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18. *Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction. Part 2 with Special Remarks on Wave Propagation in a Periodic Structure.**

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1. Introduction

In the previous paper,[1] apparent attenuation of waves through a periodic structure was studied. The method is theoretically reduced to the study of the solution of Mathieu's equation in an unstable region. The effect of the periodic structure on the wave with a certain period is expressed by the following formula ;–

$$A \propto 1/\cosh \mu z \tag{1}$$

instead of a mere exponential decay function $\exp(-\mu z)$.

From (32) in the previous paper we know that apparent attenuation like this is caused, or, we may say, balanced in energy, by generation of reflected waves. Those reflected waves are superposed and transform themselves partly into higher harmonic waves and partly into stationary waves, which, in nature, will correspond to the so-called "characteristic vibration of the ground" in an earthquake.

However, the structure that was assumed in the calculation is too much simplified to deduce from its result a conclusion on the apparent attenuation of earthquake waves effected by heterogeneous structure of the earth-crust. To approach the general theory in an irregular structure, step by step, another special case is studied.

* R. YOSHIYAMA alone is responsible for the remarks.

1) R. YOSHIYAMA, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction." *Bull. Earthq. Res. Inst.*, **38** (1960), 467-478. This is referred to in this paper as Part 1.

2. Remarks on wave propagation in a periodic structure

In the course of these studies, it was suggested[†] by a few that the difference should be made clear between the results, a) by the present writers and b) by Prof. L. Brillouin, those of the latter being published in his book, "Wave Propagation in Periodic Structure." Accordingly, some remarks are made.

It looks as though it covers all kinds of problems of waves in a periodic structure. Certainly, explanation of Mathieu's equation in relation to the problem of wave propagation is also given in Chap. VIII, "Mathieu's Equation and Related Problems." But it is not so detailed, and seems not so rigorous, as that already given by Lord Rayleigh[2] in 1887. Especially concerning the solution in an unstable region, while its interpretation and application should be the point of study at present, only a rough sketch of its properties is given, which seems misleading, in spite of elaborate explanations of the problem given in the preceding chapters. The reason why it seems misleading is as follows.

In Sec. 42, he gives Mathieu's equation[3],

$$\frac{\partial^2 u}{\partial \xi^2}+(\eta+\gamma\cos 2\xi)u=0$$

and continues, "Floquet discovered that the general solution of the equation could be written

$$u=D_1A(\xi)e^{\mu\xi}+D_2B(\xi)e^{-\mu\xi}$$

with amplitude A and B that are periodic function of ξ with period π. This solution is thus a superpositions of two waves propagated (or attenuated) in opposite directions. This is clearly seen if we retain the $e^{i\omega t}$ factor and write original ψ function of Eq. (42.1). D_1 and D_2 are arbitrary constants. If we keep only *one* of these waves, we obtain

$$\underline{u=A(\xi)e^{\mu\xi}} \qquad \underline{A(\xi)\text{ has period }\pi\,.} \tag{42.6}$$

........The frequency has been chosen as primary data, and the problem is to obtain μ, which may be

†) Discussions at the meeting of Seism. Soc. Japan, Oct. 10, 1961.

2) Lord Rayleigh, *Sci. Papers*, Vol, 3. pp. 1-14.

3) Léon BRILLOUIN, *Wave Propagation in Periodic Structure* (Dover Pub.), pp. 172.

$$\left.\begin{array}{ll} \mu = i\beta & \text{pure imaginary, unattenuated sine wave} \\ \mu = \alpha + i\beta & \text{complex or real, attenuated motion} \end{array}\right\} \qquad (42.7)$$

...."

Formula of μ in connection with η and γ is not given in this section. In the succeeding Sec. 43 "Mathieu's Function, General Discussion", it is written, together with that $A(\xi)$ and $B(\xi)$ are a certain constant respectively when $\gamma=0$, that

$$\mu = im = f(\eta, \gamma)\,, \quad \gamma \to 0\,, \quad m \to \sqrt{\eta}\,. \qquad (43.3)$$

As Brillouin also may be aware, this asymptotic formula is not available when μ is complex, because, if it is calculated by Hill's method, its imaginary part is a certain integer, as already shown by Rayleigh.

