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33. *The Leading Wave of a Tsunami.*

By Kinjiro KAJIURA,

Earthquake Research Institute.

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Based on the linear theory, the decay with distance of the first wave of a tsunami in an infinite sea of constant depth is discussed generally. In particular, for the case of a uniform source distributed in a rectangular area (horizontal dimension: major axis $2a$, minor axis $2b$) the decay is approximately proportional to the following power of the distance r, except for a distance close to the source area (for comparison, the case of one dimensional propagation is also shown).

Nature of the the source	Initial surface elevation or sudden bottom deformation		Surface impulse	
	$p_a>3$	$p_a<1$	$p_a>3$	$p_a<1$
One-dimensional propagation	0	$-1/3$	$-1/3$	$-2/3$
Two-dimensional propagation	$-2/3$	-1	-1	$-4/3$

$p_a=(6\sqrt{H/g}/t)^{1/3}(a/H)$, t: time, H: depth, g: acceleration of gravity. For a wave in the direction of the minor axis of the source, a should be replaced by b.

The ratio of the leading wave heights in the directions of the major and the minor axes varies from b/a to 1 depending on the values of p_a and p_b, so that the directivity of the leading wave height due to the elongated source distribution disappears at a very long distance from the source area. The time interval between the leading wave crest and the second one increases in proportion to $t^{1/3}$. The present theory can also be extended to the dispersive wave train in the later phases of a tsunami for an arbitrary source distribution.

1. Introduction

Many theoretical discussions have been advanced, mainly in Japan, to explain the generation of tsunami in water of uniform depth when a portion of the bottom is dislocated vertically.[1),2)] Most of these linear theories are, however, confined to a rather limited scope of wave properties because of mathematical difficulties confronted in the elucidation of formal solutions. Thus, apart from the numerical approach adopted by several authors,[3),4),5),6)] common procedures utilized are either the so-called long wave approximation or, at the other limit, the deep water wave approximation, and the analytical discussions including the intermediate range of wave lengths are quite limited because of the transcendental character of the dispersion curve. For a long distance from the source, however, the behavior of a leading wave can be handled analytically by means of an integral analogous to the Airy integral,[7),8),9),10)] where the approximation involved is essentially similar to the one used to derive the wave equation near the wave front including the effect of curvature of the water surface, and the general characteristics of waves can be estimated from the analysis of this equation.[11),12)] However, there seems to be some misunderstanding of the behavior of the leading wave at a long distance from the source. For example, the discussions of the first wave of a tsunami given by Kranzer and

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1) Japanese Organization for Tsunami Investigation, *The annotated bibliography of tsunamis* (1962), (mimeographed report).

2) B. W. WILSON, L. M. WEBB, and J. A. HENDRICKSON, "The nature of tsunamis, their generation and dispersion in water of finite depth," *NESCO Tech. Report No. SN 57-2* (1962), 146, Appendix 1, 2.

3) K. SANO, and K. HASEGAWA, "On the wave produced by the sudden depression of a small portion of the bottom of a sea of uniform depth," *Bull. Japan Centr. Met. Observ.*, **2** (1915), 30.

4) T. MATUZAWA, "On the tsunami accompanying the earthquake: Pt. 1., Two dimensional problem of incompressible water, Pt. 2., Numerical computation of integral," *Zisin* [ii], **1** (1948), 18-23, [ii], **2** (1949), 33-36, (in Japanese).

5) T. ICHIYE, "A theory on the generation of tsunamis by an impulse at the sea bottom," *J. Oceanogr. Soc. Japan*, **14** (1958), 41-44.

6) *loc. cit.*, 2).

7) K. SEZAWA and K. KANAI, "On the transmission of tsunamis in a sea of any depth," *Bull. Earthq. Res. Inst.* **20** (1942), 254-264, (in Japanese).

8) R. TAKAHASI, "On the seismic sea waves caused by deformation of the sea bottom," *Bull. Earthq. Res. Inst.* **20** (1942), 357-400, (in Japanese).

9) C. ECKART, "The approximate solution of one-dimensional wave equations," *Rev. Mod. Phys.*, **20** (1948), 399-417.

10) *loc. cit.*, 2).

11) G. H. KEULEGAN and G. W. PATTERSON, "Mathematical theory of irrotational translation waves," *J. Res., Natl. Bureau Stds.*, **24** (1940), 47-101.

12) H. JEFFREYS and B. S. JEFFREYS, *Methods of mathematical physics*, (Cambridge Press, Cambridge, England, 1956), 714.

Keller (1959),[13] and Takahasi (1961)[14] are not valid because of their incorrect application of the ordinary method of the stationary phase up to the wave where the Airy Integral should be considered, and also Van Dorn's arguments[15] on the amplitude decay of the maximum wave with distance is not accepted as evidence of good agreement between theory and observation because he did not compare the corresponding observed wave at different locations for which the theory predicts the decay law, and besides no distinction was made of the theoretical results for the one-dimensional and two-dimensional dispersion except for the factor $r^{1/2}$. Similar deficiency of understanding is obvious in Wilson's discussion[16] of the amplitude decay and period increase of the leading wave.

From a general point of view, this kind of problem is a part of the general theory related to the Cauchy-Poisson wave theory concerning the generation and propagation of water waves, and the fundamental properties of linear waves are considered to be well known[17]. However, it seems to be worthwhile to look into the problem anew with expression making use of a time dependent Green's function. By this approach, the dispersive characteristics of generated waves and their relation to the nature of the source can be clearly understood.

In particular, this paper deals with the dependence of the maximum elevation (or depression) of the water surface on the dimension of a source just after the instantaneous bottom deformation, the distinction between the one and two dimensional decay of the leading wave height at a long distance from the source, and the directional difference of wave height generated by a non-axially symmetric wave source. The limitation of the long wave approximation for the leading wave is also discussed. Finally, the method is extended to treat the dispersive wave train in the later phases of a tsunami for an arbitrary source distribution.

2. Fundamental equations and a time-dependent Green's fuction

Assume incompressible water of constant depth H and take origin of the Cartesian co-ordinate (x', y', z') at the undisturbed free surface

13) H. C. KRANZER and J. B. KELLER, "Water waves produced by explosions," *J. Appl. Phys.*, **30** (1959), 398-407.

14) R. TAKAHASI, "On some model experiments on tsunami generation," *Proc. Tsunami Hydrodynamics Conf., Univ. Hawaii* (1961), (Publication pending).

15) W. G. VAN DORN, "Some characteristics of surface gravity waves in the sea produced by nuclear explosions," *J. Geophys. Res.*, **66** (1961), 3845-3862.

16) *loc. cit.*, 2).

17) J. J. STOKER, *Water waves* (Interscience Publishers Inc., New York, 1957), 569.

with the vertical axis z' upwards. The irrotational motion in homogeneous water may be expressed by means of a velocity potential φ where the velocity vector V' is given by $V'=\text{grad}\,\varphi'$. For convenience in the later discussions, physical quantities are written in nondimensional form unless otherwise noted by putting the independent variables as follows:

$$x=x'/H\,,\quad y=y'/H\,,\quad \text{and}\quad t=t'\sqrt{g/H}\,,$$

(quantities with prime are to be the original form)

where t' is time and g is the acceleration due to gravity. The nondimensional form of the derived quantities are:

surface elevation: $\eta=\eta'/H$

velocity : $\mathrm{V}=\mathrm{V}'/\sqrt{gH}$

potential : $\varphi=\varphi'/(H\sqrt{gH})$

pressure: : $p=(p'/\rho)/(gH)$

where ρ is density of water, p' is the anomaly of the surface atmospheric pressure from the mean value, and the vertical component of V is written as w.

Within the limit of linear approximation (deformations z' at the surface and at the bottom are assumed small compared with the wave length λ' and the depth of water H together with the condition $z'\lambda'^2/H^3 \lesssim 1$, so that the boundary conditions are satisfied at the undisturbed surfaces), the kinematic and dynamic conditions at the free surface are,

$$\left.\begin{aligned}\varphi_z&=\eta_t\,, \qquad (1)\\ \varphi_t&=-\eta-p\,, \qquad (2)\end{aligned}\right\}\quad \text{for } z=0\,,$$

and the bottom condition is,

$$\varphi_z=w_B\,,\qquad \text{for } z=-1\,, \qquad (3)$$

where w_B is the assumed bottom velocity corresponding to the bottom deformation and the partial differentiations are abbreviated by letter subscripts of the respective variables.

To find a velocity potential φ satisfying (1), (2), and (3) together with suitable initial conditions, it is advantageous to derive a time-dependent Green's function G which should be a harmanic function in the variable (x, y, z) with a singularity of appropriate character at a certain point (x_0, y_0, z_0) introduced at the time $t=\tau$. Hence,

$$G \equiv G(x_0, y_0, z_0;\ \tau | x, y, z;\ t)$$

and G should be a solution of the Laplace equation,

$$\nabla^2 G = 0\ , \quad \text{for} \quad 0 > z > -1\ , \quad t \geq \tau\ , \tag{4}$$

satisfying the free surface condition

$$G_{tt} + G_z = 0\ , \quad \text{for} \quad z = 0\ , \tag{5}$$

and a bottom condition

$$G_z = 0\ , \quad \text{for} \quad z = -1\ . \tag{6}$$

At $x, y \to \infty$, we require G, G_t, and their first derivatives with respect to space coordinates to be uniformly bounded at any given time t. At the point (x_0, y_0, z_0) we require $(G - 1/R)$ to be bounded where $R^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$. As initial conditions at the time $t=\tau$, we assume

$$G = G_t = 0\ , \quad \text{for} \quad z = 0\ . \tag{7}$$

These conditions determine G uniquely.

Following the similar line described by Stoker[18], the Green's function for the case of three dimensional motion in water of finite depth can be derived: for $0 > z,\ z_0 > -1$,

$$\begin{aligned} G(x_0, y_0, z_0;\ \tau | x, y, z;\ t) &= \int_0^\infty \frac{J_0(m\bar{r})}{\cosh m} \Big[\sinh m\{1 - |z - z_0|\} - \sinh m\{1 + (z + z_0)\} \\ &+ \frac{2}{\gamma^2}\{1 - \cos\gamma(t-\tau)\} \frac{m}{\cosh m} \cosh m(1+z) \cosh m(1+z_0) \Big] dm, \end{aligned} \tag{8}$$

where $\bar{r}^2 = (x-x_0)^2 + (y-y_0)^2$ and $\gamma^2 = m \tanh m$. It is evident from (8) that G is symmetric with respect to $(x_0, y_0, z_0,)$ and $(x, y, z,)$ and also t and τ, *i. e.*:

$$G(x_0, y_0, z_0;\ \tau | x, y, z;\ t) \equiv G(x, y, z;\ t | x_0, y_0, z_0;\ \tau) \equiv G(x_0, y_0, z_0;\ t | x, y, y;\ \tau)\ .$$

Making use of the Green's formula together with the above Green's function, we may write

18) *loc. cit.*, 17) 187-196.

$$\varphi_\tau(x, y, z; \tau)=\frac{1}{4\pi}\iint_S (G\varphi_{\tau z_0}-\varphi_\tau G_{z_0})_{z_0=0}dS_0-\frac{1}{4\pi}\iint_S (G\varphi_{\tau z_0}-\varphi_\tau G_{z_0})_{z_0=-1}dS_0 , \tag{9}$$

where $dS_0=dx_0dy_0$ and integral on the lateral boundary in water vanishes because of the condition imposed on G.

