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9. *General Method of Treating Water Waves Produced by a Vibrating Bottom with an Arbitrary Form.*

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1. Introduction

The purpose of this article is to demonstrate the general method of obtaining a theoretical formula of water waves produced by a vibrating portion of the bottom with an "arbitrary form".

The methods used by many authors so far are such that theoretical formulae of waves produced by bottom displacement are derived from integration of the equation (3) in section 2 under the conditions (1) and (2) throughout the "whole domain", so their methods restrict their studies to the problems of water waves generated by a portion of the bottoms with "circular boundaries".

In our method theoretical formulae of wave heights are obtained in two parts respectively, (1) in the domain where the bottom vibrates, (2) in the domain where the bottom is at rest; but both formulae are connected smoothly by conditions at the boundary of the two domains.

Now let us develop our general method in detail in the subsequent sections.

2. Basic Equation and Boundary Conditions

We use cylindrical coordinates (r, θ, z); r and θ are taken horizontally at the undisturbed free surface of water, and z vertically upwards (Fig. 1).

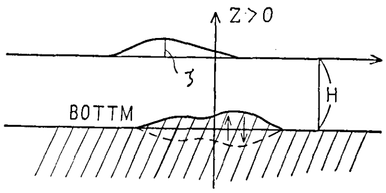

Fig. 1.

Suppose that Φ is the velocity potential, ζ the elevation of water above the undisturbed free surface (Fig. 1), g the acceleration of gravity, t the time, we have, as the surface conditions,

$$\frac{\partial \Phi}{\partial t} = -g\zeta \quad (z=0)$$

and

$$\frac{\partial\zeta}{\partial t}=\frac{\partial\Phi}{\partial z} \quad (z=0)$$

or

$$\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z}=0 \quad (z=0)\,. \tag{1}$$

Also we have, as the bottom condition,

$$\frac{\partial\Phi}{\partial z}=\eta \quad (z=-H)\,, \tag{2}$$

where η is the velocity of bottom displacement and H the depth of water (Fig. 1).

Then the basic equation is given by

$$\varDelta\Phi=0 \tag{3}$$

where

$$\varDelta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{z^2}{\partial z^2}\,.$$

3. Basic Equation and Boundary Conditions in Dimensionless Form with regard to Length

Suppose that r_0 is the radius of a circle enclosing the domain of the vibrating bottom (Fig. 2), we have the boundary conditions and the equation in dimensionless form from (1), (2) and (3) as follows;

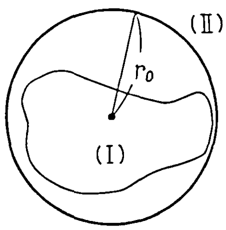

Fig. 2.

$$\frac{\partial^2\Phi'}{\partial t^2}+g'\frac{\partial\Phi'}{\partial z'}=0 \quad (z'=0)\,, \tag{4}$$

$$\frac{\partial\Phi'}{\partial z'}=\begin{cases}\eta' & (r'<1)\,,\\ 0 & (r'>1)\,,\end{cases} \quad (z'=-H')\,, \tag{5}$$

$$\varDelta'\Phi'=0\,, \tag{6}$$

where

$$\Phi'=\frac{\Phi}{r_0^{\,2}}\,, \qquad H'=\frac{H}{r_0}\,,$$

$$z'=\frac{z}{r_0}\,, \qquad \eta'=\frac{\eta}{r_0}\,,$$

$$g'=\frac{g}{r_0}\,, \qquad r'=\frac{r}{r_0}\,,$$

$$\Delta' = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z'^2} .$$

4. Reduction of Basic Equation and Boundary Conditions by Virtue of Vibration

Since we are treating the case of vibration, we may suppose that

$$\Phi' = \Phi_0' \exp(-i\omega t) ,$$
$$\eta' = \eta_0' \exp(-i\omega t) ,$$

where ω is the angular frequency of vibration.

