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42. *The Directivity of Tsunami (2): The Case of Instantaneously and Uniformly Elevated Square Wave Origin.*

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1. Introduction.

We introduced the general method of treating tsunami caused by the movement of a portion of the bottom with an arbitrary form[1] and applied this method to the tsunami produced by the uniform elevation of the elliptic wave origin.[2] In the present paper the author intends to apply this method to the tsunami generated by the instaneously and uniformly elevated square wave origin.

2. Theory.

We used the cylindrical co-ordinates (r, θ, z), (r, θ) being taken horizontally at the undisturbed free surface of water and z vertically upwards. Then assuming that ζ is the elevation of water from the undisturbed free surface of water, g the acceleration of gravity, t the variable of time, H the depth of water, Φ the velocity potential and η the velocity of the bottom displacement, we have:

as a basic equation,

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)\Phi=0\ , \tag{1}$$

as surface condition,

$$\frac{\partial^2 \Phi}{\partial t^2}+g\frac{\partial \Phi}{\partial z}=0 \quad (z=0)\ , \tag{2}$$

1) T. MOMOI, *Bull. Earthq. Res. Inst.*, **40** (1962), 309.
2) T. MOMOI, *Bull. Earthq. Res. Inst.*, **40** (1962), 297.
In the subsequent discussions, the former and the latter papers are referred to as "paper A" and "paper B" respectively.

and, as bottom condition,

$$\frac{\partial \Phi}{\partial z} = \eta \quad (z = -H) . \tag{3}$$

From the above equation (1) and the conditions (2) and (3) we obtained the wave height at the distant point from the wave origin as follows (refer to the preceding paper A);

$$\zeta' \simeq \frac{1}{-g'} \cdot \frac{1}{\sqrt{2\pi}} \cdot i \cdot \frac{\pi}{2} \cdot C^{3/2} \cdot \sum_{m=0}^{\infty} \sum_{u=1}^{\infty} [M_{m,u} \cos m\theta + N_{m,u} \sin m\theta] \cdot \\ \cdot \lambda_{m,u} \cdot J'_m(\lambda_{m,u}) , \tag{4}$$

where

$$\left.\begin{matrix} M_{m,u} \\ N_{m,u} \end{matrix}\right\} = \frac{1}{2\sqrt{2\pi}} \left[\int_{-\infty}^{t-\frac{r'+1}{c}} \left\{\begin{matrix} A_{m,u}(\alpha) \\ B_{m,u}(\alpha) \end{matrix}\right\} Q^{(1)}(\alpha) d\alpha \right. \\ \left. + \int_{t-\frac{r'+1}{c}}^{t-\frac{r'-1}{c}} \left\{\begin{matrix} A_{m,u}(\alpha) \\ B_{m,u}(\alpha) \end{matrix}\right\} Q^{(2)}(\alpha) d\alpha \right] , \tag{5}$$

$$Q^{(1)}(\alpha) = -4i \cdot \sqrt{\frac{2}{r'}} \cdot \frac{1}{\sqrt{\pi}} \int_0^{\infty} df_0 \frac{f_0^{3/2}}{f_0^2 + \lambda_{m,u}^2} \cdot J_m(f_0) \cdot e^{-f_0\{c(t-\alpha)-r'\}} , \tag{6}$$

$$Q^{(2)}(\alpha) = \pi \cdot \lambda_{m,u} \cdot N_m(\lambda_{m,u}) \cdot \sqrt{\frac{2}{\pi \lambda_{m,u} r'}} \cdot (2i) \cdot \sin\left[\lambda_{m,u}\{c(t-\alpha) - r'\} + \frac{2m+1}{4}\pi\right] \\ + i \int_0^{\infty} df_0 \frac{f_0^{3/2}}{f_0^2 + \lambda_{m,u}^2} \cdot (-4) \cdot \frac{K_m(f_0)}{\pi} \cdot \sqrt{\frac{2}{\pi r'}} \cdot e^{+f_0\{c(t-\alpha)-r'\}} , \tag{7}$$

