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22. *Method of Determining the Degree of Free Oscillation of a Radially Heterogeneous Elastic Sphere*

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Abstract

A method is proposed which gives a clue to determine the degree (m) of the free oscillation of the earth, or the azimuthal characteristic of the origin of disturbance. This method is based on the principle that the series of zero points of the guiding curve of spectrum peaks depends on the degree number m. If a specific spherical surface harmonics with a certain value of m has a series of zero points that conform to the zeroes of the guiding curve of the observed spectral peaks, the degree number m is determined.

1. Introduction

Basic solution for the free oscillation of a radially heterogeneous elastic sphere involves the spherical surface harmonics $P_n^m(\cos\theta)\cdot\exp(im\phi)$ or a somewhat modified form of the same function. (θ=co-latitudinal angle and ϕ=azimuthal angle.) In this solution the order numer n is closely connected with the period of the free oscillation, while the degree number m has nothing to do with the period. Consequently, the frequency analysis of the observed long period oscillation yields the knowledge concerning the free period of the earth, but does not give the azimuthal characteristic of the free oscillation or of the source of the disturbance.

In the following sections, however, a method is proposed which gives a useful clue for determining the degree of free oscillation, or the azimuthal characteristic of the origin. This method is based on the

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principle that the zero points of the guiding curve of spectral peaks, which are given by the spherical surface harmonics or its modified form, depend on the value m.

2. Basic theory

If we look at Fig. 1, which is the result of the spectrum analysis of the theoretical seismogram obtained by the present authors[1], we are naturally led to the questions: (1) can we draw an envelope of the spectral peaks as is given in Fig. 1, and (2) what is the meaning of such a guiding curve.

In order to make this problem clear it will be better to go back to the fundamental expression for the displacement or the strain. In a radially heterogeneous elastic sphere disturbances are expressed as

$$D=\sum_{n,m,i} C_n^m \cdot R_n(r,\, {}_ip_n) \cdot \Theta_n^m(\theta) \cdot \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi \cdot f^*({}_ip_n) \cdot \exp(j\,{}_ip_n t) \tag{2.1}$$

where

i: Radial mode number.

j: Unit of imaginary number.

n, m: Order and degree of the spherical surface harmonics respectively.

C_n^m: Coefficient due to the space distribution of the applied external force.

$f^*({}_ip_n)$: Fourier conjugate of the time function of the applied force $f(t)$.

${}_ip_n$: Frequency of the free oscillation. Since the modes with different degree m have an identical degenerated frequency, ${}_ip_n$ has no superscript m.

$R_n(r,\, {}_ip_n)$: Function giving the radial distribution of disturbance.

$\Theta_n^m(\theta) \cdot \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi$: Function giving the surface distribution of disturbance. The actual form of the function $\Theta_n^m(\theta)$ depends on (1) the kind of oscillation, whether it is spheroidal or torsional, (2) the quantity measured whether it is displacement or strain and (3) the component whether it is radial, colatitudinal or azimuthal.

The spectrum of the disturbance D obtained at a point on the

1) Y. SATÔ, T. USAMI and M. LANDISMAN, "Spectrum, Phase and Group Velocities of the Theoretical Seismogram, and the Idea of the Equivalent Surface Source of Disturbances", (in print).