The integration of (9) with respect to τ from 0 to t and the substitution of the conditions (1), (2), (5), (6), and (7) for φ and G yield,

$$\varphi(x, y, z; t)-\varphi(x, y, z; 0)$$
$$=-\frac{1}{4\pi}\iint_S\Big[(G\varphi_{z_0}-G_\tau\eta)_{\tau=0}+\int_0^t pG_{\tau\tau}d\tau\Big]_{z_0=0}dS_0-\frac{1}{4\pi}\iint_S\int_0^t (G\varphi_{z_0\tau})_{z_0=-1}d\tau dS_0 . \tag{10}$$

The first integral shows the contribution from the surface condition and the second one from the bottom condition.

At the surface $z=0$, (8) is simplified to give

$$G(x_0, y_0, z_0; \tau|x, y, 0; t)$$
$$=\int_0^\infty \frac{2}{\gamma^2}\{1-\cos\gamma(t-\tau)\}m\cosh m(1+z_0)\frac{J_0(m\bar{r})}{\cosh m}dm , \tag{11}$$

and since the surface elevation η is given by (2), the substitution of (10) into (2) yields

$$(\eta+p)=\frac{1}{4\pi}\iint_S (F_1+F_2+F_3)dS_0 , \tag{12}$$

where

$$F_1=(G_t\varphi_{z_0}-G_{\tau t}\eta)_{\tau=0} ; \qquad z_0=0 , \qquad z=0 , \tag{13}$$

and

$$F_2=\int_0^t pG_{\tau\tau t}d\tau+(pG_{\tau\tau})_{\tau=t} , \tag{14}$$

or alternately,

$$F_2=-\int_0^t p_\tau G_{\tau t}d\tau-(pG_{\tau t})_{\tau=0} ; \qquad z_0=0 , \quad z=0 , \tag{15}$$

and

$$F_3=\int_0^t G_t\varphi_{z_0\tau}d\tau , \tag{16}$$

or

$$F_3=-\int_0^t G_{t\tau}\varphi_{z_0}d\tau-(G_t\varphi_{z_0})_{\tau=0}\ ; \qquad z_0=-1\ , \quad z=0\ . \tag{17}$$

It is clear that (13) is the contribution of the initial velocity and elevation of water at the surface, (14) or (15) is the contribution of the the surface pressure, and (16) or (17) is the contribution from the bottom deformation.

The expression (12) together with (13) to (17) is quite general and, in essence, represents the application of the principle of superposition in a linear system starting from a point source solution. Since G can be computed without regard to external conditions at the source, it may be quite suitable for numerical computation.

For certain special source conditions with respect to τ, (13) to (17) can be simplified as follows with the aid of the Dirac's Delta function $\delta(\tau)$[19]:

(a) water is initially at rest with initial elevation H_S;

$$F_1=-H_S G_{\tau t}\ , \quad (\tau=0\ ,\ z=0\ , \quad z_0=0) \tag{18}$$

(b) water is initially at rest and applied pressure at the surface is impulsive at $\tau=0^+$, namely $p=I_S\delta(\tau)$;

$$F_2=I_S G_{\tau\tau t}\ , \quad (\tau=0\ , \quad z=0\ , \quad z_0=0) \tag{19}$$

(c) water is initially at rest and the deformation of the bottom is completed instantaneously at $\tau=0^+$ with the total deformation H_B, namely $w_B=H_B\delta(\tau)$;

$$F_3=-H_B G_{t\tau}\ , \quad (\tau=0\ , \quad z=0\ , \quad z_0=-1) \tag{20}$$

(d) water is initially at rest and uniform velocity of bottom deformation is given for a time interval $0<\tau<\tau^*$, with the total deformation H_B, namely $(w_B)_\tau=H_B\{\delta(0)-\delta(\tau^*)\}/\tau^*$;

$$F_3=H_B\{G_t(\tau=0)-G_t(\tau=\tau^*)\}/\tau^*\ , \quad (z=0\ , \quad z_0=-1) \tag{21}$$

(e) water is initially at rest and impulsive bottom motion is given at $\tau=0^+$ with no net deformation of the bottom after $\tau>0$, namely

$$\int_0^\tau w_B d\tau=I_B\delta(\tau)\ ,$$

19) I. N. SNEDDON, *Fourier Transform* (McGraw-Hill Book Co., Inc., New York 1951), 542.

where I_B is the maximum deformation of the bottom;
After slight modification of (17), we have

$$F_3 = I_B G_{t\tau\tau}\ ,\quad (\tau=0\ ,\quad z=0\ ,\quad z_0=-1)\ . \tag{22}$$

It may be possible to represent other kinds of source conditions in simplified forms too; for example, an atmospheric pressure disturbance such as a pressure jump line moving with the constant velocity can be simplified, but further discussions will not be attempted here.

The comparison of (a) to (e) shows that the instantaneous deformation of the bottom (c) is analogous to a given initial elevation (a), and the bottom impulse (e) can be treated as a surface pressure impulse (b). The difference lies only in the evaluation of the Green's function G at the surface ($z_0=0$) or at the bottom ($z_0=-1$) which amounts to the decrease of high frequency components by a factor $1/\cosh m$ for the bottom source. As will be shown later in the evaluation of the Green's function, the leading wave form for a large distance from the source is determined mainly by component waves of very low frequencies γ ($\gamma<\gamma^*<1$) so that $1/\cosh m$ is almost one and (a) and (c); (d) and (e) become identical and furthermore, (d) is reduced to (c) provided $\gamma^*\tau^*\ll 1$.

For simplicity, we write,

$$\eta(x,\ y,\ t) = \begin{cases} \iint_S H_S P dS_0\ , & (23) \\ \iint_S I_S Q dS_0\ , & (24) \\ \iint_S H_B R dS_0\ , & (25) \\ \iint_S H_B \{S(\tau=0)-S(\tau=\tau^*)\}/\tau^* dS_0\ , & (26) \end{cases}$$

for the cases of initial elevation H_S, initial impulse I_S, sudden elevation of the bottom H_B, uniform velocity of the bottom deformation H_B/τ^*, respectively. The functions P, Q, R, and S are considered to be water waves generated by point sources of the particular characters at $\tau=0$ and are given by evaluating the corresponding form of the Green's function (11) as follows:

$$P = \frac{1}{2\pi}\int_0^\infty \cos\gamma t\cdot m J_0(m\bar{r})dm\ , \tag{27}$$

$$Q=P_t\ , \tag{28}$$

$$R=S_t\ , \tag{29}$$

$$S=\frac{1}{2\pi}\int_0^\infty \sin\gamma t\cdot mJ_0(m\bar{r})/(\gamma\cosh m)dm\ , \tag{30}$$

where $\bar{r}=(x-x_0)^2+(y-y_0)^2$, and for $S(\tau=\tau^*)$, t should be replaced by $\bar{t}(=t-\tau^*)$.

The central problem of the wave generation theory in the present formulation is to evaluate P, Q, R, and S as well as possible. Now, returning from (23) to (26), it is easy to show that these expressions comform to the usual expressions derived on the basis of the Fourier-Bessel transform of a potential function and source conditions from the beginning (Appendix I). The advantage of the present formulation may be seen in the separation of the source distribution and the wave dispersion characteristics of the medium.

The Green's function G^* in the two-dimensional motion (x, z plane) can be derived essentially by the similar method in which the singularity imposed on a fluid should be of the type such as (G^*-lnR^*) to be bounded. In (8), $J_0(m\bar{r})$ is then replaced by $(\cos m\bar{x})/m$ where $\bar{x}^2=(x-x_0)^2$. In the application of the Green's formula in (9), the line integral should be taken instead of the surface integral and $1/(4\pi)$ is replaced by $1/(2\pi)$. The subsequent deduction is the same and the result can be expressed by

$$\eta(x,\ t)=\begin{cases}\int H_S P^* dx_0\ , & (31)\\ \int I_S Q^* dx_0\ , & (32)\\ \int H_B R^* dx_0\ , & (33)\\ \int H_B\{S^*(\tau=0)-S^*(\tau=\tau^*)\}/\tau^* dx_0\ , & (34)\end{cases}$$

where

$$P^*=\frac{1}{\pi}\int_0^\infty \cos\gamma t\cdot\cos m\bar{x}dm\ , \tag{35}$$

$$Q^*=P_t^*\ , \tag{36}$$

$$R^*=S_t^*\ , \tag{37}$$

$$S^*=\frac{1}{\pi}\int_0^\infty \sin\gamma t\cdot\cos m\bar{x}/(\gamma\cosh m)dm\ , \tag{38}$$

and for $S^*(\tau=\tau^*)$, t should be replaced by $\bar{t}(=t-\tau^*)$. It is remarked that (35) to (38) can also be derived from (27) to (30) by integration with respect to y_0 from $-\infty$ to $+\infty$ with the aid of a formula,

$$\int_{-\infty}^{\infty} J_0(m\sqrt{(x-x_0)^2+(y-y_0)^2})\,dy_0=\frac{2}{m}\cos m(x-x_0)\ .$$

3. Surface elevation at the initial time ($t\to 0$) for the case of a sudden deformation of the bottom

For $t\to 0$, (29) may be witten as

$$R=(2\pi)^{-1}\int_0^{\infty} mJ_0(m\bar{r})/\cosh m\, dm\ .$$

The expansion of $1/\cosh m$ in terms of exponential functions and the integration term by term yield,

$$R=(1/\pi)\sum_{n=0}^{\infty}(-1)^n(2n+1)\{(2n+1)^2+\bar{r}^2\}^{-3/2}\ . \tag{39}$$

In (39), n can be interpreted as the amount of reflection of the source disturbance at the surface and at the bottom, and the summation of

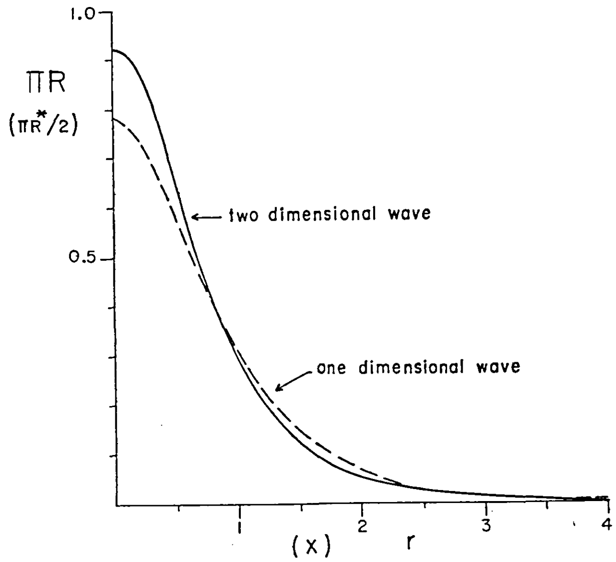

Fig. 1. The initial surface displacement due to a point source at the bottom.

reflection for infinitely many times contributes to the total deformation of the surface.

