thickness of the laminar sub-layer with the kinematic viscosity ν. Thus, (3–16) is replaced by

$$\frac{u}{u_B^*}=\frac{u_B}{u_B^*}+\frac{1}{k}\frac{Z_0(y_0e^{-i\pi/4},\, c_h)-Z_0(ye^{-i\pi/4},\, c_h)}{(y_0/2)e^{-i\pi/4}\, Z_1(y_0e^{-i\pi/4},\, c_h)}\,, \text{ for } y>y_0\,, \tag{5–1}$$

where u_B is the velocity at the top of the laminar sub-layer: $y=y_0$.

Within the laminar sub-layer, we may put approximately,

$$\nu\frac{u_B}{z_0}=\tilde{u}_B^* u_B^*\,, \quad \text{for} \quad 0\leqslant y\leqslant y_0\,, \tag{5–2}$$

so that

$$\frac{u_B}{u_B^*}=\frac{\tilde{u}_B^* z_0}{\nu}=N\,. \tag{5–3}$$

The Reynolds number N defined for the laminar sub-layer may be assumed constant and estimated to be 11.6 on the basis of an experimental formula of the velocity profile for a smooth circular tube[4].

Taking (5–3) into consideration, and neglecting the contribution of the velocity in the laminar sub-layer to the mean velocity, we may write in place of (3–20),

$$\frac{\bar{u}}{u_B^*}=(\alpha_1+N)+i\alpha_2=Be^{i\varphi}\,, \qquad (5\text{–}4)$$

where

$$\left.\begin{aligned} &B=\sqrt{(\alpha_1+N)^2+\alpha_2^2}\,,\\ &\varphi=Tan^{-1}\{\alpha_2/(\alpha_1+N)\}\leqslant 0\,. \end{aligned}\right\} \qquad (5\text{–}5)$$

and

Therefore we have, in place of (3–24), for the frictional coefficient C_s,

$$C_s=B^{-2}e^{-i\varphi}\,. \qquad (5\text{–}6)$$

Two limiting cases of y_h are easily derived as follows:

$$\left.\begin{aligned} &B=\frac{1}{k}\left\{2\log\left(\frac{y_h}{y_0}\right)-\frac{3}{2}\right\}+N\,,\\ &\varphi=0\,, \end{aligned}\right\}\ \text{for small } y_h\,, \qquad (5\text{–}7)$$

and

4) For small values of y, we may derive in place of (3-17),

$$\frac{u}{u_B{}^*}=\frac{1}{k}\log\left(\frac{z'}{z_0}\right)+N\,,\quad z'>z_0\,,$$

and substituting the relation (5-3), we have

$$\frac{u}{u_B{}^*}=5.75\log_{10}\left(\frac{z'\tilde{u}_B{}^*}{\nu}\right)+(N-5.75\log_{10}N)\,.$$

Comparing the result with the Nikuradse's experimental formula for a smooth circular tube, namely,

$$\frac{u}{u_B{}^*}=5.75\log_{10}\left(\frac{z'u_B{}^*}{\nu}\right)+5.5\,,$$

we have

$$N=11.6\quad(\tilde{u}_B{}^*\text{ is put equal to }u_B{}^*)\,.$$

Strictly speaking, N may also be a function of σ for an oscillatory flow but we assume tentatively N constant.

Returning to the definition of the eddy viscosity (2-14), we have at the top of the laminar sub-layer,

$$K_z=k\tilde{u}_B{}^*z_0=kN\nu\simeq4.64\,\nu\,.$$

Thus, we have implicitly assumed that the viscosity jumps to the eddy viscosity of about 5 times the molecular values at $z=z_0$, so that the velocity gradient is not continuous, although the velocity itself and stress are continuous at the top of the laminar sub-layer.

$$\left.\begin{aligned} B&=\left(\frac{\pi}{k}\right)\left[\left(\frac{2}{\pi}\right)^2\left\{\frac{Nk}{2}-\left(\gamma+\log\frac{y_0}{2}\right)\right\}^2+\frac{1}{4}\right]^{1/2}, \\ \varphi&=Tan^{-1}\left[-\frac{\pi}{4}\Big/\left\{\frac{Nk}{2}-\left(\gamma+\log\frac{y_0}{2}\right)\right\}\right], \end{aligned}\right\} \text{for large } y_h . \tag{5-8}$$