Hence the boundary condions (4) and (5), and the equation (6) become as follows;

$$-\omega^2 \Phi_0' + g' \frac{\partial \Phi_0'}{\partial z'} = 0 \qquad (z'=0) \tag{7}$$

$$\frac{\partial \Phi_0'}{\partial z'} = \begin{cases} \eta_0' , & (r'<1) \\ 0 , & (r'>1) \end{cases} \qquad (z' = -H') \tag{8}$$

$$\Delta' \Phi_0' = 0 \tag{9}$$

5. The Relation between the Displacement and the Velocity of the Bottom

Denoting the bottom displacement by D_{bot} or by $D'_{\text{bot}} = D_{\text{bot}}/r_0$ in dimensionless form, we have the velocity of the bottom displacement η or η' as follows;

$$\eta = \frac{\partial D_{\text{bot}}}{\partial t} \quad \text{or} \quad \eta' = \frac{\partial D'_{\text{bot}}}{\partial t} .$$

Elimination of the time factor $\exp(-i\omega t)$ from η' and D'_{bot} produces

$$\eta_0' = -i\omega (D'_{\text{bot}})_0 , \tag{10}$$

where

$$D'_{\text{bot}} = (D'_{\text{bot}})_0 \exp(-i\omega t) .$$

6. The Bottom Condition in Fourier Bessel Series

The amplitude of the bottom displacement, $(D'_{\text{bot}})_0$, with a boundary of an arbitrary form (Fig. 3) can be expressed by the Fourier Bessel Series as follows;

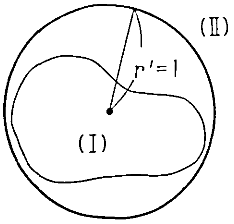

Fig. 3. (In dimensionless form).

$$(D'_{\mathrm{bot}})_0=\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}[A_{n,s}\cos n\theta+B_{n,s}\sin n\theta]\cdot J_n(\lambda_{n,s}r')\,, \tag{11}$$

where

$$\left.\begin{matrix}A_{n,s}\\B_{n,s}\end{matrix}\right\}=\frac{\varepsilon_n}{\pi[J_{n+1}(\lambda_{n,s})]^2}\cdot\int_{-\pi}^{+\pi}\int_0^1(D'_{\mathrm{bot}})_0\cdot J_n(\lambda_{n,s}r')\cdot\left\{\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}\right\}\cdot d\theta\cdot rdr\,,$$

$$\varepsilon_0=1\,,\quad \varepsilon_1=\varepsilon_2=\varepsilon_3=\cdots=2\,,$$

and $\lambda_{n,s}$ ($s=1, 2, 3,\cdots$) are positive roots of the Bessel function $J_n(\lambda_{n,s})=0$, taken in order from the smallest positive value.

Then by use of (10) and (11), the bottom condition (8) becomes

$$\frac{\partial\Phi_0'}{\partial z'}=-i\omega(D'_{\mathrm{bot}})_0\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}[A_{n,s}\cdot\cos n\theta+B_{n,s}\cdot\sin n\theta]\cdot J_n(\lambda_{n,s}r') \tag{12}$$

$$(r'<1\,,\quad z'=-H')$$

$$\frac{\partial\Phi_0'}{\partial z'}=0\qquad(r'>1\,,\quad z'=-H')\,. \tag{13}$$

Here we define the domain in the range $r'<1$ as "domain (I)" and that in the range $r'>1$ as "domain (II)" (Fig. 3).

7. The Solution in Domain (II)

In this domain the bottom condition is always zero as designated by (13).

The separation of the variables of Φ_0' in (7), (13) and (9) gives

$$-\omega^2\Phi_z'+g'\frac{d\Phi_z'}{dz'}=0\quad(z'=0)\,, \tag{14}$$

$$\frac{d\Phi_z'}{dz'}=0\quad(z'=-H')\,, \tag{15}$$

and

$$\frac{d\Phi_z'}{dz'^2}=a'^2\Phi_z'\,, \tag{16}$$

$$\left(\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2}{\partial\theta^2}+a'^2\right)\Phi'_{r\theta}=0\,, \tag{17}$$

where

$$\Phi_0'=\Phi'_{r\theta}\cdot\Phi_z'\,,$$

$\Phi'_{r\theta}$ and Φ_z' are the functions of r', θ and z' respectively, and a' is the separation constant.