For one dimensional waves, (37) becomes,

$$R^* = (2/\pi)\sum_{n=0}^{\infty}(-1)^n(2n+1)\{(2n+1)^2+\bar{x}^2\}^{-1}, \quad (t\to 0) . \tag{40}$$

The initial surface elevations (40) and (39) due to a point source in one and two dimensional waves are shown in Fig. 1, which indicates that the elevation is extended over the distance comparable to the depth of water, and no clear-cut wave front is formed.

Taking the origin of the polar-coordinate at the center of an axially symmetric bottom deformation H_B, the elevation of the water surface given by (25) at $t=0^+$, $r=0$ becomes

$$\eta = 2\sum_{n=0}^{\infty}(-1)^n(2n+1)\int_0^{\infty} H_B(r_0)r_0\{(2n+1)^2+r_0^2\}^{-3/2}dr_0, \quad (t=0^+, r=0) . \tag{41}$$

For one dimensional waves, the combination of (33) and (40) gives

$$\eta = (2/\pi)\sum_{n=0}^{\infty}(-1)^n(2n+1)\int_{-\infty}^{\infty} H_B(x_0)\{2n+1)^2+x_0^2\}^{-1}dx_0, \ (t=0^+, x=0) . \tag{42}$$

Particular examples are given below:

a) *The uniform deformation of the circular area of the bottom;*

$$\left.\begin{aligned} H_B &= \text{const.} \quad \text{for} \quad 0 \leqslant r_0 < a, \\ &= 0 \quad \text{for} \quad r_0 > a . \end{aligned}\right\} \tag{43}$$

(41) becomes

$$\eta/H_B = 2\sum_{n=0}^{\infty}(-1)^n[1-\{1+a^2/(2n+1)^2\}^{1/2}], \tag{44}$$

and for very small values of $a(a \ll 1)$,

$$\eta/H_B = a^2\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2} \sim 0.92a^2 .$$

b) *The parabolic deformation of the circular area of the bottom;*

$$\left.\begin{aligned} H_B &= H_B\{1-(r_0/a)^2\} \quad \text{for} \quad 0 \leqslant r_0 < a, \\ &= 0 \quad \text{for} \quad r_0 > a . \end{aligned}\right\} \tag{45}$$

(41) becomes

$$\eta/H_B=\eta_1/H_B-\eta_2/H_B\ , \tag{46}$$

where η_1/H_B is identical to (44) and

$$\eta_2/H_B=2\sum_{n=0}^{\infty}(-1)^n\{(2n+1)/a\}^2[\{1+a^2/(2n+1)^2\}^{1/2}+\{1+a^2/(2n+1)^2\}^{-1/2}-2]\ . \tag{47}$$

For very small values of a ($a\ll 1$),

$$\eta_2/H_B=(1/2)\eta_1/H_B\ ,$$

so that

$$\eta/H_B\sim 0.46a^2\ .$$

c) *The uniform deformation of the bottom (one dimensional propagation)*;

$$\left.\begin{aligned} H_B&=\text{const.} \quad \text{for} \quad |x_0|<a\ ,\\ &=0 \quad \text{for} \quad |x_0|>a\ .\end{aligned}\right\} \tag{48}$$

(42) becomes

$$\eta/H_B=(4/\pi)\sum_{n=0}^{\infty}(-1)^n\tan^{-1}\{a/(2n+1)\}\ , \tag{49}$$

and for small values of $a(a\ll 1)$,

$$\eta/H_B=(4a/\pi)\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-1}=a\ .$$

The initial elevation of water surface at the center of the bottom deformation, (44), (46) and (49), is shown in Fig. 2, from which it is

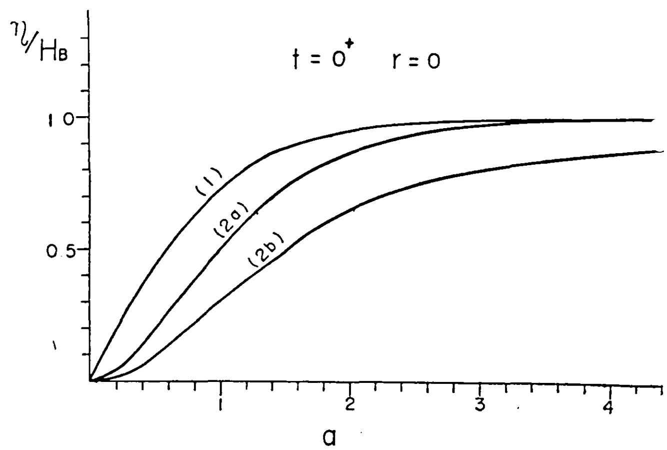

Fig. 2. The initial surface displacement at the center of the bottom deformation: One-dimensional case (1)-(c), two-dimensional case (2a)-(a), (2b)-(b).

found that the elevation of the surface at the center reaches the height of the bottom deformation if the radius of the deformation is about 3 to 4 times the depth of water. On the other hand, for small scale deformation the surface elevation is proportional to the volume of the bottom deformation.

4. Approximate evaluation of the Green's function

a) *Shallow water waves* ($m \ll 1$, $\gamma = m$):

In shallow water, it is usual to adopt the long wave approximation which assumes that the pressure in water is hydrostatic. In the present formulation, the assumption corresponding to the long wave is $m \ll 1$ and $\gamma = m$.

The evalution of the Green's function for this case is rather easy and the results are,

$$S = (2\pi)^{-1}\int_0^\infty \sin mt J_0(m\bar{r})dm$$
$$= \begin{cases} (2\pi)^{-1}(t^2-\bar{r}^2)^{-1/2}, & \text{for} \quad t > \bar{r}, \\ 0, & \text{for} \quad t < \bar{r}, \end{cases} \tag{50}$$

and

$$P = R = S_t; \qquad Q = P_t = S_{tt}. \tag{51}$$

If we shart from the beginning on the assumption of a long wave by expanding φ in power series of z and retaining the first order terms only, we arrive at a two dimensional long wave equation with respect to ζ,

$$\zeta_{tt} - \nabla^2_{xy}\zeta = w_t - p_{tt}, \tag{52}$$

where w is the bottom velocity, $\zeta = \eta + p$, and $\nabla^2_{xy} = \partial^2/\partial x^2 + \partial^2/\partial y^2$. It should be noticed that S given by (50), where t is replaced by $\bar{t}(=t-\tau)$, is nothing but a Green's function of the wave equation (52) and can be used with advantage for the studies of storm surges[20] and tsunamis[21].

Near the wave front for a long distance from the source, we may

20) K. KAJIURA, "A theoretical and empirical study of storm induced water level anomalies," *Tech. Report, Ref. 59-23F, Dep't of Ocn. and Met., Texas A and M*, (1959), 97.

21) L. N. SRETENSKY and A. S. STAVROVSKY, "Computation of the height of tsunami waves along the coast," *Trans. Marine Hydro. Inst., Acad. Sci. USSR*, **24** (1961), 23-43. (Transl. scripta technica, inc., for the A. G. U.)

approximate

$$t^2-\bar{r}^2\simeq 2\bar{r}(t-\bar{r})$$

so that (50) and (51) are reduced to

$$S=(2\pi)^{-1}\bar{r}^{-1/2}\{2(t-\bar{r})\}^{-1/2}\ , \quad t>\bar{r}\ , \tag{53}$$

$$P=R=-(2\pi)^{-1}\bar{r}^{-1/2}\{2(t-\bar{r})\}^{-3/2}\ , \quad t>\bar{r}\ , \tag{54}$$

and

$$Q=(2\pi)^{-1}3\bar{r}^{-1/2}\{2(t-\bar{r})\}^{-5/2}\ , \quad t>\bar{r}\ . \tag{55}$$

These expressions show that, for a fixed value of $(t-\bar{r})$, P, Q, R, and S are all proportional to $\bar{r}^{-1/2}$ irrespective of the source characteristics and at the front, $t=\bar{r}$, the degree of singularity increases from S to P or R and P to Q. However, for a long distance from the source, the usual long wave approximation presented here is not valid near the wave front because the curvature of the water surface plays a role in the dispersion. In other words, it is necessary to assume $\gamma=m-m^3/6$ in the neighborhood of $\bar{r}/t\simeq 1$ and $t\gg 1$.

For one dimensional waves, the same long wave assumption gives

$$\begin{aligned} P^* &= (1/\pi)\int_0^\infty \cos mt \cos m\bar{x} dm \\ &= (1/2)\{\delta(t+\bar{x})+\delta(t-\bar{x})\} \end{aligned} \tag{56}$$

and

$$R^*=P^*\ ; \quad Q^*=P_t^*\ . \tag{57}$$

(56) indicates that one half of the initial surface elevation moves without change of form in the positive and the negative directions respectively.

b) *Wave form near the wave front* $(m<1, \gamma=m-m^3/6)$:

For a long distance from the source, (30) may be replaced by

$$S=\frac{1}{2\pi}\int_0^\infty \frac{m \sin \gamma t}{\gamma \cosh m}\sqrt{\frac{2}{\pi m\bar{r}}}\cos(m\bar{r}-\pi/4)dm\ ,$$

and then transformed into

$$\begin{aligned} S=(2\pi)^{-1}(\pi\bar{r})^{-1/2}\int_0^\infty &\{\sin(m\bar{r}+\gamma t-\pi/4) \\ &-\sin(m\bar{r}-\gamma t-\pi/4)\}/\sqrt{\sinh 2m}dm\ . \end{aligned} \tag{58}$$

It can be shown that the use of the asymptotic expression for $J_0(mr)$ results in the error of the order r^{-1} for S. Now, since $\bar{r}, t$ and γ, m are positive, the contribution of $\sin(\gamma t+m\bar{r}-\pi/4)$ term to the integral for a long distance is of the order r^{-1}. Thus, (58) may be approximated by

$$S=-(2\pi)^{-1}(\pi\bar{r})^{-1/2}\int_0^\infty \sin(m\bar{r}-\gamma t-\pi/4)/\sqrt{\sinh 2m}dm\ . \tag{59}$$

For the leading wave of a tsunami ($\bar{r}/t\simeq 1$) for a long distance from the source, it can be shown that the main contribution to S comes from small m so that it is possible to assume $\gamma=m-m^3/6$ and $\sinh 2m\simeq 2m$. Therefore, by replacing $m^3t/6=u^6$, (59) becomes,

$$S=(2\pi)^{-1}(\pi\bar{r})^{-1/2}(6/t)^{1/6}T(p)\ , \tag{60}$$

and

$$T(p)=Re\left[[(1+i)\int_0^\infty \exp i(u^6+pu^2)du\right]\ , \tag{61}$$

where $Re[z]$ means the real part of z and

$$p=(6/t)^{1/3}(\bar{r}-t)\ .$$

If we neglect u^6 term in the power of the exponential in (61), we have,

$$\begin{aligned} T(p)&=\sqrt{\pi/(2|p|)}\ , &\quad \text{for } p<0\ ,\\ &=0\ , &\quad \text{for } p>0\ ,\end{aligned}$$

and S is reduced to (53).