$$D' = \sum_{m=0}^{\infty} \sum_{u=1}^{\infty} \{A_{m,u}(t) \cos m\theta + B_{m,u}(t) \sin m\theta\} \cdot J_m(\lambda_{m,u} r') , \tag{8}$$

$$\left.\begin{matrix} A_{m,u}(t) \\ B_{m,u}(t) \end{matrix}\right\} = \frac{\varepsilon_m}{\pi [J_{m+1}(\lambda_{m,u})]^2} \cdot \int_{-\pi}^{\pi} \int_0^1 D' \cdot \left\{\begin{matrix} \cos m\theta \\ \sin m\theta \end{matrix}\right\} \cdot J_m(\lambda_{m,u} r') \cdot d\theta \cdot r' dr' \tag{9}$$

$$(\varepsilon_0 = 1, \ \varepsilon_2 = \varepsilon_3 = \cdots = 2) ,$$

$H' = H/r_0$, $\zeta' = \zeta/r_0$, $D' = D/r_0$, $r' = r/r_0$, $g' = g/r_0$, $C = \sqrt{g'H'}$, r_0 denotes the radius of the circumscribed circle of a displaced portion of the bottom, D and D_0 stand for the displacement and the amplitude of the displacement of the bottom, $A_{m,u}(t)$ and $B_{m,u}(t)$ ($m=0, 1, 2, 3, \cdots$; $u=1, 2, 3, \cdots$) are the Fourier Bessel coefficients in Fourier Bessel expansion of the dimensionless displacement $D'(=D/r_0)$, $\lambda_{m,u}$ ($m=0, 1, 2, \cdots$; $u=1, 2, 3, \cdots$) are the positive roots of $J_m(\lambda_{m,u})=0$ taken in order from the smallest positive value.

In the case of the square wave origin, the expression of the bottom (8) degenerates into the following form by the symmetry of the square wave origin, i. e., $D'(r', \theta)=D'(r', -\theta)=D'(r', \pi-\theta)=D'(r', \pi/2-\theta)$ (refer to Fig. 1);

$$D'=\sum_{m=0}^{\infty}\sum_{u=1}^{\infty} A_{4m,u}(t)\cdot\cos 4m\theta\cdot J_{4m}(\lambda_{4m,u}r'), \quad (8')$$

where, from (9),

$$A_{4m,u}(t)=\frac{8\varepsilon_{4m}}{\pi[J_{4m+1}(\lambda_{4m,u})]^2}\cdot\int_0^{\pi/4}\int_0^1 D'\cdot\cos 4m\theta\cdot J_{4m}(\lambda_{4m,u}r')\cdot d\theta\cdot r'dr' \quad (9')$$

$$(\varepsilon_0=1,\ \varepsilon_4=\varepsilon_8=\cdots=2)\ .$$

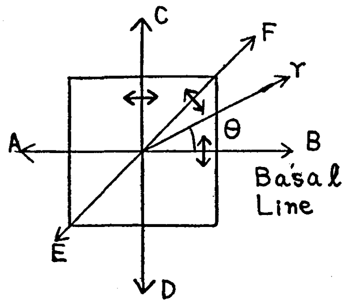

Fig. 1. The lines AB, CD and EF denote the symmetric axes of the square wave origin.

Then the expressions (4), (5), (6) and (7) are reduced to

$$\zeta'\simeq\frac{1}{-g'}\cdot\frac{1}{\sqrt{2\pi}}\cdot i\cdot\frac{\pi}{2}\cdot C^{3/2}\cdot\sum_{m=0}^{\infty}\sum_{u=1}^{\infty}[M_{4m,u}\cos 4m\theta]\cdot\lambda_{4m,u}\cdot J'_{4m}(\lambda_{4m,u})\ , \quad (4')$$

$$M_{4m,u}=\frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{t-\frac{r'+1}{c}} A_{4m,u}(\alpha)Q^{(1)}(\alpha)d\alpha+\int_{t-\frac{r'+1}{c}}^{t-\frac{r'-1}{c}} A_{4m,u}(\alpha)Q^{(2)}(\alpha)d\alpha\right], \quad (5')$$