Solving the equation (16) under the conditions (14) and (15), we obtain a group of solutions

$$\Phi_z' = \cosh a_0'(H'+z')$$

and

$$\cos a_s'(H+z')\,, \qquad (s=1,\ 2,\ 3,\cdots) \tag{18}$$

where a_0' and ia_s' are the real and imaginary roots of Airy's relation

$$\omega^2 = a'g' \tanh a'H'\,.$$

By this equation the separation constant a' in equation (17) can be determind as a_0' and ia_s' $(s=1,\ 2,\ 3,\cdots)$.

Taking into account that Φ_0' has periodicity of 2π with regard to θ and damps out at the infinite point of r, the solutions of the equation (17) become

$$\Phi'_{r\theta} = (C_{n,0}\cos n\theta + D_{n,0}\sin n\theta)H_n^{(\mathrm{I})}(a_0'r')$$

and

$$(C_{n,s}\cos n\theta + D_{n,s}\sin n\theta)K_n(a_s'r')\,. \qquad \begin{pmatrix} n=0,\ 1,\ 2,\ 3,\cdots \\ s=1,\ 2,\ 3,\cdots \end{pmatrix} \tag{19}$$

From (18) and (19), we finally obtain Φ_0' as follows;

$$\Phi_0^{(\mathrm{II})\prime} = \sum_{n=0}^{\infty}(C_{n,0}^{(\mathrm{II})}\cos n\theta + D_{n,0}^{(\mathrm{II})}\sin n\theta)H_n^{(\mathrm{I})}(a_0'r')\cdot\cosh a_0'(H'+z')$$
$$+\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(\mathrm{II})}\cos n\theta + D_{n,s}^{(\mathrm{II})}\sin n\theta)\cdot K_n(a_s'r')\cdot\cos a_s'(H'+z')\,, \tag{20}$$

where the values relevant to domain (II) are denoted by surersuffix (II), and $C_{n,s}$, $D_{n,s}$ $(n=0,\ 1,\ 2,\cdots;\ s=0,\ 1,\ 2,\cdots)$ are arbitrary constants.

8. The Solution in Domain (I)

Supposing that $\Phi'_{(h)}$ satisfies the "homogeneous" boundary conditions, let us put $\eta_0' \equiv 0$, i. e.,

$$-\omega^2 \Phi'_{(h)} + g' \frac{\partial \Phi'_{(h)}}{\partial z'} = 0 \quad (z'=0) ,$$

$$\frac{\partial \Phi'_{(h)}}{\partial z'} = 0 \quad (z' = -H') ,$$

and

$$\Delta' \Phi'_{(h)} = 0 ,$$

which are identical in form with the conditions and the equation in domain (II). We have then the solution of $\Phi'_{(h)}$ in the same manner as in section 7,

$$\Phi'_{(h)} = \sum_{n=0}^{\infty} (C_{n,0} \cos n\theta + D_{n,0} \sin n\theta) \cdot J_n(a_0' r') \cdot \cosh a_0'(H'+z')$$
$$+ \sum_{s=1}^{\infty} \sum_{n=0}^{\infty} (C_{n,s} \cos n\theta + D_{n,s} \sin n\theta) \cdot I_n(a_s' r') \cdot \cos a_s'(H'+z') . \qquad (21)$$

Since the pressure (or velocity potential) must be finite at the origin ($r'=0$), the terms including the Bessel function $Y_n(a_0' r')$ have been eliminated, and $C_{n,u}$, $D_{n,u}$ ($n=0, 1, 2, 3, \cdots ;\ u=0, 1, 2, 3, \cdots$) are arbitrary constants.