Since t is large and the variation of $(6/t)^{1/3}$ with respect to t is small, we may replace the derivative with respect to t or r by the derivative with respect to p, and we may write,

$$P=R=(2\pi)^{-1}(\pi\bar{r})^{-1/2}(6/t)^{1/2}(-T_p)\ , \tag{62}$$

and

$$Q=(2\pi)^{-1}(\pi\bar{r})^{-1/2}(6/t)^{5/6}T_{pp}\ . \tag{63}$$

For one dimensional waves, the asymptotic expression for R^* can be derived parallel to the case of two dimensional waves. Thus, for large positive values of t and $\bar{x}$, (37) is reduced to

$$R^*=(2\pi)^{-1}\int_0^\infty \cos\{(t/6)m^3+(\bar{x}-t)m\}dm+O(\bar{x}^{-1})\ . \tag{64}$$

Putting $q=p/3=(1/3)(6/t)^{1/3}(\bar{x}-t)$, and $m=(6/t)^{1/3}v$,
we have

$$R^*=(2\pi)^{-1}(6/t)^{1/3}\int_0^\infty \cos(v^3+3qv)dv \ .$$

The integral can be identified as an Airy Integral $Ci_3(q)$ where

$$Ci_3(q)=\sqrt{q/3}K_{1/3}(2q^{3/2}) \ ,$$

and

$$Ci_3(-q)=(\pi/3)\sqrt{q}\{J_{1/3}(2q^{3/2})+J_{-1/3}(2q^{3/2})\} \ ; \quad q>0 \ .$$

Here, $K_{1/3}(x)$ is a modified Bessel function and $J_{1/3}(x)$, $J_{-1/3}(x)$ are Bessel functions[22]. (64) is essentially similar to the integral discussed by Eckart[23], and Hendrickson[24], who treated the asymptotic behavior of the leading wave of the one dimensional tsunami generated by an initial elevation and a bottom deformation of the small scale respectively.

For convenience in later discussions, we define $T^*(p)$,

$$T^*(p)=\int_p^\infty Ci_3(p/3)dp \ , \tag{65}$$

Then, it follows,

$$R^*=(2\pi)^{-1}(6/t)^{1/3}(-T_p^*) \ , \tag{66}$$

$$S^*=(2\pi)^{-1}T^* \ , \tag{67}$$

$$P^*=R^* \ ,$$

and

$$Q^*=(2\pi)^{-1}(6/t)^{2/3}T_{pp}^* \ . \tag{69}$$

Comparing S and S^*, R and R^* (P and P^*), Q and Q^*, it is evident that, even if we take the factor $r^{-1/2}$ due to geometrical spreading effect out of consideration, the decay law of amplitude with time for the leading wave is different for the one dimensional and two dimensional waves. Furthermore, the decay laws of the leading waves for the initial elevation and initial impulse are different. A mathematical reason for these differences lies in the fact that, in the evaluation of the integral, the factor in front of the oscillatory term, say, the

22) S. MORIGUCHI, *et al*, *Mathematical Formulas, III* (Iwanami Book Co., Ltd., Tokyo, 1959), 231-232, (in Japanese).

23) *loc. cit.*, 9).

24) *loc. cit.*, 2) Appendix 2.

amplitude spectrum for small values of m plays a deciding role.

For example, the decay law of the leading wave amplitude for a long distance from the source of the type of initial elevation or the sudden deformation of the bottom; namely the variation of the first maximum values of R and R^* are $(rt)^{-1/2}$ and $t^{-1/3}$ respectively. The decay law of $(rt)^{-1/2}$ for two dimensional waves was first noticed by Takahasi[25] but was somehow abandoned in his later paper[26], and the decay law of $t^{-1/3}$ for one dimensional waves is taken for granted for two dimensional waves as well[27),28], except for the factor $r^{-1/2}$. In the Appendix II, the decay law for one dimensional waves is derived by means of the superposition of two dimensional waves to show the difference of the decay laws clearly.

For the leading wave generated by a surface impulse, the amplitude decays proportionally to $r^{-1/2}t^{-5/6}$ and $t^{-2/3}$ for the two dimensional and one dimensional cases, respectively.

In both one and two dimensional waves, the wave form of the leading part of the wave train is completely determined by a parameter p so that, with the increase of t, the crest of the wave is retarded with respect to the reference point moving with the velocity of ordinary long wave $\sqrt{gH}$, by a factor $t^{1/3}$ and the surging part in front of this crest spreads outward. Furthermore, the time interval between the first crest and the second one increases proportionally to $t^{1/3}$.

T, T_p, and T_{pp} for two dimensional waves and T^*, T_p^*, and T_{pp}^* for one dimensional waves are shown as a function of p in Fig. 3 and Fig. 4 respectively. Numerical values of $T(p)$ are originally given by Takahasi[29], but the re-computation is carried out by a different method. The result shows a slight modification of the Takahasi's values. Apart from the decay factor already mentioned, the wave forms for the two dimensional waves and the one dimensional waves are qualitatively similar provided the position of the first maximum is a little later for the one dimensional wave than for the two dimensional wave. As will be shown in the later section, the wave form for the leading wave of a tsunami at a very long distance from the source is considered to be well represented by T_p and T_p^* for the cases of the initial surface elevation or the bottom deformation and by T_{pp} and T_{pp}^* for the case of

25) *loc. cit.*, 8).
26) *loc. cit.*, 14).
27) *loc. cit.*, 15).
28) *loc. cit.*, 2).
29) *loc. cit.*, 8) Fig. 13.

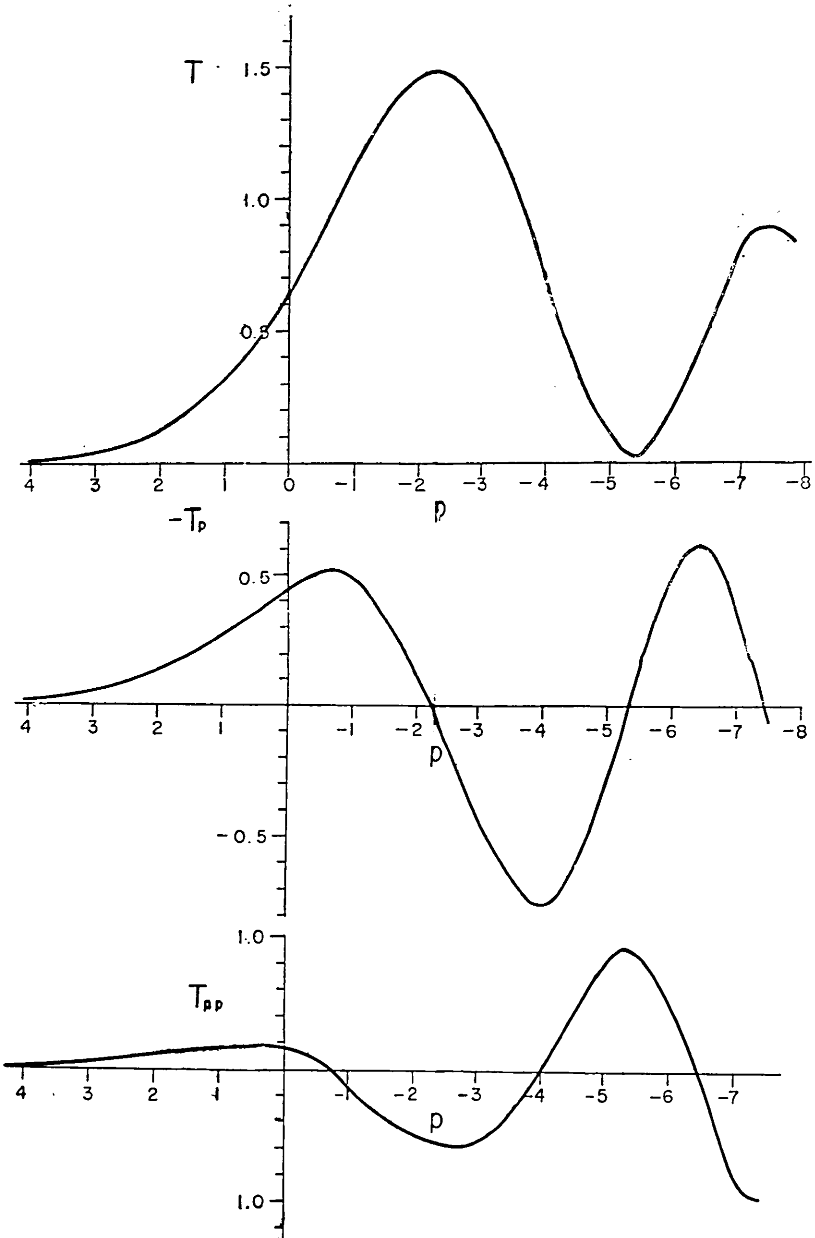

Fig. 3. T, T_p, and T_{pp} as a function of p.