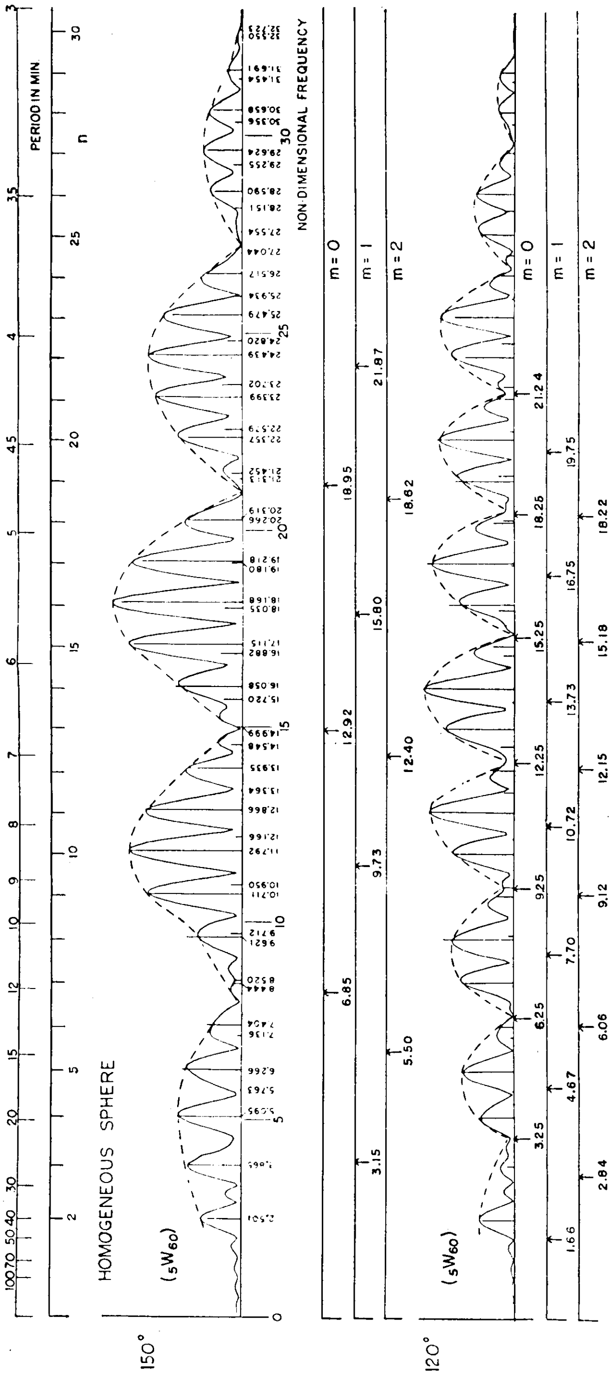

Fig. 1. Spectrum computed from a theoretical seismogram at $\theta=150°$ and $120°$ showing the azimuthal component of displacement of torsional oscillation when $m=0$. Broken line shows guiding curve of spectral peaks. Arrows show the theoretical zero points n of the guiding curve for $m=0$, 1 and 2.

surface with an epicentral distance $\theta=\theta_0$ and azimuth $\phi=\phi_0$ is

$$ {}_iS_n^m = C_n^m \cdot R_n(a, {}_ip_n) \cdot \Theta_n^m(\theta_0) \cdot \frac{\cos}{\sin} m\phi_0 \cdot f^*({}_ip_n) \qquad (2.2)$$

where a is the radius of the earth. If the following notation (2.4) is introduced the spectrum is given by the form

$$ {}_iS_n^m = {}_i(\text{common spectrum})_n^m \cdot \Theta_n^m(\theta_0) \cdot \frac{\cos}{\sin} m\phi_0 \qquad (2.3)$$

in which

$$ {}_i(\text{common spectrum})_n^m = C_n^m \cdot R_n(a, {}_ip_n) \cdot f^*({}_ip_n) \qquad (2.4)$$

Spectrum obtained at a certain point on the surface is, therefore, the product of this common spectrum and the function $\Theta_n^m(\theta) \cdot \frac{\cos}{\sin} m\phi$, which is determined only by the coordinates of the point. Of the three factors, of which the common spectrum consists, the first one C_n^m depends on the geographical distribution of the force applied. If the force is localized within a small range of area this quantity is a slowly varying function of n. The second function $R_n(a, {}_ip_n)$ also varies slowly if the vertical distribution of the material is not exceptionally complicated. The last function $f^*({}_ip_n)$ does not change rapidly, either, if the force is an impulsive one, which is the usual case. If, therefore, the guiding curve of the spectrum peaks is drawn, its zero points should be mostly those of the function $\Theta_n^m(\theta)$.

The solution for each kind of problem is given in Table 1, and the actual form of the function $\Theta_n^m(\theta)$ can be found in the corresponding section. For example, if the radial displacement of the spheroidal oscillation is observed, the guiding curve of its spectrum should approach to zero at the zero points of P_n^m, while for the azimuthal displacement of the torsional oscillation at those of the function $dP_n^m/d\theta$.