In any case, if we expect to learn something in this chapter, the point of great consequence lies in how to understand the passage underlined.

Though definite qualification of each of the waves which he names "two waves", is not given, one might generally understand as it implicates a wave slightly, assuming γ is not large, changed in form from what might propagate when $\gamma=0$. From the viewpoint of seismological application, it is desirable that studies be conducted holding an image of a wave on that implication. As "*one* of these waves" would mean a wave in one of two directions which are implied in the "opposite directions", either propagated (unattenuated) or attenuated, the passage in question would mean that, by putting one of two arbitrary constants $D_2=0$ in the general solution of the unstable region, we can obtain at once a mathematical representation of an attenuated wave progressing in one direction through a periodic structure, amplitude of which would vary slightly and periodically, ruled by $A(\xi)$, and would die down exponentially, ruled by $\exp(\alpha\xi)$, along x-axis, in the direction of propagation. After all, in spite of the statement concerning μ when it is complex, one might be led from this passage automatically to suppose that, even in an unstable region, the main features of the "two waves" in opposite directions with decreasing, though not monotonous, amplitude are given by $e^{\mu\xi}e^{i\omega t}$ and $e^{-\mu\xi}e^{i\omega t}$ respectivly, and that these two waves are propagated, which means, of course, that they satisfy the equation of motion, independently of each other.

If this is the case, the problem of wave propagation in a periodic structure could be solved without difficulty. However, the solution of Mathieu's equation in an unstable region can not be obtained in an expression which is fitted to those interpretations, adopted, it seems, by Prof. Brillouin. It seems, furthermore, unreasonable from the law of energy conservation that an interpretation like that should hold, because the medium is at any rate a perfectly elastic one.

Apparent attenuation of wave in a perfectly elastic medium, if any, may come from a mechanism like a resonator, and will be quite different in nature from the attenuation by viscosity or any other consuming mechanism.

To refer to the results by Rayleigh may be another objection probably most easy to understand. Rayleigh introduces Mathieu's equation

$$\frac{d^2W}{d\xi^2}+(\Theta_0+2\Theta_1\cos 2\xi)W=0\ ,$$

practically the same as that by Brillouin, and obtained the general solution in one of the unstable regions, retaining time factor e^{ipt},

$$\begin{aligned} W=&Re^{-s\xi}\{\Theta_1e^{i(pt-\xi)}+(1-\Theta_0-2is)e^{i(pt+\xi)}\} \\ &+Se^{+s\xi}\{\Theta_1e^{i(pt-\xi)}+(1-\Theta_0+2is)e^{i(pt+\xi)}\} \end{aligned} \tag{72}$$

R and S are arbitrary constants, but s is not arbitrary. It should be determined so that W satisfies Mathieu's equation, and he gives

$$4s^2=\Theta_1^2-(\Theta_0-1)^2\ . \tag{43}$$

Therefore he says, "Whatever may be the relative values of R and S, the first solution preponderates when x is large and negative, and the second preponderates when x is large and positive. In either extreme case the motion is composed of two progressive waves moving in opposite directions, *whose amplitudes are equal in virtue of (43).*"

The important point in these accounts by Rayleigh compared with the foregoing by Brillouin lies clearly in the passages underlined, a part of which is written in italics by Rayleigh himself. It seems the statements by these two eminent authors concerning one of fundamental properties of waves in a periodic structure are inconsistent with each other.

According to Rayleigh, we are unable, despite Brillouin's assumption, to "keep only one of two waves", because their amplitudes are necessarily equal. In the region of unstable solution, that is when we think about an attenuated wave, s is real. And, whatever we may assume to be the value of arbitrary constants R and S, we cannot build up from (72) a simply decaying function which readily represents a wave progressing in one direction. It will be still more clear, if we refer to the results of recent studies of Mathieu's equation and Mathieu's function.