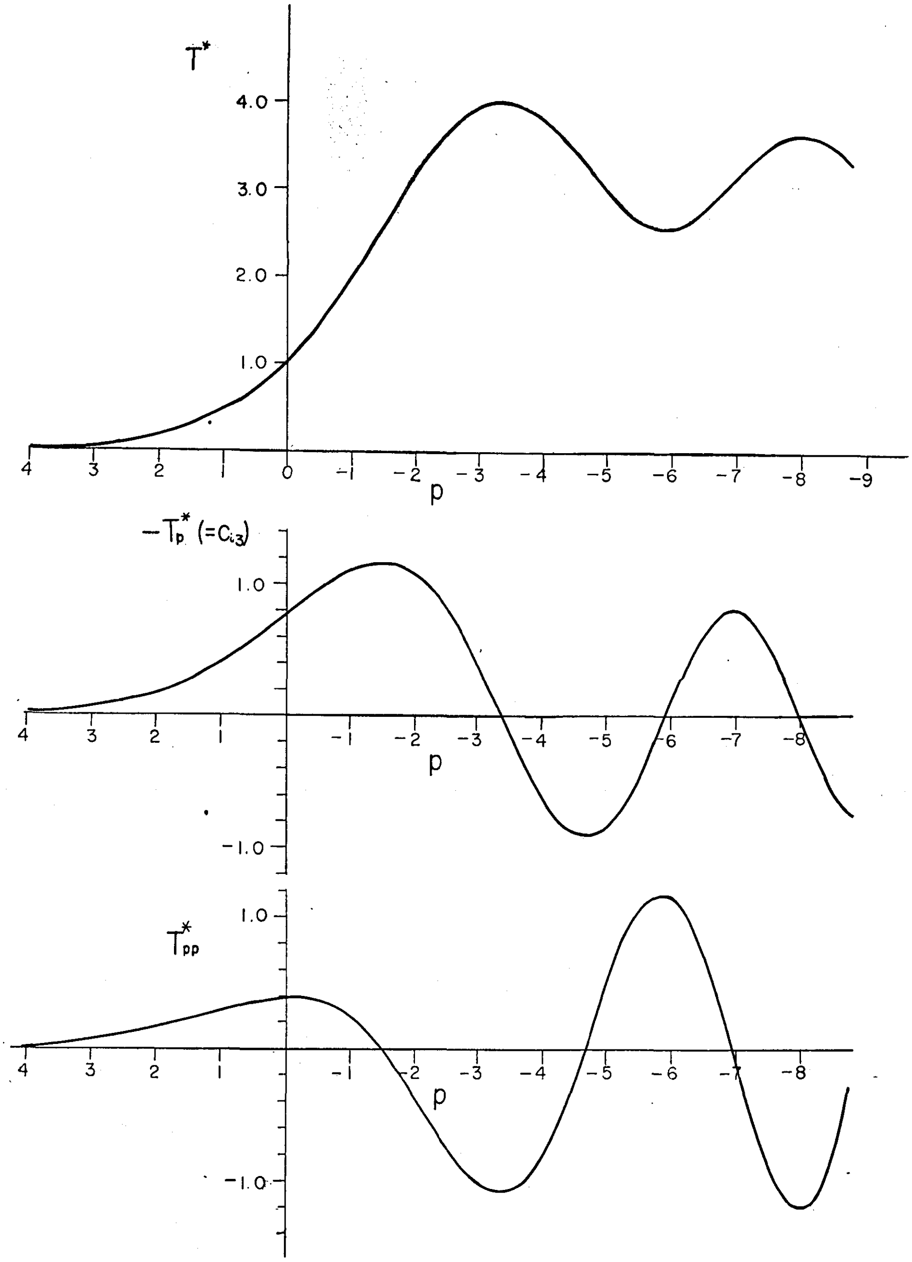

Fig. 4. T^*, T_p^*, and T_{pp}^* as a function of p.

the initial surface impulse. From the figures, it is found that for the one dimensional wave started from the initial elevation or the bottom deformation (T_p^*), the leading wave has the maximum height for a very long distance from the source, but for waves started from the initial impulse (T_{pp}^*), this is not the case, and in many cases of practical interest, the leading wave may not be recognized because of its low amplitude and long wave length. For the two dimensional wave (T_p, T_{pp}), the leading wave may not be the wave with the maximum height for very long distances from the source area, and the second or the later crest will have the maximum height. The intervals between the first maximum and the second one are the same for all T_p, T_p^*, T_{pp}, T_{pp}^*, and the numerical value in terms of p is 5.7.

Qualitatively similar conclusions concerning the change of the wave form can be obtained on the basis of the one dimensional long wave equation including the effect of curvature of the water surface. According to Keulegan and Patterson[30], the velocity ω of propagation of an element of volume of an intumescence satisfies the relation,

$$\eta_t + (\eta\omega)_x = 0 \ ,$$

and the wave equation including the second order terms is given by

$$\eta_{tt} - (\eta + \eta_{xx}/3)_{xx} = 0 \ ,$$

provided $(3/2)\eta^2 \ll (1/3)\eta_{xx}$ or $\eta_0\lambda'^2/H^3 \lesssim 1$ (η_0: wave amplitude, λ': wave length, H: depth of water). This wave equation is essentially similar to the assumption $\gamma = m - m^3/6$. The combination of these two equations and the integration with respect to x give,

$$\omega = 1 + \eta_{xx}/(6\eta) \ .$$

From this relation for the velocity of propagation of a volume element of an intumesence moving in still water (say $\eta > 0$), we can easily find that the wave front moves faster than the "long wave velocity" and the first crest (maximum point) is retarded with respect to the point of inflection where $\eta_{xx} = 0$ and $\omega = 1$ as shown graphically in Fig. 5.

As for the time interval between the first crest and the second, the different approach made by Munk[31] about the period increase of the

30) *loc. cit.*, 11).

31) W. H. MUNK, "Increase in period of waves travelling over large distance, with application to tsunamis, swell and seismic surface waves," *Trans. Amer. Geopys. Un.*, **28** (1947), 198-217.

conservative waves in general may be applicable. A simple solution for the tsunami given by Munk can be further simplified for the case of a constant depth as follows:

$$\tau=(\xi-J/\tau^2)/a \ ,$$

$$\xi=t-x \ ,$$

and

$$J=2\pi^2 x \ ,$$

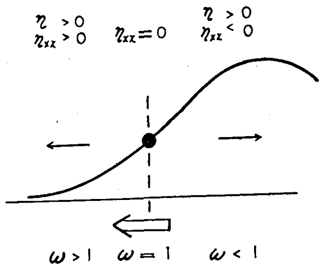

Fig. 5. The behavior of the leading wave.

where τ is the wave period at the time t and the distance x (all quantities are in the non-dimensional form), and a is an arbitrary constant. For a long distance from the source, the time interval between the first and the second crests is approximately equal to $t-x$ and also to the wave period τ, so that we may put $\tau\simeq t-x$. Thus we have

$$\tau=\{2\pi^2/(1-a)\}^{1/3}x^{1/3} \ .$$

Furthermore, since $x/t\simeq 1$, we can conclude that the period increase is proportional to $t^{1/3}$.

The effect of finite duration of the deforming motion at the bottom can be examined by means of (26), which may be written as

$$\eta=\iint_S H_B I dS_0$$

where

$$I=\{S(\tau=0)-S(\tau=\tau^*)\}/\tau^*$$

In terms of p, I is replaced by

$$I=-(2\pi)^{-1}(\pi\bar{r})^{-1/2}(6/t)^{1/2}\{T(p)-T(p_0)\}/(p-p_0) \ ,$$

where

$$p=(6/t)^{1/3}(\bar{r}-t+\tau^*) \text{ and } p_0=(6/t)^{1/3}(\bar{r}-t) \ .$$

Judging from Fig. 4, the variation of T with respect to p in the neighborhood of $(T_p)_{\max}$ is almost linear in the interval $(p-p_0)<1$, so that

$$I\simeq-(2\pi)^{-1}(\pi\bar{r})^{-1/2}(6/t)^{1/2}T_p=R \ .$$

Thus the assumption of the instantaneous deformation is valid in the evaluation of the first crest for the deformation with the time interval τ^* provided $(6/t)^{1/3}\tau^*<1$. For one dimensional waves too, the same condition approximately holds.

c) *The wave train in the later phase*:

The asymptotic solution of (30) or (56) within the limit of applicability of the stationary phase method is given by,

$$S=A\sin(\gamma_0 t-m_0\bar{r})/(\gamma_0\cosh m_0)\,,\quad \text{for } t>\bar{r}\gg 1\,, \tag{70}$$

where

$$\gamma=\sqrt{m\tanh m}\,,$$

and

$$A=(2\pi)^{-1}\bar{r}^{-1}(m_0\gamma_0'/|\gamma_0''|)^{1/2}\,. \tag{71}$$

In (71), primes of γ_0 indicate differentiation with respect to m and the suffix 0 shows the values to be evaluated at $m=m_0$, which is a real root of the equation, $\gamma'=t/\bar{r}$.

Neglecting the time derivative of the slowly varying amplitude compared with that of the carrier wave, we have by differentiating S with respect to t,

$$P=A\cos(\gamma_0 t-m_0\bar{r})\,, \tag{72}$$

$$Q=-A\gamma_0\sin(\gamma_0 t-m_0\bar{r})\,, \tag{73}$$

and

$$R=A\cos(\gamma_0 t-m_0\bar{r})/\cosh m_0\,. \tag{74}$$

For one dimensional waves, (38) is reduced to

$$S^*=A^*\sin(\gamma_0 t-m_0\bar{x}-\pi/4)/(\gamma_0\cosh m_0)\,,\ \text{for } t>\bar{x}\gg 1\,, \tag{75}$$

where

$$A^*=(2\pi)^{-1/2}\bar{x}^{-1/2}(\gamma_0'/|\gamma_0''|)^{1/2}\,, \tag{76}$$

and the other functions can be derived straight-forwardly:

$$P^*=A^*\cos(\gamma_0 t-m_0\bar{x}-\pi/4)\,, \tag{77}$$

$$Q^*=-A^*\gamma_0\sin(\gamma_0 t-m_0\bar{x}-\pi/4)\,, \tag{78}$$

and

$$R^*=A^*\cos(\gamma_0 t-m_0\bar{x}-\pi/4)/\cosh m_0\,. \tag{79}$$

For the case of deep water waves, *i. e.* when the wave number m

is much larger than unity and γ is approximated by $m^{1/2}$, the asymptotic solutions are reduced to well known formulas where

$$A=(1/\sqrt{2})(\pi\bar{r})^{-1}m_0\ , \tag{80}$$

$$A^*=(\pi\bar{x})^{-1/2}m_0^{1/2}\ , \tag{81}$$

$$m_0=t^2/(4\bar{r}^2)\ \text{or}\ t^2/(4\bar{x}^2)\ , \tag{82}$$

$$\gamma_0=t/(2\bar{r})\ \text{or}\ t/(2\bar{x})\ . \tag{83}$$

The frequency and wave number of the individual wave is fixed by $\gamma_0=t/(2\bar{r})$, and $m_0=t^2/(4\bar{r}^2)$ respectively. The decay laws with distance of the amplitude for a fixed wave number are $\bar{r}^{-1}$ and $\bar{x}^{-1/2}$ for the two dimensional and one dimensional waves, respectively.