Making use of the Reynolds numbers such as

$$L=\frac{\tilde{u}_B^* z_h}{\nu}\,; \qquad M=\frac{\tilde{u}_B^*\delta}{\nu}\,, \tag{5-9}$$

with

$$\beta=\delta^{-1}=\sqrt{\sigma/(2\nu)}\,, \tag{5-10}$$

we have

$$y_0=2\sqrt{2N/k}\,\beta z_h/L=2\sqrt{2N/k}/M\,,$$

and

$$y_h=2\sqrt{2/k}\,\beta z_h/\sqrt{L}\,,$$

so that (5–7) becomes

$$\left.\begin{aligned} B&=5.75\log_{10}L+1.75\,, \\ \varphi&=0\,, \end{aligned}\right\} \tag{5-11}$$

and (5–8) becomes

$$\left.\begin{aligned} B&=\left(\frac{\pi}{k}\right)\left[\left(\frac{2}{\pi}\right)^2\left\{\frac{Nk}{2}-\gamma-\log\sqrt{2N/k}+\log M\right\}^2+\frac{1}{4}\right]^{1/2}, \\ \varphi&=Tan^{-1}\left[-\frac{\pi}{4}\left\{\frac{Nk}{2}-\gamma-\log\sqrt{2N/k}+\log M\right\}^{-1}\right]. \end{aligned}\right\} \tag{5-12}$$

(5–11) together with (5–6) shows that for small values of y_h, say $\beta z_h<0.2\,L^{1/2}$, the frictional coefficient C_s is a function of a Reynolds number L only, and is equivalent to the case of the steady turbulent flow over a smooth bottom. On the other hand (5–12) shows that for large values of y_h, say $\beta z_h>0.6\,L^{1/2}$, the frictional coefficient C_s is a function of a Reynolds number M only[5] and the depth of water z_h is not an important factor in the determination of C_s.

---

5) A different definition of a Reynolds number is possible, for example,

$$R=\tilde{u}_B{}^*l/\nu\,, \quad l=2\tilde{u}_B{}^*/\sigma\,,$$

where l is a kind of excursion distance defined by the friction velocity. Then, it is easily seen that

$$R=M^2\,.$$

From a practical standpoint, it seems to be easier to understand the relation by taking a Reynolds number based on the mean velocity instead of the friction velocity. From (5–9) and (5–4) we may put

$$\left.\begin{aligned} L^* &= \tilde{\bar{u}} z_h/\nu = BL\,, \\ \text{and} \qquad & \\ M^* &= \tilde{\bar{u}}\delta/\nu = BM\,. \end{aligned}\right\} \tag{5–13}$$

In particular, for two limiting cases of y_h, this conversion can be made without regard to βz_h.

In Fig. 5 a and Fig. 5 b, $\hat{C}_s$ and φ are shown as a function of L^* with βz_h as a parameter. For $\beta z_h < L^{*1/2}$, the curves of different βz_h converges to a single curve and $|\varphi|$ approaches zero, indicating the dependence of C_s only on L^*. For $\beta z_h > L^{*1/2}$ the effect of the period of oscillation (βz_h) shows up distinctly in this representation. Another representation of the relation is made with the aid of the Reynolds number M^* in Figs. 6 a, 6 b which do not depend on βz_h explicitly for large values of $\beta z_h/L^{*1/2}$ and are simpler than Figs. 5 a and 5 b.