Next let us consider the particular solution $\Phi'_{(p)}$ which satisfies

$$-\omega^2 \Phi'_{(p)} + g' \frac{\partial \Phi'_{(p)}}{\partial z'} = 0 \quad (z'=0) ,$$

$$\frac{\partial \Phi'_{(p)}}{\partial z'} = \eta_0' \quad (z' = -H') ,$$

$$\Delta' \Phi'_{(p)} = 0 ,$$

where, from (12),

$$\eta_0' = -i\omega (D'_{\text{bot}})_0 \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} [A_{n,s} \cdot \cos n\theta + B_{n,s} \cdot \sin n\theta] \cdot J_n(\lambda_{n,s} r') .$$

By inspection we can take the particular solution $\Phi'_{(p)}$ to be in the following form;

$$\Phi'_{(p)} = -i\omega \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} [A_{n,s} \cdot \cos n\theta + B_{n,s} \cdot \sin n\theta] \cdot J_n(\lambda_{n,s} r') \cdot (E_{n,s} e^{\lambda_{n,s} z'} + F_{n,s} e^{-\lambda_{n,s} z'}) \qquad (22)$$

where

$$E_{n,s} = \frac{1}{M_{n,s}} \cdot \frac{\omega^2 + g' \lambda_{n,s}}{\lambda_{n,s}} ,$$

$$F_{n,s} = \frac{1}{M_{n,s}} \cdot \frac{-\omega^2 + g' \lambda_{n,s}}{\lambda_{n,s}} ,$$

$$M_{n,s}=\begin{vmatrix} (-\omega^2+g'\lambda_{n,s}), & -(\omega^2+g'\lambda_{n,s}) \\ e^{-\lambda_{n,s}H'}, & -e^{\lambda_{n,s}H'} \end{vmatrix}.$$

Thus finally we get the general solution (I) from (21) and (22) as follows;

$$\begin{aligned}\Phi_0^{(\mathrm{I})'}=\Phi'_{(h)}+\Phi'_{(p)}=&\sum_{n=0}^{\infty}(C_{n,0}^{(\mathrm{I})}\cos n\theta+D_{n,0}^{(\mathrm{I})}\sin n\theta)\cdot J_n(a_0'r')\cdot\cosh a_0'(H'+z')\\ &+\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(\mathrm{I})}\cos n\theta+D_{n,s}^{(\mathrm{I})}\sin n\theta)\cdot I_n(a_s'r')\cdot\cos a_s'(H'+z')\\ &-i\omega\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}[A_{n,s}\cos n\theta+B_{n,s}\sin n\theta]\cdot J_n(\lambda_{n,s}r')\cdot(E_{n,s}e^{\lambda_{n,s}z'}+F_{n,s}e^{-\lambda_{n,s}z'}),\end{aligned} \tag{23}$$

where super-suffix (I) stands for the values relevant to domain (I).

9. The Determination of the Arbitrary Constants $C_{n,u}^{(\mathrm{I})}$, $D_{n,u}^{(\mathrm{I})}$, $C_{n,u}^{(\mathrm{II})}$, and $D_{n,u}^{(\mathrm{II})}$

For determining arbitrary constants $C_{n,u}^{(l)}$, $D_{n,u}^{(l)}$ (l=I, II; $n=0, 1, 2, 3,\cdots$; $u=0, 1, 2, 3,\cdots$), we need two "independent" conditions at $r'=1$. For these conditions the next three equations are available;

$\Phi_0^{(\mathrm{I})}=\Phi_0^{(\mathrm{II})'}$ (continuity of pressure),

$$\left.\begin{aligned}\frac{\partial\Phi_0^{(\mathrm{I})'}}{\partial\theta}&=\frac{\partial\Phi_0^{(\mathrm{II})'}}{\partial\theta}\\ \frac{\partial\Phi_0^{(\mathrm{I})'}}{\partial r'}&=\frac{\partial\Phi_0^{(\mathrm{II})'}}{\partial r'}\end{aligned}\right\}$$ (continuity of velocity of water particle),

but the first two of these equations are found to be equivalent. Consequently, the conditions

$$\Phi_0^{(\mathrm{I})'}=\Phi_0^{(\mathrm{II})'} \tag{24}$$

and

$$\frac{\partial\Phi_0^{(\mathrm{I})'}}{\partial r'}=\frac{\partial\Phi_0^{(\mathrm{II})'}}{\partial r'} \tag{25}$$

are enough to determine the arbitrary constants.