In both one and two dimensional waves, it is evident that for a fixed location the amplitudes of P and Q increase with time but Q increases more rapidly than P because $Q=P_t$ and for Q weight is placed on high frequency components which arrive at the fixed location later. In contrast, R has a maximum at some intermediate time, because, for very high frequency components of waves, the factor $1/\cosh m_0$ dominates and the amplitudes of high frequency components are surpressed by the factor $\exp(-m_0)$.

5. Leading wave of a tsunami

Consider the leading wave of a tsunami for a long distance from the source area enclosed by $-a<x_0<a$, and $-b<y_0<b$ where waves may be generated either by the deformation of the bottom or by the initial elevation of the surface. Taking the origin of the co-ordinate (x, y) at the center of the source area, we may assume $\sqrt{a^2+b^2}/r\ll 1$ where

$$r^2=x^2+y^2\ .$$

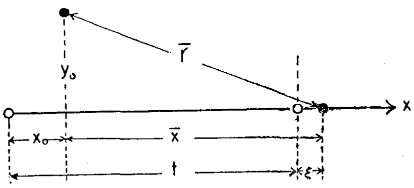

Fig. 6. Geometry of the source point (x_0, y_0) and the observing point (x, y).

Geometry of the source point (x_0, y_0) and the observing point (x, y) is illustrated in Fig. 6, where, for simplicity, the observing point is placed on the x-axis, and ξ is taken as the distance of the observing point relative to the moving reference point where $x=t$. Assuming $\bar{r}\simeq\bar{x}$, we may write

$$p=(6/t)^{1/3}(\bar{r}-t)=(6/t)^{1/3}(\xi-x_0) \tag{84}$$
$$=p^*-p_0$$

where

$$p_0=(6/t)^{1/3}x_0\ , \qquad p^*=(6/t)^{1/3}\xi\ .$$

Thus, p is independent of y_0 for an observing point on the x-axis.

From (25) together with (62), waves generated by the bottom deformation are given by

$$\eta=(2\pi)^{-1}(\pi r)^{-1/2}(6/t)^{1/2}\iint_S H_B(-T_p)dS_0 \tag{85}$$

Since T_p is independent of y_0 for a long distance in the x-direction from the source, we may rewrite (85) with the aid of (84) into the form:

$$\eta=(2\pi)^{-1}(\pi r)^{-1/2}(6/t)^{1/6}\int_{-p_a}^{p_a} W(-T_p)dp_0\ , \tag{86}$$

where

$$W=\int_{-b}^{b}H_B dy_0 \quad \text{and} \quad p_a=(6/t)^{1/3}a\ .$$

In general, we may put $H_B=0$ at the outer edge of the source area, $p_0=p_a$ and $-p_a$, so that the partial integration of (86) gives,

$$\eta=(2\pi)^{-1}(\pi r)^{-1/2}(6/t)^{1/6}U\ , \tag{87}$$

where

$$U(p^*)=\int_{-p_a}^{p_a} W_{p_0}[-T(p^*-p_0)]dp_0\ . \tag{88}$$

If we introduce an explicit distribution of H_B, $U(p^*)$ can be computed as a function of p_a and p^* and the surface elevation η is determined from (87).

For small values of p_a, T_p does not change significantly near the maximum of $-T_p$ so that we may approximate (86) by

$$\eta_{\max}=(2\pi)^{-1}(\pi r)^{-1/2}(6/t)^{1/2}(-T_p)_{\max}V\ , \tag{89}$$

where

$$V=\int_{-a}^{a}W dx_0=\int_{-a}^{a}\int_{-b}^{b}H_B dy_0 dx_0\ .$$

(89) shows that the shape of the source does not affect the height of the crest or trough of the leading wave for a very long distance from the source ($p_a \ll 1$). The wave height is proportional to the total volume of the original deformation of the bottom and decays approximately proportional to $(rt)^{-1/2}$ which is already expected from the

analysis of R. If the total volume V of the deformation of the bottom is zero, the leading wave at a very long distance from the origin decays inversely proportional to a higher power of the time than 1/2.

For one dimensional waves, we may derive similar equations. From (33) together with (66), the wave generated by bottom deformation is given by

$$\eta=(2\pi)^{-1}\int_{-p_a}^{p_a} H_B(-T_p^*)dp_0 , \tag{90}$$

and (90) may be transformed into

$$\eta=-(2\pi)^{-1}\int_{-p_a}^{p_a} (H_B)_{p_0}T^* dp_0 , \tag{91}$$

since

$$T_{p_0}^*=-T_p^* \text{ and } H_B=0 \text{ for } |p_0|\geqslant p_a .$$

For very small values of p_a, (90) is reduced to

$$\eta_{\max}=(2\pi)^{-1}V^*(6/t)^{1/3}(-T_p^*)_{\max} , \tag{92}$$

where

$$V^*=\int_{-a}^{a} H_B dx_0 .$$

Particular examples are given in the follows:

a) *The uniform deformation of a rectangular area of the bottom*;

$$\begin{aligned} H_B&=\text{const.} \quad \text{for } |x|<a , \quad |y|<b , \\ &=0 \quad \text{for } |x|\geqslant a , \text{ or } |y|\geqslant b . \end{aligned} \tag{93}$$

For waves in the x-direction, we may put

$$W=2bH_B , \quad \text{for} \quad -p_a<p_0<p_a ,$$

so that

$$W_{p_0}=W\{\delta(-p_a)-\delta(p_a)\} .$$

(89) gives,

$$U=W\{T(p^*-p_a)-T(p^*+p_a)\} . \tag{94}$$

For small values of p_a,

$$U_{\max}=(4abH_B)(6/t)^{1/3}(-T_p)_{\max} , \tag{95}$$

and for very large values of p_a,

$$U_{\max}=W[T(p^*-p_a)]_{\max} , \tag{96}$$

since $T(p^*+p_a)\rightarrow 0$ for large values of p^*+p_a. Here, $(-T_p)_{\max}$ is located at $p^*=-0.75$ and $T_{\max}$ is located at $p^*=p_a-2.3$.

b) *The elliptic deformation of an elliptic area of the bottom;*

$$H_B=H_{B0}\{1-(y_0/b)^2-(x_0/a)^2\} , \tag{97}$$

within the region enclosed by

$$(x_0/a)^2+(y_0/b)^2=1 .$$

For waves in the x-direction, we may write,

$$W=(4/3)H_{B0}b\{1-(x_0/a)^2\}^{3/2} ,$$

so that

$$W_{p_0}=(4H_{B0}b)(-p_0/p_a^2)\{1-(p_0/p_a)^2\}^{1/2} .$$

(89) gives

$$U=(4H_{B0}b)\int_{-1}^{1}T(1-\alpha^2)^{1/2}\alpha d\alpha , \tag{98}$$

where $\alpha=p_0/p_a$ and $T=T(p^*-p_a\alpha)$.

For small values of p_a, we have

$$U_{\max}=(\pi/2)H_{B0}ab(6/t)^{1/3}(-T_p)_{\max} , \tag{99}$$

and for large values of p_a, we cannot reduce (98) into a simple relation as in the case of a rectangular source.

c) *The uniform deformation of the bottom (one dimensional propagation);*

$$\left.\begin{aligned} H_B&=\text{const.} &&\text{for } |x_0|<a \\ &=0 &&\text{for } |x_0|\geqslant a . \end{aligned}\right\} \tag{100}$$

We may write,

$$(H_B)_{p_0}=H_B\{\delta(-p_a)-\delta(p_a)\} ,$$

and

$$\eta/H_B=(2\pi)^{-1}\{T^*(p^*-p_a)-T^*(p^*+p_a)\} . \tag{101}$$

For the one demensional leading wave according to (101), $\eta_{\max}/H_B$ and

p^* for which $\eta_{\max}$ is attained are shown as a function of p_a in Fig. 7, which indicates that $\eta_{\max}/H_B$ is considered to be a linear function of p_a for small values of p_a, say, for $p_a<1$, so that the leading wave amplitude varies proportional to a and $t^{-1/3}$. On the other hand, for large values of p_a, say for $p_a>3$, the leading wave height reaches $0.635H_B$ and does not change with the scale a and the time t. The time interval between the reference point (moving with the long wave velocity $\sqrt{gH}$ from the origin) and the arrival time of the leading wave crest given

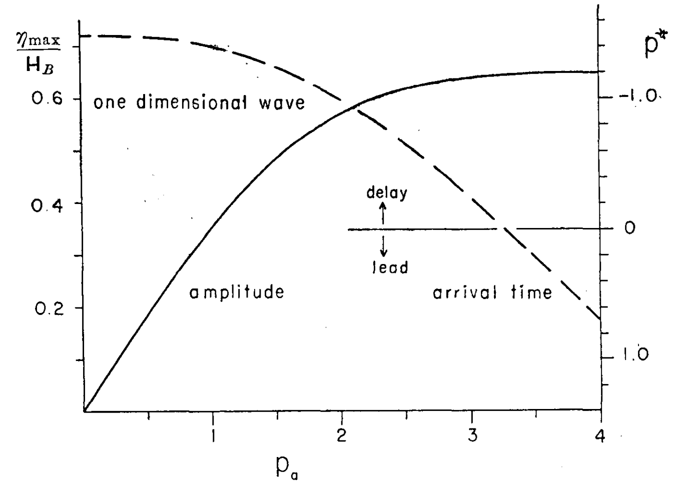

Fig. 7. η_{max}/H_B and p^* as a function of p_a: (c).

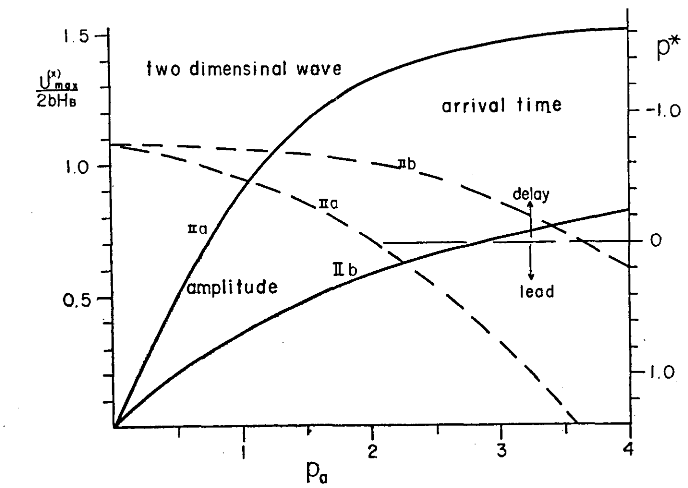

Fig. 8. $U_{\max}/(2bH_B)$ and P^* as a function of p_a: IIa-(a), IIb-(b).

by p^* implies that the crest is retarded with respect to the reference point as the wave travels further, showing the velocity of the crest movement to be smaller than $\sqrt{gH}$.