$$Q^{(1)}(\alpha)=-4i\cdot\sqrt{\frac{2}{r'}}\cdot\frac{1}{\sqrt{\pi}}\int_0^{\infty}df_0\frac{f_0^{3/2}}{f_0^2+\lambda_{4m,u}^2}\cdot J_{4m}(f_0)\cdot e^{-f_0\{c(t-\alpha)-r'\}}\ , \quad (6')$$

$$Q^{(2)}(\alpha)=\pi\cdot\lambda_{4m,u}\cdot N_{4m}(\lambda_{4m,u})\cdot\sqrt{\frac{2}{\pi\lambda_{4m,u}r'}}\cdot(2i)\cdot\sin\left[\lambda_{4m,u}\{c(t-\alpha)-r'\}+\frac{8m+1}{4}\pi\right]$$

$$+i\int_0^{\infty}df_0\frac{f_0^{3/2}}{f_0^2+\lambda_{4m,u}^2}\cdot(-4)\cdot\frac{K_{4m}(f_0)}{\pi}\sqrt{\frac{2}{\pi r'}}\cdot e^{+f_0\{c(t-\alpha)-r'\}}\ . \quad (7')$$

Since we are considering the instaneously and uniformly elevated wave origin, the time factor of the Fourier Bessel coefficients $A_{4m,u}(t)$ in the expression (8′) can be taken out of the Fourier Bessel series, that is to say, $D'=T(t)\cdot D_0'\cdot E(r, \theta)$, $D_0'=D_0/r_0$, $E(r, \theta)=1$ and 0 inside and outside the displaced bottom respectively, $A_{4m,u}(t)=T(t)\cdot D_0'\cdot G_{4m\ u}$ in (9′) or

$$E(r, \theta)=\sum_{m=0}^{\infty}\sum_{u=1}^{\infty} G_{4m,u}\cos 4m\theta\cdot J_{4m}(\lambda_{4m,u}r')\ , \quad (10)$$

and hence the expression (9′) becomes

$$G_{4m,u}=\frac{8\varepsilon_{4m}}{\pi[J_{4m+1}(\lambda_{4m,u})]^2}\cdot\int_0^{\pi/4}\int_0^1 E\cdot\cos 4m\theta\cdot J_{4m}(\lambda_{4m,u}r')\cdot d\theta\cdot r'dr'\ . \quad (11)$$

And also the expression (5′) is transformed to

$$M_{4m,u}=\frac{D_0'\cdot G_{4m,u}}{2\sqrt{2\pi}}\left[\int_{-\infty}^{t-\frac{r'+1}{c}} T(\alpha)\cdot Q^{(1)}(\alpha)d\alpha+\int_{t-\frac{r'+1}{c}}^{t-\frac{r'-1}{c}} T(\alpha)\cdot Q^{(2)}(\alpha)d\alpha\right]. \quad (12)$$

Here the time factor $T(t)$ is expressed as

$$\left.\begin{aligned} T(t)&=0 \quad (t<0)\ ,\\ &=1 \quad (t>0)\ ,\end{aligned}\right\} \quad (13)$$

Putting (12) into (4′) leads to

$$\zeta'\simeq\frac{1}{-g'}\cdot\frac{D_0'}{8}\cdot C^{3/2}\cdot\sum_{m=0}^{\infty}\sum_{u=1}^{\infty}G_{4m,u}\cos 4m\theta\cdot\lambda_{4m,u}\cdot J_{4m}'(\lambda_{4m,u})$$

$$\cdot\left[\int_{-\infty}^{t-\frac{r'+1}{c}} T(\alpha)\cdot Q^{(1)}(\alpha)d\alpha+\int_{t-\frac{r'+1}{c}}^{t-\frac{r'-1}{c}} T(\alpha)\cdot Q^{(2)}(\alpha)d\alpha\right]. \quad (14)$$