3. Tables and figures

Table 1 gives the basic solutions for a number of typical problems. These solutions are chosen because they are well observed by long period inertia instruments and strain seismographs. In this table the underlined part expresses function $\Theta_n^m(\theta)$.

In Figs. 2a-2i the zeroes of the function $\Theta_n^m(\theta)$ given in Table 2 are drawn with the abscissa θ and the ordinate n. Since the above functions are either even or odd function of θ about $\theta=90°$, θ is assumed to be between 0° and 90°. The degree number m is assumed to be 0,

Table 1. Solution of various types of problems relating to the polar coordinates. Underlined part is the function $\Theta_n^m(\theta)$.

Component	Displacement		Strain	
	Spheroidal	Torsional	Spheroidal	Torsional
Radial	$U_n(r)\underline{P_n^m(\cos\theta)}\,{\cos \atop \sin}\, m\phi$	0	$\frac{d}{dr}U_n\cdot \underline{P_n^m(\cos\theta)}\,{\cos\atop\sin}\, m\phi$	0
Colatitudinal	$V_n(r)\underline{\frac{d}{d\theta}P_n^m(\cos\theta)}\,{\cos\atop\sin}\, m\phi$	$W_n(r)\cdot m\,\underline{\frac{P_n^m(\cos\theta)}{\sin\theta}}\,{\cos\atop\sin}\, m\phi$	$\frac{V_n}{r}\underline{\frac{d^2}{d\theta^2}P_n^m(\cos\theta)}\,{\cos\atop\sin}\, m\phi$	$\frac{W_n}{r}m\underline{\frac{d}{d\theta}\left(\frac{P_n^m(\cos\theta)}{\sin\theta}\right)}\,{\cos\atop\sin}\, m\phi$
Azimuthal	$V_n(r)\cdot m\,\underline{\frac{P_n^m(\cos\theta)}{\sin\theta}}\,{\sin\atop -\cos}\, m\phi$	$-W_n(r)\underline{\frac{d}{d\theta}P_n^m(\cos\theta)}\,{\sin\atop -\cos}\, m\phi$	$\frac{V_n}{r}m^2\underline{\frac{P_n^m(\cos\theta)}{\sin^2\theta}}\,{\cos\atop\sin}\, m\phi$	$-\frac{W_n}{r}m\underline{\frac{dP_n^m(\cos\theta)/d\theta}{\sin\theta}}\,{\cos\atop\sin}\, m\phi$

Table 2. Function $\Theta_n^m(\theta)$ treated in this paper.

m				
0	$P_n(\cos\theta)$ (Fig. 2a)	$\frac{d}{d\theta}P_n(\cos\theta)=-P_n^1(\cos\theta)$ (Fig. 2b)	$\frac{d^2}{d\theta^2}P_n(\cos\theta)=-\frac{d}{d\theta}P_n^1(\cos\theta)$ (Fig. 2c)	
1	$P_n^1(\cos\theta)$ (Fig. 2b)	$\frac{d}{d\theta}P_n^1(\cos\theta)$ (Fig. 2c)	$\frac{d^2}{d\theta^2}P_n^1(\cos\theta)$ (Fig. 2d)	$\frac{d}{d\theta}\left(\frac{P_n^1(\cos\theta)}{\sin\theta}\right)$ (Fig. 2e)
2	$P_n^2(\cos\theta)$ (Fig. 2f)	$\frac{d}{d\theta}P_n^2(\cos\theta)$ (Fig. 2g)	$\frac{d^2}{d\theta^2}P_n^2(\cos\theta)$ (Fig. 2h)	$\frac{d}{d\theta}\left(\frac{P_n^2(\cos\theta)}{\sin\theta}\right)$ (Fig. 2i)