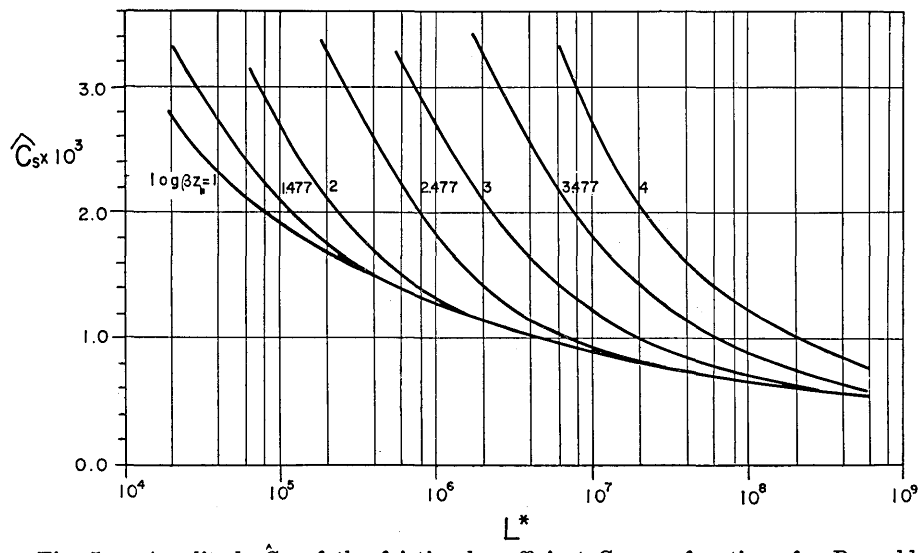

Fig. 5 a. Amplitude $\hat{C}_s$ of the frictional coefficient C_s as a function of a Reynolds number L^* for the oscillating turbulent flow over a smooth surface.

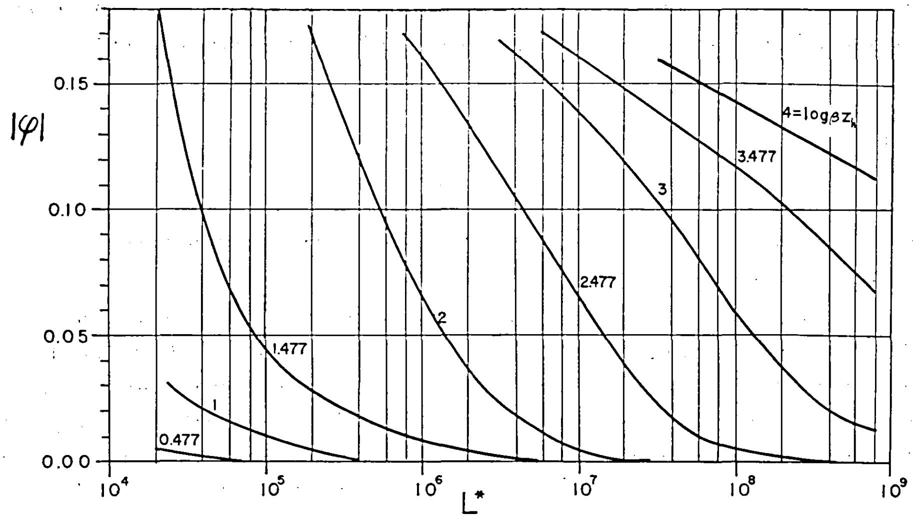

Fig. 5b. Phase φ of the frictional coefficient C_s as a function of a Reynolds number L^* for the oscillating turbulent flow over a smooth surface.

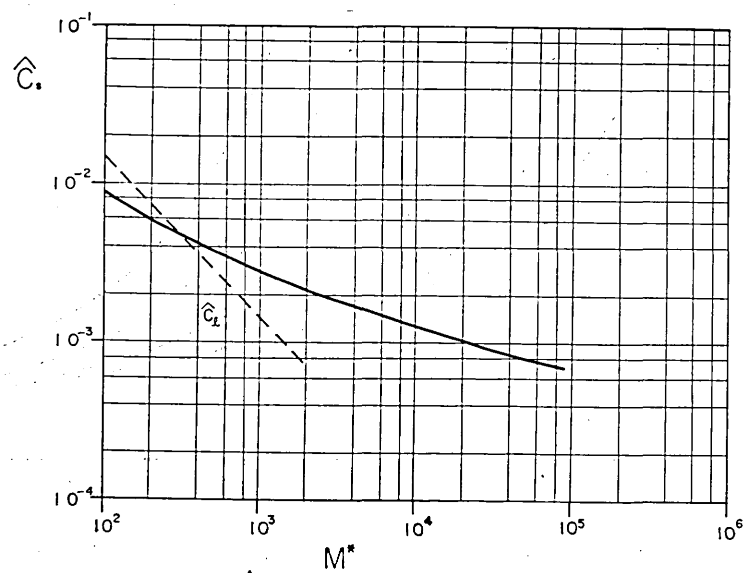

Fig. 6a. Amplitude $\hat{C}_s$ of the frictional coefficient C_s as a function of a Reynolds number M^* for large values of $\beta z_h(>0.6\sqrt{L})$. Broken line indicates the amplitude of the laminar frictional coefficient C_l.