This portion may be a trivial one in his book, because the chief interest of Prof. Brillouin might still have been in the solution in a stable region, though it is remarked in Sec. 42 that "Here the discussion is conducted in the opposite way. The frequency has been chosen as primary data, and......" The circumstances are quite different with us in the studies of earthquake waves. Moreover, the interpretation by Rayleigh is not sufficient for our study, as far as it is explicitly given.

3. Effect of intermediate homogeneous medium

The medium is divided into five parts, from I to V, at $x=x_1$, x_2, x_3, and x_4, where $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4$. The wave is propagated from medium I to V. Density ρ is assumed constant, and expressed as ρ_0; velocity c in each part is assumed as follows:

I : $x \leqslant x_1$, homogeneous structure, $c_1=\text{const.}$,
II : $x_1 \leqslant x \leqslant x_2$, periodic str., $c_2=a\{1+b\cos\gamma(x-x_1)\}$,
III : $x_2 \leqslant x \leqslant x_3$, homogeneous str., $c_3=\text{const.}$,
IV : $x_3 \leqslant x \leqslant x_4$, periodic str., $c_4=a\{1+b\cos\gamma(x-x_3)\}$,
V : $x_4 \leqslant x$, homogeneous str., $c_5=\text{const.}$

Then the wave motion in each part is obtained by the method explained in Part 1; it is, removing common time factor $\exp(ipt)$ and assuming $|b| \ll 1$,

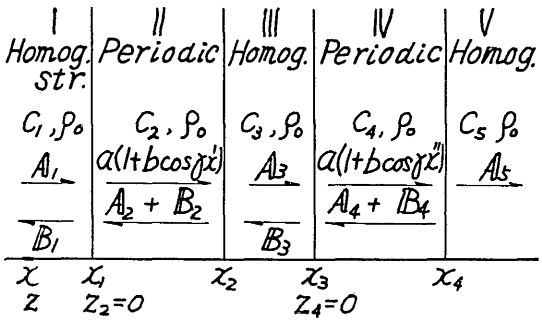

Fig. 1. Schematic illustration of the assumption and the notations. $x'=x-x_1$, $x''=x-x_3$.

$$\text{I}: \ \mathbf{A}_1=\frac{A_1}{\sqrt{\rho_0 c_1}}\exp\{-ik_1(x-x_1)\},$$

$$\mathbf{B}_1=\frac{B_1}{\sqrt{\rho_0 c_1}}\exp\{ik(x-x_1)\},$$

$$k_1=\frac{p}{c_1}.$$

$$\text{II}: \ \mathbf{A}_2+\mathbf{B}_2=\frac{1}{\sqrt{\rho_0 c_2}}\{A_2\psi(-z_2)\exp(-\mu z_2)+B_2\psi(z_2)\exp(\mu z_2)\},$$

$$t_2=\int_{x_1}^{x}\frac{dx}{c_2}, \qquad z_2=\frac{\gamma a t_2\sqrt{1-b^2}}{2}\longrightarrow\frac{\gamma(x-x_1)}{2} \text{ as } |b|\longrightarrow 0.$$

$$\text{III}:\quad \mathbf{A}_3=\frac{A_3}{\sqrt{\rho_0 c_3}}\exp\{-ik_3(x-x_3)\}\,,$$

$$\mathbf{B}_3=\frac{B_3}{\sqrt{\rho_0 c_3}}\exp\{ik_3(x-x_3)\}\,,\quad k_3=\frac{p}{c_3}\,.$$

$$\text{IV}:\quad \mathbf{A}_4+\mathbf{B}_4=\frac{1}{\sqrt{\rho_0 c_4}}\{A_4\psi(-z_4)\exp(-\mu z_4)+B_4\psi(z_4)\exp(\mu z_4)\}\,,$$

$$t_4=\int_{x_3}^{x}\frac{dx}{c_4}\,,\quad z_4=\frac{\gamma a t_4\sqrt{1-b^2}}{2}\rightarrow\frac{\gamma(x-x_4)}{2}\quad \text{as } |b|\rightarrow 0\,.$$

$$\text{V}:\quad \mathbf{A}_5=\frac{A_5}{\sqrt{\rho_0 c_5}}\exp\{-ik_5(x-x_4)\}\,,\qquad k_5=\frac{p}{c_5}\,.$$