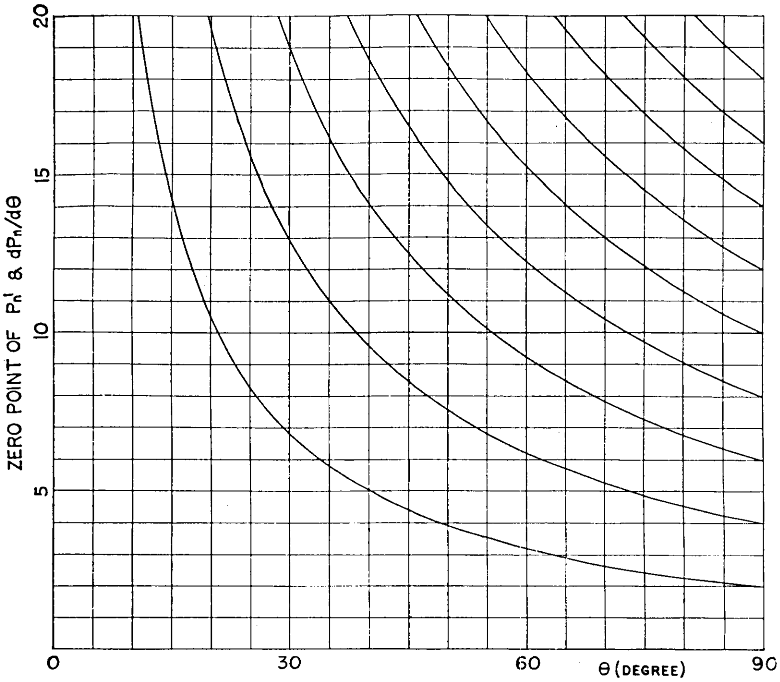
ZERO POINT OF P_n^1 & $dP_n/d\theta$
5
10
15
20
0
30
60
θ (DEGREE)
90

Fig. 2b.

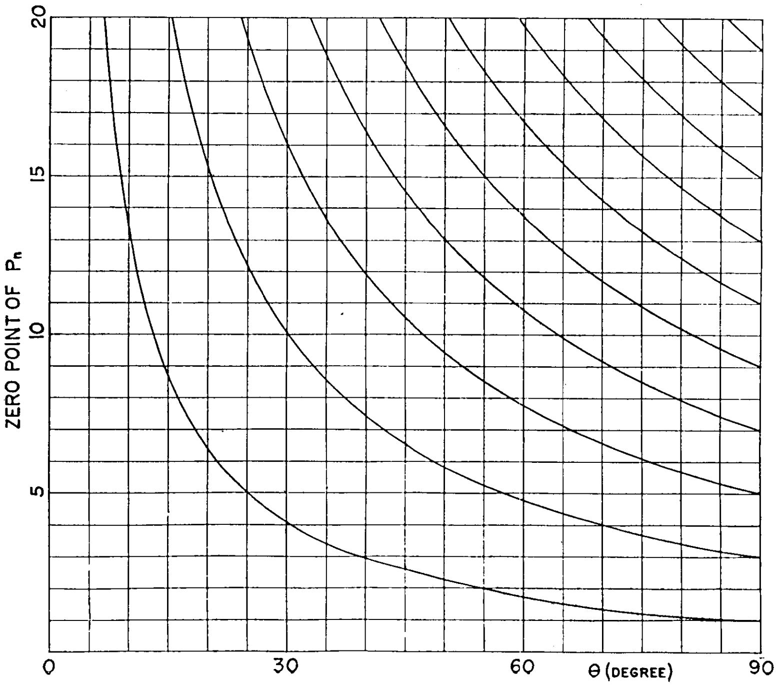
ZERO POINT OF P_n
5
10
15
20
0
30
60
θ (DEGREE)
90

Fig. 2a.

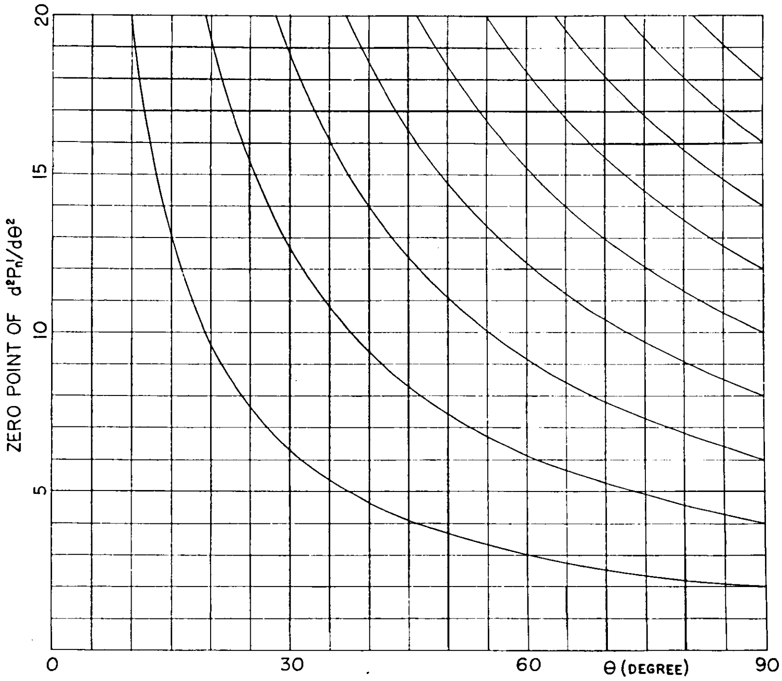
ZERO POINT OF $d^2P_n^1/d\theta^2$
0
5
10
15
20
30
60
90
θ (DEGREE)