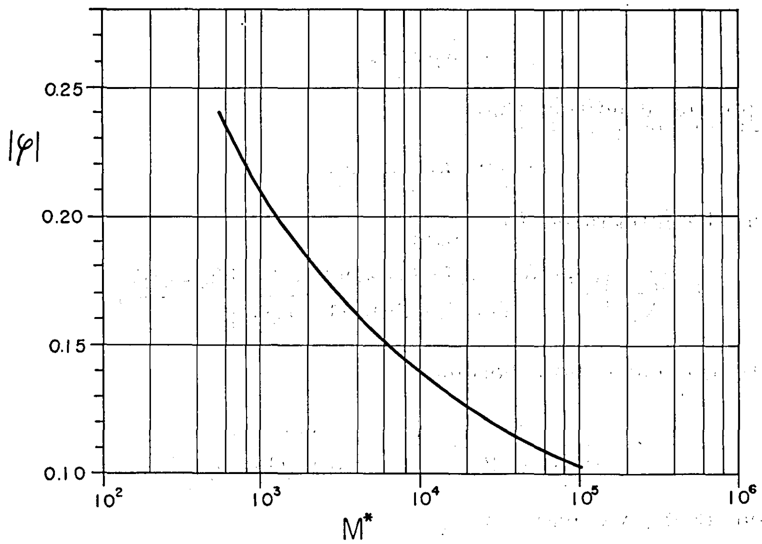

Fig. 6 b. Phase φ of the frictional coefficient C_s as a function of a Reynolds number M^* for large values of $\beta z_h(>0.6\sqrt{L})$.

6. The frictional coefficient for the case of a laminar flow

Although the frictional damping of water waves for the case of a laminar flow has been well investigated theoretically (Proudman and Doodson, 1924; Biesel, 1949), it may be worth-while to discuss the problem from a viewpoint of the frictional coefficient.

For the case of a laminar flow, with the kinematic viscosity ν we may put in place of (2–16),

$$\frac{\partial u}{\partial z}=\frac{\tau}{\nu}\,. \tag{6–1}$$

Then, the equation corresponding to (3–2) is

$$\frac{\partial^2\tau}{\partial z^2}-\frac{i\sigma}{\nu}\tau=0\,. \tag{6–2}$$

The solution of (6–2) under the stress free boundary condition at the surface ($z=z_h$) is given by

$$\frac{\tau}{\tau_B}=\frac{\sinh\{(1+i)\beta(z_h-z)\}}{\sinh\{(1+i)\beta z_h\}}\,, \tag{6–3}$$

where

$$\beta=\sqrt{\sigma/(2\nu)}\,.$$

Under the condition that

$$u=0 \quad \text{at} \quad z=0\,,$$

(6–1) may be integrated to yield

$$u=\left(\frac{\tau_B}{\nu}\right)\frac{\cosh\{(1+i)\beta z_h\}-\cosh\{(1+i)\beta(z_h-z)\}}{(1+i)\beta\sinh\{(1+i)\beta z_h\}}\,. \tag{6–4}$$

Therefore the mean flow becomes,

$$\bar{u}=\frac{i\tau_B}{(\nu\sigma)^{1/2}\sqrt{2}\,\beta z_h}\left[1-\frac{(1+i)\beta z_h}{\tanh\{(1+i)\beta z_h\}}\right]. \tag{6–5}$$

Now from (2–4), we may put

$$U=\bar{u}+\frac{\tau_B}{i\sigma z_h}\,, \tag{6–6}$$

so that by substitution of (6–5) into (6–6) we have

$$\tau_B=(\nu\sigma)^{1/2}\tanh\{(1+i)\beta z_h\}e^{i\pi/4}U\,. \tag{6–7}$$

Therefore, from (6–5) the mean velocity is given by

$$\bar{u}=\left[1-\frac{\tanh\{(1+i)\beta z_h\}}{(1+i)\beta z_h}\right]U\,, \tag{6–8}$$

and from (6–4) the surface velocity u_S becomes

$$u_S=\left[1-\frac{1}{\cosh\{(1+i)\beta z_h\}}\right]U\,. \tag{6–9}$$

Formally, we may write (6–7), (6–8) and (6–9) in the form :