Substituting (20) and (23) into (24) and (25) we have

$$\sum_{n=0}^{\infty}(C_{n,0}^{(I)}\cos n\theta+D_{n,0}^{(I)}\sin n\theta)\cdot J_n(a_0')\cdot\cosh a_0'(H'+z')$$
$$+\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(I)}\cos n\theta+D_{n,s}^{(I)}\sin n\theta)\cdot I_n(a_s')\cdot\cos a_s'(H'+z')$$
$$=\sum_{n=0}^{\infty}(C_{n,0}^{(II)}\cos n\theta+D_{n,0}^{(II)}\sin n\theta)\cdot H_n^{(1)}(a_0')\cdot\cosh a_0'(H'+z')$$
$$+\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(II)}\cos n\theta+D_{n,s}^{(II)}\sin n\theta)\cdot K_n(a_s')\cdot\cos a_s'(H'+z')\ , \qquad (26)$$

$$\sum_{n=0}^{\infty}(C_{n,0}^{(I)}\cos n\theta+D_{n,0}^{(I)}\sin n\theta)\cdot a_0'\cdot J_n'(a_0')\cdot\cosh a_0'(H'+z')$$
$$+\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(I)}\cos n\theta+D_{n,s}^{(I)}\sin n\theta)\cdot a_s'\cdot I_n'(a_0')\cdot\cos a_s'(H'+z')$$
$$-i\omega\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}(A_{n,s}\cos n\theta+B_{n,s}\sin n\theta)\cdot\lambda_{n,s}\cdot J_n'(\lambda_{n,s})(E_{n,s}e^{\lambda_{n,s}z'}+F_{n,s}{}^{-\lambda_{n,s}z'})$$
$$=\sum_{n=0}^{\infty}(C_{n,0}^{(II)}\cos n\theta+D_{n,0}^{(II)}\sin n\theta)\cdot a_0'\cdot H_n^{(1)\prime}(a_0')\cdot\cosh a_0'(H'+z')$$
$$+\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(II)}\cos n\theta+D_{n,s}^{(II)}\sin n\theta)\cdot a_s'\cdot K_n'(a_s')\cdot\cos a_s'(H'+z')\ . \qquad (27)$$

Orthogonalities of functions $\{\cosh a_0'(H'+z'),\ \cos a_s'(H'+z')\}$ $(s=1, 2, 3, \cdots)$ in the range $0\geq z'\geq -H'$ [1] and of functions $\{\cos n\theta,\ \sin n\theta\}$ $(n=0, 1, 2, 3,\cdots)$ in the range $2\pi\geq\theta\geq 0$ reduce the relation (26) and (27) into the following simultaneous equations

$$\left.\begin{array}{ll}
C_{n,0}^{(I)}J_n(a_0')-C_{n,0}^{(II)}H_n^{(1)}(a_0')=0 & (n=0,\ 1,\ 2,\ 3,\cdots)\ ,\\
D_{n,0}^{(I)}J_n(a_0')-D_{n,0}^{(II)}H_n^{(1)}(a_0')=0 & (n=1,\ 2,\ 3,\cdots)\ ,\\
C_{n,s}^{(I)}I_n(a_s')-C_{n,s}^{(II)}K_n(a_s')=0 & \begin{pmatrix}n=0,\ 1,\ 2,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix},\\
D_{n,s}^{(I)}I_n(a_s')-D_{n,s}^{(II)}K_n(a_s')=0 & \begin{pmatrix}n=1,\ 2,\ 3,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix},\\
C_{n,0}^{(I)}J_n'(a_0')-C_{n,0}^{(II)}H_n^{(1)\prime}(a_0')=R_{n,0}^{A_n} & (n=0,\ 1,\ 2,\ 3,\cdots)\ ,\\
D_{n,0}^{(I)}J_n'(a_0')-D_{n,0}^{(II)}H_n^{(1)\prime}(a_0')=R_{n,0}^{B_n} & (n=1,\ 2,\ 3,\cdots)\ ,\\
C_{n,s}^{(I)}I_n'(a_s')-C_{n,s}^{(II)}K_n'(a_s')=R_{n,s}^{A_n} & \begin{pmatrix}n=0,\ 1,\ 2,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix},\\
D_{n,s}^{(I)}I_n'(a_s')-D_{n,s}^{(II)}K_n'(a_s')=R_{n,s}^{B_n} & \begin{pmatrix}n=1,\ 2,\ 3,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix},
\end{array}\right\} \qquad (28)$$