For the two dimensional leading waves according to (94) and (98), $U_{max}/(2bH_B)$ and p^* for with U becomes maximum are shown in Fig. 8, which indicates a trend with respect to p_a which is qualitatively similar to the case of the one dimensional wave. As for the directional difference of the leading wave height, the ratio of the leading wave heights in the x- and y-directions for the same time and the distance is given by

$$\eta^{(x)}_{max}/\eta^{(y)}_{max}=U^{(x)}_{max}/U^{(y)}_{max} . \tag{102}$$

Thus, it is easy to compute the ratio from Fig. 8 if p_a and p_b are known. For large values of p_a and p_b, say p_a and $p_b>3$, the leading wave amplitude decreases as $r^{-1/2}t^{-1/6}$ and the amplitude ratio approaches approximately to b/a. On the other hand, for small values of p_a and p_b, say p_a and $p_b<1$, the amplitude decreases as $(rt)^{-1/2}$ and the directional difference of the leading wave amplitude disappears.

The leading wave generated by the surface impulse can be discussed along a similar line of argument and the surface elevation derived from (24) and (63) is given by

$$\eta=(2\pi)^{-1}(\pi r)^{-1/2}(6/t)^{1/2}\int_{-p_a}^{p_a} J\cdot T_{pp}dp_0 , \tag{103}$$

where

$$J=\int_{-b}^{b} I_S dy_0 ,$$

and for small values of p_a, we have

$$\eta_{max}=(2\pi)^{-1}(\pi r)^{-1/2}(6/t)^{5/6}K\cdot[T_{pp}]_{max} , \tag{104}$$

where

$$K=\int_{-a}^{a}\int_{-b}^{b} I_S dy_0 dx_0 .$$

For the one dimensional case, the elevation may be derived from (32) and (67) as follows:

$$\eta=(2\pi)^{-1}(6/t)^{1/3}\int_{-p_a}^{p_a} I_S\cdot T^*_{pp}dp_0 , \tag{105}$$

and for small values of p_a, we have

$$\eta_{\max}=(2\pi)^{-1}(6/t)^{2/3}K^*\cdot[T^*_{pp}]_{\max}\ , \tag{106}$$

where

$$K^*=\int_{-a}^{a} I_S dx_0\ .$$

Thus the decay of the leading wave height for the case of the surface impulse is greater by the factor $t^{-1/3}$ than that for the case of the initial surface elevation or the sudden deformation of the bottom. However, the directional difference of the leading wave heights follows a rule similar to the case of the initial surface elevation.

6. Dispersion of a wave train in the later phase

Making use of the approximate representation of the Green's function, it is straight-forward to derive the expression of a wave train in the later phase, originating from an extended source area ($|x_0|<a$, $|y_0|<b$).

For a large distance from the source, ($\sqrt{a^2+b^2}/r\ll 1$), we may approximate

$$\bar{r}=r-r_0\cos\theta_0\ ,$$

Therefore, the substitution of (72) into (23), which represents the waves started from an initial surface elevation H_S, yields,

$$\eta(r,\ t)=A\Big[\cos(\gamma_0 t-m_0 r)\iint_S H_S(r_0,\ \theta_0)\cos(m_0 r_0\cos\theta_0)r_0 dr_0 d\theta_0 - \sin(\gamma_0 t-m_0 r)\iint_S H_S(r_0,\ \theta_0)\sin(m_0 r_0\cos\theta_0)r_0 dr_0 d\theta_0\Big]\ . \tag{107}$$

For the special case of an axially symmetric source, $H_S\equiv H_S(r_0)$, it is easy to show that

$$\eta(r,\ t)=A\overline{H_S(m_0)}\cos(\gamma_0 t-m_0 r) \tag{108}$$

where

$$\overline{H_S(m_0)}=2\pi\int_0^a H_S(r_0)J_0(m_0 r_0)r_0 dr_0\ .$$

In the derivation of (108), the following formulas are used:

$$\int_0^{2\pi}\cos(z\cos\theta)d\theta=2\pi J_0(z)\ ,$$

and

$$\int_0^{2\pi} \sin (z \cos \theta) d\theta = 0 \; .$$

In a similar way, the waves originating from the axially symmetric surface impulse, $I_S(r_0)$, are given by

$$\eta(r, \; t) = -A\gamma_0 \overline{I_S(m_0)} \sin (\gamma_0 t - m_0 r) \; , \tag{109}$$

where

$$\overline{I_S(m_0)} = 2\pi \int_0^a I_S(r_0) J_0(m_0 r_0) r_0 dr_0 \; .$$

(108) and (109) are identical to the solutions given by Kranzer and Keller[32] who gave the detailed discussions of the wave characteristics derived from these solutions. However, it may be remarked that their conclusions related to the leading wave should be understood with some reservations because the leading wave should be treated as an asymptotic solution of the Airy phase.

It is noticed by examining (107) that for a source function different from axial symmetry, the computation of the elevation becomes complicated in the polar co-ordinate. On the other hand, in the rectangular co-ordinate for η in the x-direction, (107) can be integrated with respect to y_0 without regard to the sine or cosine term since $r_0 \cos \theta_0$ may be replaced by x_0.

Hence, (107) may be written as

$$\eta(x, \; t) = A\Big[\cos (\gamma_0 t - m_0 r) \int_{-a}^{a} W \cos m_0 x_0 dx_0 - \sin (\gamma_0 t - m_0 r) \int_{-a}^{a} W \sin m_0 x_0 dx_0 \Big] \; , \tag{110}$$

where

$$W(x_0) = \int_{-b}^{b} H_S(x_0, \; y_0) dy_0 \; .$$

The integrals in the right hand side of (110) are nothing but a Fourier cosine and sine transform of W, and represent the amplitude modulation for the carrier wave. For η in the y-direction, the integration with respect to x can be performed first irrespective of the sine or cosine term since we may put $r_0 \cos \theta_0 = y_0$.

Some specific examples are given below :

32) *loc. cit.*, 13).

a) *The uniform initial elevation of a rectangular area of the surface;*

$$\left.\begin{aligned} &H_S=\text{constant} &&\text{for } |x_0|<a,\ |y_0|<b\ , \\ \text{and}\quad &H_S=0 &&\text{for } |x_0|\geqslant a \text{ or } |y_0|\geqslant b\ , \end{aligned}\right\} \tag{111}$$

We have

$$\eta(x,\ t)=A(2bH_S)(2\sin m_0a/m_0)\cos(\gamma_0t-m_0r)\ . \tag{112}$$

Replacing x by y and a by b, the elevation in the y-direction can be obtained,

$$\eta(y,\ t)=A(2aH_S)(2\sin m_0b/m_0)\cos(\gamma_0t-m_0r)\ . \tag{113}$$

Thus, for the amplitude of the carrier wave, the ratio in the x- and the y-directions is

$$\eta(x,\ t)/\eta(y,\ t)\sim(A^{(x)}bm_0^{(y)}\sin m_0^{(x)}a)/(A^{(y)}am_0^{(x)}\sin m_0^{(y)}b)\ . \tag{114}$$

If we follow the same wave length m_0, the amplitude ratio becomes $(b\sin m_0a)/(a\sin m_0b)$. In particular, if a and b are very small so that $m_0a,\ m_0b\ll1$, the directional difference of wave amplitudes in the x- and y-directions disappears. For deep water waves ($\gamma\simeq m^{1/2}$) and for the same distance $x=y$, the ratio of the maximums of the modulation amplitudes in the x- and y-directions becomes b/a and the arrival times of the corresponding maximum amplitudes are given by

$$t^{(x)}/t^{(y)}\sim(b/a)^{1/2}\ .$$

Therefore, in the direction of the shorter axis, say in the y-direction if $b<a$, the arrival of the modulation maximum is later and the wave length of the carrier wave of the maximum amplitude is smaller than those in the x-direction. In other words, the modulation has a larger amplitude and wave length in the direction of the shorter axis than those in the longer axis.

b) *The uniform deformation of an elliptic area of the bottom;*

$$H_S=\text{constant}\ ,$$

within the area enclosed by

$$(x_0/a)^2+(y_0/b)^2=1\ . \tag{115}$$

We have,

$$W=2H_S(b/a)\sqrt{a^2-x_0^2}\ , \tag{116}$$

and the substitution of (116) into (110) yields

$$\eta(x,\ t)=A(2bH_S)\{\pi J_1(m_0a)/m_0\}\cos(\gamma_0t-m_0r)\ , \tag{117}$$

where the following formula is used:

$$\int_0^a \sqrt{a^2-x_0^2}\cos m_0x_0dx_0=\pi aJ_1(m_0a)/(2m_0)\ .$$

In the y-direction, the wave train is given by

$$\eta(y,\ t)=A(2aH_S)\{\pi J_1(m_0b)/m_0\}\cos(\gamma_0t-m_0r)\ . \tag{118}$$

It is shown that the above expression is reduced to (108) if $a=b$, because the Hankel Transform of the circular source with constant H_S is given by

$$\overline{H_S(m_0)}/(2\pi)=H_SaJ_1(m_0a)/m_0\ .$$

Now, the ratio of the amplitude of the carrier waves is given by

$$\eta(x,\ t)/\eta(y,\ t)\sim\{A^{(x)}bm_0^{(y)}J_1(m_0^{(x)}a)\}/\{A^{(y)}am_0^{(x)}J_1(m_0^{(y)}b)\}\ . \tag{119}$$

This expression is qualitativery similar to the case of a rectangular source provided the sine term is now replaced by the Bessel function. Therefore the qualitative conclusions regarding the difference of the modulation in the x- and the y-directions are similar.

Similar treatments may be applicable to the cases of bottom deformation or with the initial impulse at the surface. For the case of bottom deformation, the result is obtained simply by replacing H_S by H_B and adding the factor $exp(-m_0)$. For the case of the impulsive generation of waves, the result is obtained by replacing H_S by I_S, changing the carrier waves from cosine to sine, and multiplying the factor $\gamma_0(\simeq m_0^{1/2})$.

7. Comparison with experimental data

a) One dimensional propagation:

Prins[33] investigated waves generated by an initial local elevation or depression of uniform height in a two dimensional model. Within the

33) J. E. PRINS, "Characteristics of waves generated by a local disturbance," *Trans. Amer. Geophys. Union*, **39** (1956), 865-874.

range of $H'_S a'^2/H^3 < 1$ (H'_S: the initial height, a': the half length of the elevation, H: the depth of water), the leading wave height $\eta'_{\max}$ is found to be proportional to H'_S and the proportionality factor may be read off from Fig. 7 of his paper.