$$\tau_B/U=(\nu\sigma)^{1/2}ae^{i\theta_1}\,, \tag{6–10}$$

$$\bar{u}/U=be^{i\theta_2}\,, \tag{6–11}$$

$$u_S/U=ce^{i\theta_3}\,, \tag{6–12}$$

and, it follows

$$\tau_B/\bar{u}=(\nu\sigma)^{1/2}De^{i\theta_4}\,, \tag{6–13}$$

with

$$D=a/b\,, \quad \text{and} \quad \theta_4=\theta_1-\theta_2\,. \tag{6–14}$$

For large values of βz_h, it is found that

$$\theta_1=\pi/4\,, \quad \text{and} \quad a=1\,; \quad \theta_2=0\,, \quad \text{and} \quad b=1\,, \tag{6–15}$$

so that (6–10) becomes

$$\tau_B/U=(\nu\sigma)^{1/2}e^{i\pi/4}\,, \tag{6–16}$$

and (6–14) becomes

$$D=1\,, \quad \text{and} \quad \theta_4=\pi/4\,. \tag{6–17}$$

For small values of βz_h, we have

$$\theta_1=\pi/2\,, \quad \text{and} \quad a=\sqrt{2}\,\beta z_h\,; \quad \theta_2=\pi/2\,, \quad \text{and} \quad b=3/2(\beta z_h)^2 \tag{6–18}$$

so that (6–10) becomes

$$\tau_B/U=(\sigma z_h)e^{i\pi/2}\,, \tag{6–19}$$

and (6–13) becomes

$$\tau_B/\bar{u}=(\nu\sigma)^{1/2}3/(\sqrt{2}\,\beta z_h)=3\nu/z_h\,, \tag{6–20}$$

and

$$D=3/\{\sqrt{2}\,\beta z_h\}\,, \quad \text{and} \quad \theta_4=0\,.$$

Fig. 7 gives a and θ_1 of (6–10) as a function of βz_h which shows

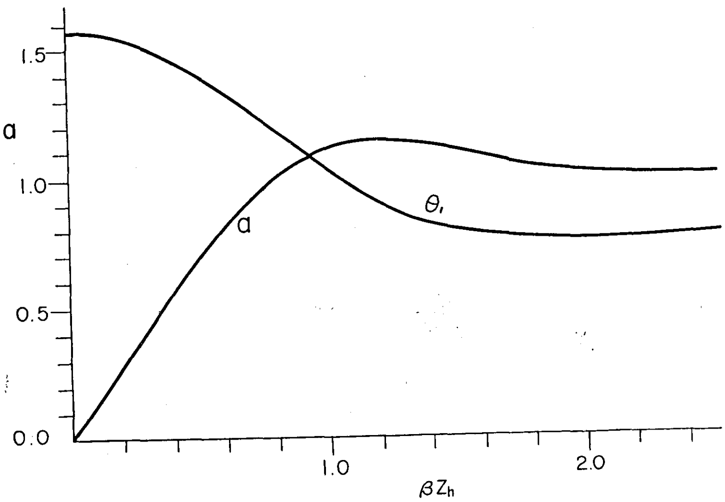

Fig. 7. a and θ_1 as a function of βz_h for the laminar oscillating flow.

that a increases from 0 to 1 and the phase angle decreases from $\pi/2$ to $\pi/4$ with increasing βz_h. Fig. 8 shows b, c and θ_2, θ_3 as functions of βz_h in which $|\bar{u}/u_s|$ is also shown. From the figure, it is found that the ratio of the mean current to the surface current varies from 1 to 0.66 and the minimum value lies somewhere around $\beta z_h \sim 1.0$. The