Nine arbitrary constants, A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4 and A_5, instead of the five in Part 1 are reserved for boundary conditions. Direction of propagation of each wave is shown in Fig. 1. In a homogeneous medium, primary wave and reflected wave are set separately, because either of the two is respectively a particular solution of equation of motion. On the other hand, those in a periodic structure are not. $\psi(z)$ is a periodic function of z which enters into the general solution of Mathieu's equation.

$$\frac{d^2\phi_1}{dz^2}+\{q^2+2b\cos 2z\}\phi_1=0\,,$$

$$\phi_1=A\psi(-z)\exp(-\mu z)+B\psi(z)\exp(\mu z)\,,$$

q is a parameter that depends on the period of the wave and the constant of the structure,

$$q=\frac{2p}{a\gamma\sqrt{1-b^2}}\fallingdotseq\frac{2p}{a\gamma}\,.$$

If wave length λ is defined in a homogeneous medium by $\lambda=2\pi a/p$, $q=2L/\lambda$, where $L=2\pi/\gamma$ is the wave length of the structure. As it was explained in Part 1, the most important case in our studies is given by $q\fallingdotseq 1$, i. e., $\lambda=2L$ or $p\fallingdotseq a\gamma/2$. Then, as it was also already explained,

$$\psi(z)\fallingdotseq\sin(z-\sigma)+s_3\sin(3z-\sigma)\,,\quad \sigma=\pi/4\,,\quad s_3=b/8\,,$$

$\mu\fallingdotseq b/2$, regardless of the sign of b.

$\psi(-z)$ is obtained by putting $\sigma=-\pi/4$. We put in this paper, assuming $|b|\ll 1$, $\psi(z)=\sin(z-\pi/4)$, $\psi(-z)=\sin(z+\pi/4)$ and $\mu=b/2$.

From the conditions of continuity of displacement and stress at four boundaries, we have eight simultaneous algebraic equations of $A_1, B_1, \cdots \cdots, A_5$ of a type similar to (22) in Part 1, from which A_5/A_1 is obtained. To minimize the effect of boundary reflection and to clarify the effect of periodic structure, velocity and its gradient are assumed continuous at four boundaries and $c_1=c_3=c_5$; i. e., i) $c_1=c_3=c_5=c_0=a(1+b)$, ii) $x_2-x_1=2l\pi/\gamma$, and $x_4-x_3=2m\pi/\gamma$, where l and m are integers respectively. Four boundaries in z-coordinates are $z_2=0$, $z_2=l\pi$, $z_4=0$ and $z_4=m\pi$; velocity gradient at the boundary vanishes. Then we have, putting $|b|\ll 1$,

$$k_1=k_3=k_5=k_0=p/a\ , \text{ and } q\doteqdot 1 \text{ gives } k_0\doteqdot\gamma/2\ ,$$

$$\psi(m\pi)=-\psi(-m\pi)=(-1)^{m+1}/\sqrt{2}\ ,$$

$$\frac{d}{dz}\psi(z)\exp(\mu z)\doteqdot\exp(\mu z)\cdot\cos(z-\sigma)\ ,\quad \mu=b/2$$

$$=1/\sqrt{2}\ ,\quad \text{when}\quad z=0$$

$$=(-1)^m\exp(\mu m\pi)/\sqrt{2}\ ,\quad \text{when}\quad z=m\pi\ .$$

$$\frac{d}{dz}\psi(-z)\exp(-\mu z)=1/\sqrt{2}\ ,\quad \text{when } z=0$$

$$=(-1)^m\exp(-\mu m\pi)/\sqrt{2}\ ,\quad \text{when}\quad z=m\pi\ .$$

The eight algebraic equations above stated are much simplified as follows;