Fig. 2d.

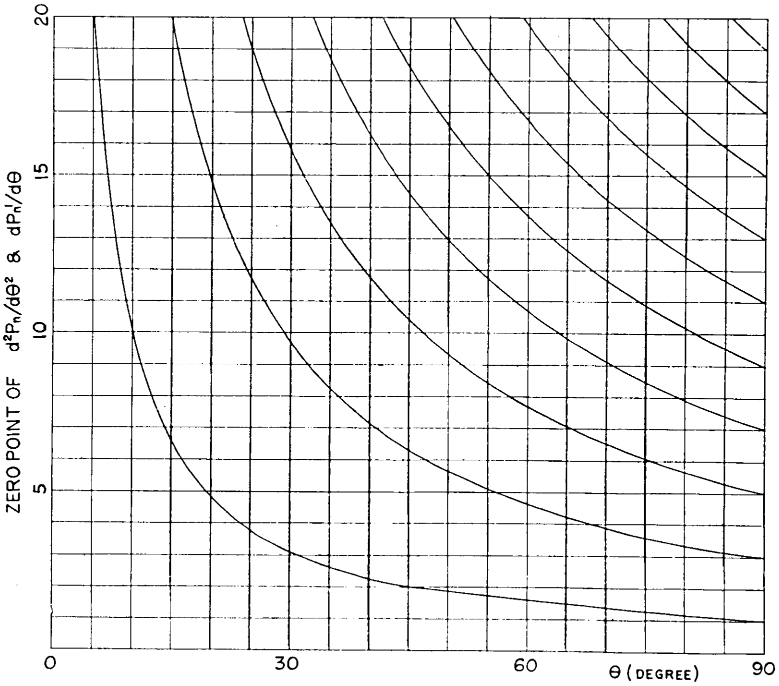
ZERO POINT OF $d^2P_n/d\theta^2$ & $dP_n^1/d\theta$
0
5
10
15
20
30
60
90
θ (DEGREE)

Fig. 2c.

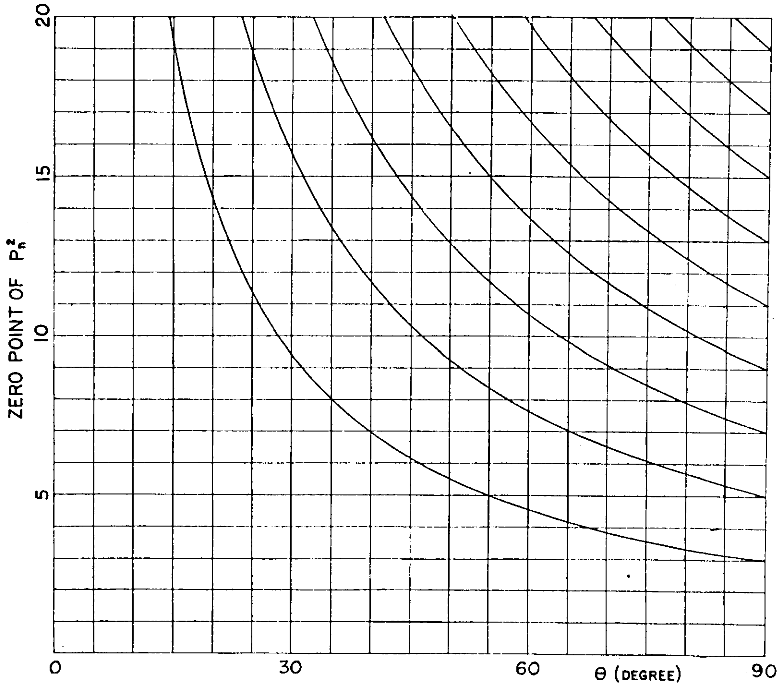
ZERO POINT OF P_n^2
20
15
10
5
0
30
60
θ (DEGREE)
90

Fig. 2f.