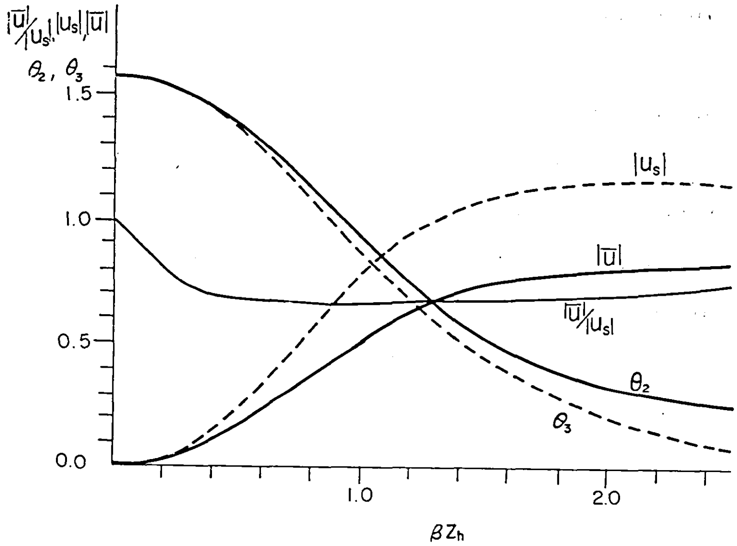

Fig. 8. Amplitudes (non-dimensional) and phases of the mean velocity and surface velocity, and the ratio of the mean velocity to the surface velocity as a function of βz_h for the laminar oscillating flow.

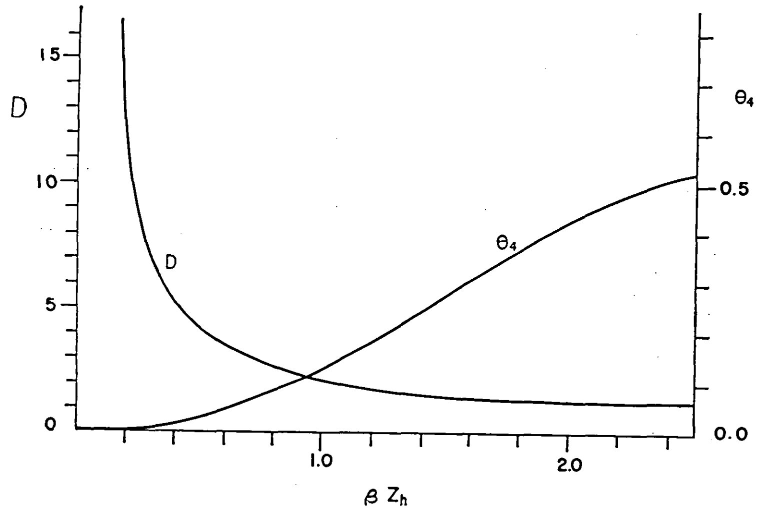

Fig. 9. D and θ_4 as a function of βz_h for the laminar oscillating flow.

phase angle relative to U decreases from $\pi/2$ to 0, but the difference of θ_2 and θ_3 is always small. Fig. 9 is a graph of D and θ_4 of (6–13) as a function of βz_h which shows that D decreases and θ_4 increases with increasing values of βz_h.

The frictional coefficient C_l defined by

$$\tau_B = C_l\left(\frac{8}{3\pi}\right)\hat{\bar{u}}\bar{u}\,, \tag{6–21}$$

can be written with the aid of (6–13) in the form,

$$C_l = \left(\frac{3\pi}{8}\right)\frac{(\nu\sigma)^{1/2}}{\hat{\bar{u}}}De^{i\theta_4}\,. \tag{6–22}$$

Making use of the Reynolds number L^* or M^*, defined in (5–13), we have

$$C_l = \sqrt{2}\,\beta z_h D e^{i\theta_4}/L^*\,, \tag{6–23}$$

or

$$C_l = \sqrt{2}\,De^{i\theta_4}/M^*\,. \tag{6–24}$$

Therefore, the frictional coefficient is inversely proportional to the Reynolds number and also a function of βz_h. In particular, for large values of βz_h, $C_l = \sqrt{2}\,M^{*-1}e^{i\pi/4}$, (c. f., Fig. 6) and for small values of βz_h, $c_l = 3/L^*$.

7. Application of the laminar frictional formula

The advantage of writing the bottom stress in the form (6–10) or (6–13) may be seen in the following discussions.