Table 1. $\eta'_{\max}/H'_S$: one dimensional propagation.

x' (ft) / a' (ft)	5′			25′		
	obs.	calc.	p_a	obs.	calc.	p_a
2′	0.49	0.42	1.22	0.25	0.26	0.71
1′	0.29	0.23	0.61	0.13	0.13	0.36
1/3′	0.09	0.08	0.20	0.045	0.045	0.12

Depth of water H: 2.3′
Elevation H'_s : 0.1′, 0.2′, 0.3′

Table 1 shows the comparison between the results of the theory and the observation and it may be said that good agreement is obtained for the distance $x'=25'$. For $x'=5'$, the agreement is not as good as for $x'=25'$, because the approximation made in the theory is not accurate for a short distance from the source.

b) *Two dimensional propagation*:

Takahasi and Hatori[34] carried out a model experiment for the generation of waves by the deformation of a bottom portion of the elliptic shape. The conditions of the experiment are:

Depth H: 5 cm and 17.3 cm
Dimension of the source: $2a'=90$ cm, $2b'=30$ cm

The final form of the displaced bottom surface is approximately parabolic with the maximum height H_{B0} at the center, and the duration of the bottom motion is of the order of 0.1 sec. (the bottom motion is simulated by the deformation of a rubber memberance). The results which are of interest in the present discussion are shown in Table 2.

The theoretical values for the maximum elevation at the center are estimated approximately by taking $(a'+b')/2$ as the representative dia-

34) R. TAKAHASI, and T. HATORI, "A model experiment of the tsunami generation from a bottom deformation area of elliptic shape," *Bull. Earthq. Res. Inst.*, **40** (1962), 873-883, (in Japanese).

Table 2. η/H_{B0} and $\eta^{(y)}/\eta^{(x)}$: two dimensional propagation.

Depth H (cm)	Maximum elevation at the center, η/H_{B0}		Ratio of the leading wave heights in the directions of the minor axis and the major axis, $\eta^{(y)}/\eta^{(x)}$	
	obs.	calc.	obs.	calc.
5	0.55	1.00	1.8	1.6
17.3	0.27	0.58	1.6	1.3

Bottom deformation H_{B0}: variable.

meter of the deformation of the type 2b in Fig. 2. The observed values for the ratio of the leading wave heights in the directions of the major and the minor axes are the average over the distance $1m$ to $4m$ but the theoretical values are computed for the distance of $4m$ on the basis of Fig. 8 curve IIb.

The observed elevation at the center is about half of the theoretical value and the ratio of the observed wave heights in the major and minor axes for the distance of $4m$ is larger than the theoretical values. The reason for this disagreement between theory and observation may lie partly in the inadequate representation of the theoretical model for the actual experimental model conditions, particularly with respect to the time dependence of the bottom motion, and partly in the uncertainty of the model data which show considerable scattering when several simulations are recorded.

8. Concluding remark

The theory developed in the present paper may be severely restricted in application from the practical point of view, because of the various assumptions made in the course of study, such as 1) the linear approximation in the equations, 2) constant depth and no lateral boundary, 3) the leading wave at long distances from the source, 4) time dependence of the source to be of the Delta function type. Since the area of the tsunami generation lies mainly on the continental slope along the Pacific Ocean, the assumption (2) should be removed to have a little more realistic picture of the tsunami. The coastal boundary and the continental shelf produce reflected waves and waves travelling along the boundary, so that the wave-train of the tsunami observed along the open coast will be quite different from that expected from the theory with no boundary. Moreover, the irregularities of the ocean bottom and the existence of islands would change the wave-form significantly during the course of propagation by refraction and diffraction. Unless these factors are adequately taken into account, it seems impossible to understand the wave train of a tsunami clearly. For example, the prevalent period of about one hour observed along the coast of Japan at the time of the tsunami of the Chilean Earthquake cannot be considered to be of local origin, say the oscillation of the bay and the shelf nearby, but of some distant origin. However, it is unlikely that the dispersion is responsible for the wave train of this long period. The reflected waves along the coast of South America and North America, the boundary waves propagated along the coasts and/or the interference of the refracted waves along the course of propagation of the direct waves might be the cause.

As for the assumptions (3) and (4), the numerical computation with the aid of an electronic computor would remove the restriction. Here, the application of the principle of superposition would be very useful to examine various cases of the source condition.

Lastly, the assumption (1) may be justified for the ordinary cases when the deformation of the bottom is not so large compared with the depth of water and also the lateral extent of the source. However, in shallow seas, the condition may not be satisfied and the non-linear effect comes into play. At the coast, the run-up of the tsunami should be discussed separately.

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Appendix I

For example, (23) may be integrated first with respect to dS_0 as follows:

$$\eta(r,\ t)=(2\pi)^{-1}\int_0^\infty m\cos\gamma t\left[\int_0^\infty\int_0^{2\pi} H_S(r_0,\ \theta_0)J_0(m\bar{r})r_0dr_0d\theta_0\right]dm\,. \qquad \text{(I-1)}$$

Now, since $\bar{r}^2=r^2+r_0^2-2rr_0\cos\theta_0$, where $(r,\ 0)$ are the co-ordinates of the point in question and $(r_0,\ \theta_0)$ those of the source point, in the polar co-ordinate with the origin at the center of the source area, we may expand,

$$J_0(m\bar{r})=J_0(mr)J_0(mr_0)+2\sum_{n=1}^{\infty}J_n(mr_0)\cos n\theta_0 \ . \tag{I-2}$$

Thus, (I–1) is reduced to

$$\eta(r,\ t)=\int_0^\infty m\cos\gamma t\left[J_0(mr)\overline{I_0(m)}+2\sum_{n=1}^{\infty}J_n(mr)\overline{I_n(m)}\right], \tag{I-3}$$

where

$$\overline{I_n(m)}=(2\pi)^{-1}\int_0^{2\pi}\int_0^\infty H_S(r_0,\ \theta_0)J_n(mr_0)r_0dr_0d\theta_0\ ,\quad (n=0,\ 1,\ 2,\ \cdots)\ , \tag{I-4}$$

If $H_S(r_0,\ \theta_0)\equiv H_S(r_0)$: namely for an axially symmetric source, it is easily shown that

$$\overline{I_0(m)}=\int_0^\infty H_S(r_0)J_0(m_0r_0)r_0dr_0\ ,\qquad \text{for}\quad n=0\ , \tag{I-5}$$

and

$$\overline{I_n(m)}=0\ ,\qquad \text{for}\quad n\neq 0\ . \tag{I-6}$$

Appendix II

From the definition of R^* and R, it is evident that

$$R^*=\int_{-\infty}^{\infty}Rdy_0=2\int_x^\infty Rr(r^2-x^2)^{-1/2}dr\ , \tag{II-1}$$

where x_0 is put zero for simplicity and $y_0^2=r^2-x^2$.

Writing r and x in terms of p and p^* for a given time t;

$$r=(t/6)^{1/3}p+t\ ,\quad \text{and}\quad x=(t/6)^{1/3}p^*+t\ , \tag{II-2}$$

we have

$$(r^2-x^2)=(t/6)^{1/3}(p-p^*)\{(t/6)^{1/3}(p+p^*)+2t\}\ , \tag{II-3}$$

and

$$dr=(t/6)^{1/3}dp\ . \tag{II-4}$$

Since R is very small for large positive values of p and the contribution of R to R^* is confined to small values of p only, we may put approximately,

$$r\simeq t\ ,\quad \text{and}\quad (r^2-x^2)\simeq 2t(t/6)^{1/3}(p-p^*)\ . \tag{II-5}$$

And the substitution of (II-4), (II-5), and (62) into (II-1) yields,

$$R^*(p^*)=(2\pi)^{-1}(2/\pi)^{1/2}(6/t)^{1/3}\int_{p^*}^{\infty}-T_p/(p-p^*)^{1/2}dp\ . \qquad \text{(II-6)}$$

Thus, R^* is proportional to $t^{-1/3}$.

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33. 津波の第一波について

(津波の発生，伝播に関する古典的理論)

地震研究所　梶浦欣二郎

線型近似の範囲内において，海面あるいは海底に与えられたある種の外的じよう乱に原因しておこる水の波の問題は古くから研究されているが，ここでは時間に関係する型のグリーン函数を利用して，一様な深さの海におこる波の問題を統一的に考える．特に外的じよう乱が時間的に簡単な形で与えられる場合（瞬間的な変動）には波の表現が簡単になり，波の性質を波源の特性（初期水位，瞬間的な衝撃，瞬間的な海底変動等）と波源のひろがりによるものとに分けて考えることができる．

今 r を波源中心からの水平距離，t を経過時間とすると，波源から極めて遠方における第一波の減衰は次の量に比例する．

	一次元伝播	二次元伝播
初期水位 海底の瞬間的隆起	$t^{-1/3}$	$r^{-1/2}t^{-1/2}$
水面衝撃	$t^{-2/3}$	$r^{-1/2}t^{-5/6}$

これでみて判る通り，第一波については，一次元・二次元という伝播の相異，および波源の特性の相異が減衰にも関係している．これに反して，第一波と第二波との峯の時間間隔はすべて $t^{1/3}$ に比例して伸びる．

一方，波源のひろがりの影響については，$2a$ を波源の大きさとすると $p_a=(6/t)^{1/3}a$（t および a は水深 H および重力加速度 g を用いて無次元化した量）というパラメータが重要となり，p_a の大小によつて第一波の減衰の有様が異なる．$p_a<1$ では，波は波源から極めて遠方と考えることができて，減衰は上述の通りであるが，$p_a>3$ では波源にかなり近いところとなり，減衰は次のようになる．

	一次元伝播	二次元伝播
初期水位 海底の瞬間的隆起	t^{0}	$r^{-1/2}t^{-1/6}$
水面衝撃	$t^{-1/3}$	$r^{-1/2}t^{-1/2}$

今，長軸 $2a$，短軸 $2b$ という矩形の波源から出る波についてその第一波の波高の方向性を調べると，近距離では長軸方向の波高は短軸方向の波高の b/a であるが，遠距離では差がなくなる．

定常位相法の利用できる後続波については，円形以外の波源からのものも簡単に調べることができて，たとえば楕円波源の場合には，波高のモジュレーションの長さおよび最大振巾は長軸方向よりも短軸方向の方が大きくなることが示される．

得られた結果を実験値と比較すると，一次元伝播の場合には一致が極めて良いが，二次元伝播の場合にはそれほど良くない．