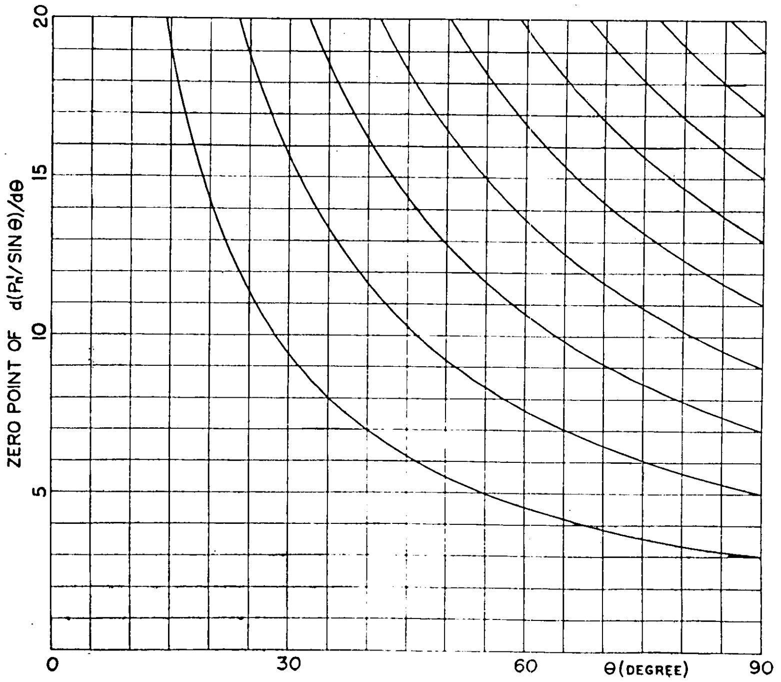
ZERO POINT OF $d(P_n^1/\sin\theta)/d\theta$
20
15
10
5
0
30
60
θ(DEGREE)
90

Fig. 2e.

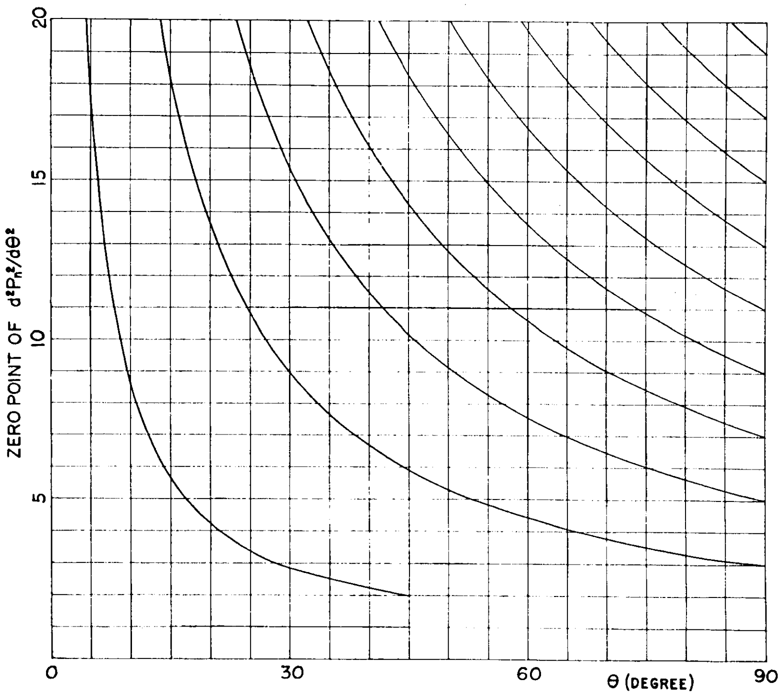
ZERO POINT OF $d^2P_n^2/d\theta^2$
20
15
10
5
0
30
60
θ (DEGREE)
90

Fig. 2h.