Now substituting (6′), (7′), (13) into (14) and integrating, we get:

i) when $t<\dfrac{r'-1}{c}$,

$$\zeta'\simeq 0\ ; \quad (15)$$

ii) when $\dfrac{r'-1}{c}<t<\dfrac{r'+1}{c}$,

$$\zeta'\simeq\frac{1}{g'}\cdot\frac{1}{\sqrt{2\pi}}\cdot C^{1/2}\cdot\sqrt{\frac{1}{r'}}\cdot D_0'\cdot\zeta_r'\ ,$$

$$\zeta_r'=\sum_{m=0}^{\infty}\sum_{u=1}^{\infty}G_{4m,u}\cdot\lambda_{4m,u}\cdot J_{4m}'(\lambda_{4m,u})\cdot\cos 4m\theta$$

$$\cdot\left[\pi\cdot N_{4m}(\lambda_{4m,u})\cdot\frac{1}{\sqrt{\lambda_{4m,u}}}\cdot\sin\left\{\frac{1}{2}\lambda_{4m,u}(ct-r'-1)+\frac{8m+1}{4}\pi\right\}\right.$$

$$\left.\cdot\sin\left\{\frac{1}{2}\lambda_{4m,u}(ct-r'+1)\right\}+\frac{1}{\pi}\{L_{4m,u}(-1)-L_{4m,u}(ct-r')\}\right]\ ; \quad (16)$$

iii) when $\dfrac{r'+1}{c}<t$,

$$\zeta'\simeq\frac{1}{g'}\cdot\frac{1}{\sqrt{2\pi}}C^{1/2}\cdot\sqrt{\frac{1}{r'}}\cdot D_0'\cdot\zeta_r'\ ,$$

$$\zeta'_r=\sum_{m=0}^{\infty}\sum_{u=1}^{\infty} G_{4m,u}\cdot\lambda_{4m,u}\cdot J'_{4m}(\lambda_{4m,u})\cdot\cos 4m\theta$$

$$\cdot\Big[-M_{4m,u}(-1)+M_{4m,u}\{-(ct-r')\}+\frac{1}{\pi}\{L_{4m,u}(-1)-L_{4m,u}(+1)\}$$

$$+\pi\cdot N_{4m}(\lambda_{4m,u})\cdot\frac{1}{\sqrt{\lambda_{4m,u}}}\cdot\sin\frac{8m+1}{4}\pi\cdot\sin\lambda_{4m,u}\Big]\,; \tag{17}$$

where

$$L_{4m,u}(x)=\int_0^\infty df_0\frac{\sqrt{f_0}}{f_0^2+\lambda_{4m,u}^2}K_{4m}(f_0)\cdot e^{f_0 x}\,,$$

and

$$M_{4m,u}(x)=\int_0^\infty df_0\frac{\sqrt{f_0}}{f_0^2+\lambda_{4m,u}^2}J_{4m}(f_0)\cdot e^{f_0 x}\,. \tag{18}$$

3. Numerical Analysis.

In order to integrate (11), we divided the domain of the integration into small parts, i. e.,

$$G_{4m,u}=\frac{8\varepsilon_{4m}}{\pi[J_{4m+1}(\lambda_{4m,u})]^2}\cdot\sum_{ij}E\cdot\cos 4m\theta_{ij}\cdot J_{4m}(\lambda_{4m,u}r'_{ij})\cdot\Delta x'\cdot\Delta y'\,, \tag{19}$$

where

$E=1$ inside the square wave origin,
$=0$ outside 〃 ;
$\varepsilon_0=1$, $\varepsilon_4=\varepsilon_8=\cdots=2$;
$\lambda_{4m,u}$ ($m=0, 1, 2,\cdots$; $u=1, 2, 3,\cdots$) are the positive roots of $J_{4m}(x)=0$, being taken in order from the smallest positive value ;
x and y are related with r and θ such as $\tan\theta=\frac{y}{x}$, $r^2=x^2+y^2$;
$x'=x/r_0$, $y'=y/r_0$;
the lengths of the small intervals $\Delta x'$ and $\Delta y'$ are taken as $\Delta x'=\Delta y'=0.02$;
$x'_i=i\cdot\Delta x'$, $y'_j=j\cdot\Delta y'$ ($i, j=0, 1, 2,\cdots 49$) ;
$\theta_{ij}=\tan^{-1}\frac{y'_j}{x'_i}$, $r'_{ij}=\sqrt{x'^2_j+y'^2_j}$:

For the integration (18), we used the method of *graphical integration.* Here since we derived the general theory within the scope of the long wave approximation, the number of terms in the series (16) and (17) must be restricted in accordance with such approximation.

If we take the ratio between the depth and the radius of the circumscribed circle of the square wave origin as 1/10, the restriction of long wave becomes

$$a_0'H' \ll 1 \quad \text{or} \quad a_0' \ll \frac{1}{H'} = \frac{r_0}{H} = 10\ , \tag{20}$$

where $a_0' = a_0/r_0$, a_0 is the wave number of the progressive wave.

As in paper B, the restriction (20) becomes

$$\lambda_{m,u} \ll 10\ .$$

And hence the necessary and sufficient terms in the series (16) and (17) are obtained, after paper B, as follows;

$$\left.\begin{aligned} &\lambda_{0,u} \quad (u=1,2,3)\ , \\ &\lambda_{4,u} \quad (u=1,\quad\ )\ . \end{aligned}\right\} \tag{21}$$

Now we can draw the curves of the relative wave height ζ_r' in each direction according to the above-mentioned methods and restriction, which is shown in Fig. 2.

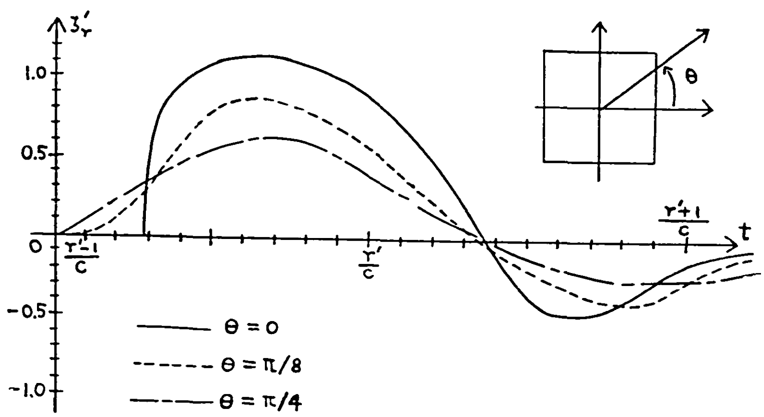

Fig. 2. The variation of the relative wave height ζ_r' in each direction versus time variation.

The maximum values of the relative wave height ζ_r' in each direction read from Fig. 2 are given in Table 1, which shows that the wave height in the direction parallel to the side of the square wave origin is higher than that in the direction of the diagonal by $(1-0.54)\times 100\%$ $=46\%$ and that the form of the former is different from that of the latter (refer to Fig. 2).

Table 1. The ratios between the wave height in each direction and that in the direction of $\theta=0$.

θ	θ	$\pi/8$	$\pi/4$
ratio	1	0.77	0.54

4. Acknowledgment.

The author wishes to offer his thanks to Professor R. Takahasi and Assistant Professor K. Kajiura of this Institute and to Dr. K. Takano of the Geophysical Department of Tokyo University for their kind discussions.

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42. 津波の方向性について (2)：瞬間的に，かつ一様にもち上げられた正方形浪源の場合

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著者はさきに任意の形の海底変位によつておこされる津波の解法を論じ，それを瞬間的に上る楕円浪源の場合に適用して，津波の方向性を論じた．本論文において著者はさらにこの解法を正方形浪源の場合にも適用し，Fig. 2 および Table 1 に示すごとき津波の方向性を得た．