If we are to solve (2–8) for the periodic wave in time (progressive waves), $\zeta = \zeta_0 e^{i(\sigma t - mx)}$, by introducing the relations (2–2) and (6–10) we have the characteristic equation as follows:

$$m_0^2 = m^2(1 + iEe^{i\theta_1})\,, \tag{7–1}$$

where

$$gz_h = c_0^2\,,\quad m_0 = \sigma/c_0\,,\quad \text{and}\quad E = a/(\sqrt{2}\,\beta z_h).$$

Since σ is assumed real, we have

$$m = m_1 + im_2 = m_0(1 + iEe^{i\theta_1})^{-1/2}\,. \tag{7–2}$$

For small friction, the solution may be reduced to

$$m_1 = m_0\left(1 + \frac{1}{2}E \sin\theta_1\right), \tag{7-3}$$

and

$$m_2 = -\frac{m_0}{2}E\cos\theta_1 . \tag{7-4}$$

Therefore, the phase velocity c is given by

$$c = c_0\left(1 - \frac{1}{2}E \sin\theta_1\right). \tag{7-5}$$

In particular, for large values of βz_h, $a=1$ and $\theta_1 = \pi/4$ so that

$$\frac{m_2}{m_0} = -\frac{1}{4\beta z_h}, \tag{7-6}$$

and

$$\frac{c}{c_0} = 1 - \frac{1}{4\beta z_h}. \tag{7-7}$$

For free seiches in an enclosed basin, the wave decays with time so that the discussion should be modified by putting $\sigma = \sigma_1 + i\sigma_2 = \sigma^* e^{i\varepsilon}$. The essential change lies in the estimation of tanh $(1+i)\beta z_h$ in which β is no longer real but $\beta^* e^{i\varepsilon/2}$ with $\beta^* = \sqrt{\sigma^*/(2\nu)}$ and $\varepsilon = \tan^{-1}(\sigma_2/\sigma_1)$. However, for the case of small friction, we may assume

$$(\beta^* z_h)\cdot\frac{1}{2}\tan^{-1}(\sigma_2/\sigma_1) \ll 1,$$

and may use β^* in place of β. Then, for waves periodic in space (standing waves), $\zeta = \zeta_0 \cos mx\, e^{i\sigma t}$, the characteristic equation becomes

$$\sigma^2 = \sigma_0^2(1 + iE^* e^{i\theta_1}), \tag{7-8}$$

where $\sigma_0 = c_0 m$ and $E^* = a/(\sqrt{2}\beta^* z_h)$. For small values of E^*, we have

$$\sigma_1 = \sigma_0\left(1 - \frac{1}{2}E^* \sin\theta_1\right), \tag{7-9}$$

and

$$\sigma_2 = \frac{\sigma_0}{2}E^*\cos\theta_1 . \tag{7-10}$$

For large values of $\beta^* z_h$, $a=1$ and $\theta_1 = \pi/4$, so that

$$\frac{\sigma_1}{\sigma_0}=1-\frac{1}{4\beta^* z_h}, \tag{7-11}$$

and

$$\frac{\sigma_2}{\sigma_0}=\frac{1}{4\beta^* z_h}. \tag{7-12}$$

These results are the same as the results given by Biesel (1949), at least for shallow water waves with $\beta z_h \gg 1$[6]. Furthermore, for $\beta z_h \gtrsim 2$, (7–11) and (7–12) coincide with the result given by Proudman and Doodson (1924) by a complicated numerical method (See also Defant; Dynamical Oceanography, Vol. 2, p. 159, 1961) as shown by Platzmann and Rao (1964).

The essential parameter for the damping and period increase of seiches is βz_h as in the case of progressive waves and an important characteristic of the laminar friction is that the bottom frictional stress τ_B is not in phase with the mean velocity $\bar{u}$ except for very small values of βz_h, so that, if we put the stress in the form $-f z_h \bar{u}$, the coefficient f is related to the period of oscillation and includes the term of phase difference.

In terms of the horizontal displacement of the water mass ξ, (2–7) can be written as

$$\frac{\partial^2 \xi}{\partial t^2}-g z_h \frac{\partial^2 \xi}{\partial x^2}=-f\frac{\partial \xi}{\partial t}, \tag{7-13}$$

and if f is real and independent of the period T of a seiche, which seems to be realized in the case of a turbulent flow with $\bar{u}/(\sigma z_h)>10^2$, the rate of period increase $\Delta T/T$ is proportional to T^2 and $\bar{u}^2$. However, for the case of a laminar flow with large values of βz_h, $f=(\nu\sigma)^{1/2}e^{i\pi/4}/z_h$[7], and the period increase $\Delta T/T$ is proportional to $T^{1/2}$. The difference of the period increase due to friction for the laminar and turbulent flow is already mentioned by Platzman and Rao (1964).