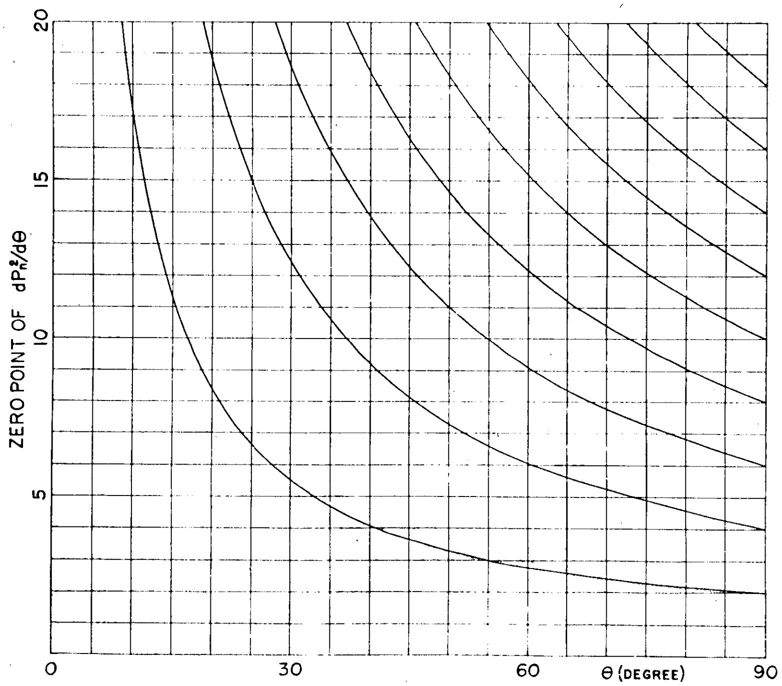
ZERO POINT OF $dP_n^2/d\theta$
20
15
10
5
0
30
60
θ (DEGREE)
90

Fig. 2g.

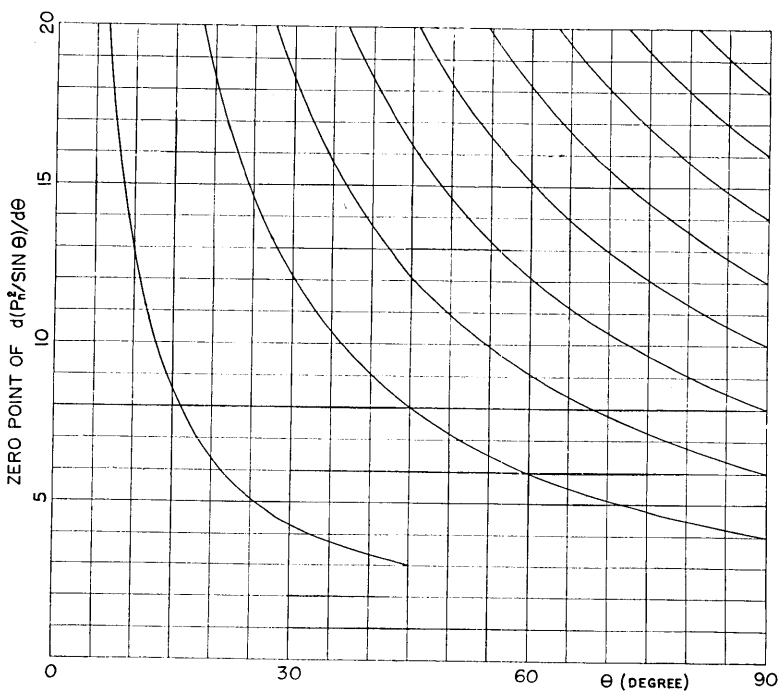

Fig. 2i.

1 or 2, since these are the cases in which we are interested.