B_1,	$\frac{+A_2}{\sqrt{2}}$,	$\frac{+B_2}{\sqrt{2}}$,	$+(A_3+B_3)$,	$+i(A_3-B_3)$,	$\frac{+A_4}{\sqrt{2}}$,	$\frac{+B_4}{\sqrt{2}}$,	$+A_5$	$=A_1$
-1	1	-1	0	0	0	0	0	1
1	i	i	0	0	0	0	0	1
0	$\zeta(-l)$	$-\zeta(l)$	-1	0	0	0	0	0
0	$\zeta(-l)$	$\zeta(l)$	0	1	0	0	0	0
0	0	0	$-\cos k_0x_0$	$\sin k_0x_0$	1	-1	0	0
0	0	0	$\sin k_0x_0$	$\cos k_0x_0$	1	1	0	0
0	0	0	0	0	$\zeta(-m)$	$-\zeta(m)$	-1	0
0	0	0	0	0	$\zeta(-m)$	$\zeta(m)$	i	0

where $\zeta(n)=(-1)^n\exp(\mu n\pi)$, n being an integer, and $x_0=x_3-x_2$ is thickness of the intermediate homogeneous medium. For convenience of comparison with the formula (1), put $l\pi=z_2$ and $m\pi=z_4$, $2z_2/\gamma$ and $2z_4/\gamma$ are approximately equal respectively to the thickness of the zone where structure is periodic. From these equations, we have

$$\frac{A_5}{A_1}=\frac{1}{\cos k_0x_0 \cosh \mu(z_2+z_4)+i \sin k_0x_0 \cosh \mu(z_2-z_4)}$$

so that

$$\left|\frac{A_5}{A_1}\right|=\frac{1}{\{\cosh^2 \mu(z_2+z_4)-\sin^2 k_0x_0 \sinh 2\mu z_2 \sinh 2\mu z_4\}^{1/2}}.$$

Strictly speaking, these formulae are available only when z_2 and z_4 are respectively a multiple of π, but effect of periodic structure will be deduced therefrom. The important point of the formula is that the effect of periodic structures at intervals is not always additive, though always $|A_5/A_1| \ngtr 1$. $k_0=p/a=\gamma/2$, so that $\cos\gamma x=2\cos^2 k_0x-1$. When $\sin k_0x_0 =0$, i. e. $\cos k_0x_0=\pm 1$ and $\cos\gamma x_0=1$, $|A_5/A_1|=1/\cosh \mu(z_2+z_4)$; the effect is perfectly additive. When $\sin k_0x_0=\pm 1$, i. e., $\cos k_0x_0=0$ and $\cos\gamma x_0=-1$, $|A_5/A_1|=1/\cosh \mu(z_2\sim z_4)$; the effect is subtractive. And it is natural to presume some correlation between the effect and the magnitude of Fourier's coefficient of the mathematical expression of the whole structure, through which the wave is transmitted.

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18. 不均質な媒質内を伝わる弾性波の安定性について 第2報
付．周期構造をもつ媒質を伝わる波動

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周期構造をもつ媒質を伝わる波の研究には本来数学的に困難な点がある．過去の研究にはそれを顧慮しないため，非常に不明確な解釈に導かれているものがある．それは古い Rayleigh の研究結果とも明かに相容れないことを示すため，それらの一部を引用して注意すべき点を述べた．これは現在の研究過程において不必要なことではあるが，そのような不明確な結果を無批判に受け入れるならば当然筆者等の計算に対して起ると思われる討論えの応答としてつけ加えたものである．

さて前報告のつづきとして周期構造をもつ二層の間に一つの均質構造がある場合を計算した．周期構造の影響はいつも加算的とは限つておらず，ある場合には完全な引き算とさえなり，

$$\frac{A_5}{A_1}=\frac{1}{\cos k_0x_0 \cosh \mu(z_2+z_4)+i \sin k_0x_0 \cosh \mu(z_2\sim z_4)}$$

で表わされる．この結果は波に最大の影響を与えるものは，その源から観測される現在位置まで途中の不規則な構造をどのようにして平均したものであるかを考えるに際して適当な指示を与えるものであろう．しかし同時に地震波の観測結果とむすびつけるためには数学的にもまたその解釈の仕方にも困難な点が多く残されていることを示している．