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6) For small βz_h, his method of approximation is not accurate, because of the development in terms of $\nu^{1/2}$.

7) According to experiments, the actual value of f seems to be several times greater than that derived on the basis of the linearized laminar theory. The discrepancy of the f value between theory and experiment is not yet resolved, although an attempt is made by Grosch (1962) without success to explain the discrepancy by introducing non-linear terms in the equation of the boundary layer.

8. Discussion

The approximation of long waves which is assumed throughout this paper requires that the wave length should be large compared with the depth of water, namely, $mz_h \ll 1$ and $\sigma/m = \sqrt{gz_h}$, where $m = 2\pi/\lambda$. Taking these conditions into consideration,

$$\frac{\bar{u}}{\sigma z_h} = \left(\frac{\bar{u}}{\sqrt{gz_h}}\right)\frac{1}{mz_h} \gtrsim \frac{10\bar{u}}{\sqrt{gz_h}} .$$

Or, in terms of the wave amplitude a, we may write for the case of $\bar{u} \simeq U$,

$$\bar{u} \simeq (a\sigma)/(mz_h) ,$$

and

$$\frac{\bar{u}}{\sigma z_h} = \left(\frac{a}{z_h}\right)\frac{1}{mz_h} \gtrsim \frac{10a}{z_h} .$$

This shows that the approximation of long wave is roughly satisfied for $\bar{u}/(\sigma z_h) > 1$ if $(a/z_h) = 0.1$ (c. f., Fig. 2a, and Fig. 2b).

The critical Reynolds number for the transition from laminar to turbulent boundary layer in an oscillating wave motion is investigated experimentally by several authors. Li (1954) studied the case of an oscillating plate and proposed a Reynolds number for a smooth surface,

$$R_{\text{crit}} = (\sigma/\nu)^{1/2} d' = 800 ,$$

in which d' is the total excursion of water particles relative to the plate just outside the bottom boundary layer. For long waves, this condition is approximately equivalent to $(3\pi/8)M^* = 566$. Vincent (1957) argued on the basis of his experimental results for progressive waves that Li overestimated the range of laminar conditions at least for the case of a rough bottom. Both of these results are estimated from the observation of dye streaks, watching the transition from laminar to turbulent appearance. Recently, based on a different principle to determine the critical Reynolds number, Collins (1963) gave $(3\pi/8)M^* = 160$.

For $\nu = 10^{-2}\,\text{cm}^2\,\text{sec}^{-1}$, $g = 980\,\text{cm sec}^{-2}$, the criterion of Collins can be written as $a \gtrsim 0.75(z_h/T)^{1/2}$ (*c. g. s.* unit). Therefore, a wave of, say, 10 minutes in period may be considered to have a turbulent boundary layer if $a > 1$ cm for the depth of 10 m. This shows that tsunamis and seiches would have a turbulent boundary layer in shallow water even if the bottom is smooth. It may be of some interest to note that $\hat{C}_l$

(laminar) and $\hat{C}_s$ (turbulent) curves (c. f., Fig. 6 a) intersects at about $M^*=300$ for large values of $\beta z_h(\beta z_h>0.6\,L)$.

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7. 時間的に変化する流れに伴なう水底摩擦

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渦動粘性係数を，水底摩擦速度と水底からの高さに比例すると仮定して摩擦係数を計算し，底面が粗および滑らかな場合のそれぞれについて，摩擦係数が流れの振幅，周期，水深に関してどう変るかを適当な無次元量をつかつて図示した．浅海の潮流については，摩擦係数は z_0/z (z: 水深，z_0: 粗度長さ）によつてきまり，z_0/z のもつともらしい値に対して摩擦係数は 2×10^{-3} 程度となるが，周期の短かいセイシュなどでは摩擦係数が大きくなる．層流の場合の摩擦係数も簡単に議論してあり，底の滑らかな場合に，水深がかなりあると，摩擦係数に関係するレイノルズ数は水深 z_h をつかつた $L^*(=\bar{u}z_h/\nu)$ ではなくて境界層の厚さ δ をつかつた M^* $(=\bar{u}\delta/\nu)$ が適当であることが示され，層流摩擦係数と乱流摩擦係数とがほぼ等しくなるのは $M^*=300$ のあたりになる．