When n is fairly large and $\sin\theta$ is not very small, the asymptotic expansion of the associated Legendre function

$$P_n^m(\cos\theta)\sim(-)^m n^m\sqrt{\frac{2}{n\pi\sin\theta}}\cos\left\{\left(n+\frac{1}{2}\right)\theta+\frac{m\pi}{2}-\frac{\pi}{4}\right\} \quad (3.1)$$

is useful, because it holds with a good degree of accuracy.

Most of the numerical work was carried out using the IBM 7090, and the rest by OKITAC 5090. Some of the zero points near $\theta=90°$ were obtained from the above asymptotic expression. This formula is accurate enough for preparing the figures, which are, considering the precision of the present observational work, more than enough for the actual work.

4. Example of the application of the method

When a seismogram is given, the first step is the spectrum analysis of the observed disturbance. Then the zero points of the guiding curve of the spectral peaks are compared with those of the specific $\Theta_n^m(\theta)$-function corresponding to that problem. If one of the series of zeroes, picked up from the figures for $m=0$, 1 and 2, agrees well with the observed ones, the degree number m, or the azimuthal characteristic of the source is determined.

The spectrum given in Fig. 1 was obtained analysing the theoretical seismogram for the azimuthal displacement of the torsional oscillation at the epicentral distance $\theta=150°$[1]. Therefore in Fig. 2b, which is for the case $m=0$, drawing a straight line parallel to the ordinate at $\theta=30°$ $(=180°-150°)$, zero points are found as follows.

$$n=6.85,\ 12.92,\ 18.95 \tag{4.1}$$

These values are given by the arrows in the same figure exactly showing the zeroes of the guiding curve. If another value is assumed for m, the zero points come out as follows;

$$\left.\begin{array}{ll} m=1; & n=3.15,\ 9.73,\ 15.80,\ 21.87 \\ m=2; & n=5.50,\ 12.40,\ 18.62 \end{array}\right\} \tag{4.2}$$

These values are plotted in the same figure, but the discrepancy from the zeroes of the guiding curve is large and it is hard to adopt either of the two values of m.

For $\theta=60°$ the same method was applied and the zero points came out as follows.

$$\left.\begin{array}{ll} m=0; & n=3.25,\ 6.25,\ 9.25,\ 12.25,\ 15.25,\ 18.25,\ 21.24 \\ m=1; & n=1.66,\ 4.67,\ 7.70,\ 10.72,\ 13.73,\ 16.75,\ 19.75 \\ m=2; & n=2.84,\ 6.06,\ 9.12,\ 12.15,\ 15.18,\ 18.22 \end{array}\right\} \tag{4.3}$$

The result is given in Fig. 1, from which we may conclude that (1) the observation near the pole is preferable to the one near the equator, and (2) the long range of the time of observation is not always good for this kind of analysis.

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22. 半径方向に不均質な弾性球の自由振動のデグリーを決める方法

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気象庁地震課 宇佐美龍夫

半径方向に不均質な弾性球の自由振動の解は球面調和函数 $P_n^m(\cos\theta)\cdot\exp(im\phi)$ または，これを多少変形した函数を含んでいる．この解で自由振動のオーダーは固有周期と密接な関係があるが，デグリー m は周期と無関係である．したがつて，自由振動のオーダーは長周期波の周期分析の結果と理論とを比較すれば知ることができる．しかし，デグリー m は周期分析をしても知ることができない．

一方デグリー m は震源における力の方位特性と関係があるから，観測結果から m を求めることができれば，震源のメカニズムの解明にも役に立つと思われる．

周期分析の結果，いくつかのピークが出るのが普通である．このピークの先端を結ぶ線を引くと，この線の零点の列は，球の自由振動のデグリー m と密接に関係している．もし力の加わる面積が小さく，弾性球の物質があまり複雑でなく，かつ外力が衝撃的なものなら，この零点列は主として，表面球函数 P_n^m または，それを多少変形した函数 Θ_n^m の零点となる．したがつて，各種の問題に対して，理論的に Θ_n^m の零点列を求めておけば，それと観測の分析結果から得られる零点列とを比べることにより，自由振動のデグリー m をきめることができる．

筆者らは，先に等質な弾性球の捩れ振動による表面での振巾の時間的変化を示す理論的な地震記象を求めた．これに対して上の理論を適用した結果が第1図に示してある．この図は，理想的な場合には，われわれの方法が満足すべき結果を与えることを示している．