1) T. H. HAVELOCK, "Forced Surface Waves on Water", *Phil. Mag.*, **8** (1929).

where

$$\left.\begin{matrix} R^{A_n}_{n,0} \\ R^{B_n}_{n,0} \end{matrix}\right\} = \frac{i\omega \cdot 4}{\sinh 2a_0'H' + 2a_0'H'} \cdot \sum_{u=0}^{\infty} \left\{ \begin{matrix} A_{n,u} \\ B_{n,u} \end{matrix} \right\} \lambda_{n,u} \cdot J_n'(\lambda_{n,u}) \cdot L^{(0)}_{n,u} \,,$$

$$L^{(0)}_{n,u} = \int_{-H}^{0} (E_{n,u} e^{\lambda_{n,u} z'} + F_{n,u} e^{-\lambda_{n,u} z'}) \cosh a_0'(H'+z') dz'$$

$$= \frac{1}{M_{n,u}} \cdot \frac{2}{\lambda^2_{n,u} - a_0'^2} \cdot \{\omega^2 (\cosh a_0'H' - \cosh \lambda_{n,u} H')$$

$$+ g'(-a_0' \sinh a_0'H' + \lambda_{n,u} \sinh \lambda_{n,u} H')\} \,,$$

$$\left.\begin{matrix} R^{A_n}_{n,s} \\ R^{B_n}_{n,s} \end{matrix}\right\} = \frac{i\omega \cdot 4}{\sin 2a_s'H' + 2a_s'H'} \cdot \sum_{u=0}^{\infty} \left\{ \begin{matrix} A_{n,u} \\ B_{n,u} \end{matrix} \right\} \cdot \lambda_{n,u} \cdot J_n'(\lambda_{n,u}) \cdot L^{(s)}_{n,u}$$

$$L^{(s)}_{n,u} = \int_{-H}^{0} (E_{n,u} e^{\lambda_{n,u} z'} + F_{n,u} e^{-\lambda_{n,u} z'}) \cos a_s'(H'+z') dz'$$

$$= \frac{1}{M_{n,u}} \cdot \frac{2}{\lambda^2_{n,u} + a_s'^2} \cdot \{\omega^2 (\cos a_s'H' - \cosh \lambda_{n,u} H')$$

$$+ g'(a_s' \sin a_s'H' + \lambda_{n,u} \sinh \lambda_{n,u} H')\} \,,$$

$$M_{n,u} = \begin{vmatrix} (-\omega^2 + g'\lambda_{n,u}) \,, & -(\omega^2 + g'\lambda_{n,u}) \\ e^{-\lambda_{n,u} H'} \,, & -e^{\lambda_{n,u} H'} \end{vmatrix} .$$

From (28) we finally obtain the arbitrary costants as follows;

$$\left.\begin{aligned}
&\left.\begin{matrix} C^{(\mathrm{I})}_{n,0} \\ C^{(\mathrm{II})}_{n,0} \end{matrix}\right\} = \frac{i \cdot \pi a_0'}{2} \cdot R^{A_n}_{n,0} \left\{ \begin{matrix} H^{(1)}_n(a_0') \\ J_n(a_0') \end{matrix} \right. , \quad (n=0,\ 1,\ 2, \cdots) \,, \\
&\left.\begin{matrix} D^{(\mathrm{I})}_{n,0} \\ D^{(\mathrm{II})}_{n,0} \end{matrix}\right\} = \frac{i \cdot \pi a_0'}{2} \cdot R^{B_n}_{n,0} \left\{ \begin{matrix} H^{(1)}_n(a_0') \\ J_n(a_0') \end{matrix} \right. , \quad (n=1,\ 2,\ 3, \cdots) \,, \\
&\left.\begin{matrix} C^{(\mathrm{I})}_{n,s} \\ C^{(\mathrm{II})}_{n,s} \end{matrix}\right\} = a_s' \cdot R^{A_n}_{n,s} \left\{ \begin{matrix} K_n(a_s') \\ I_n(a_s') \end{matrix} \right. , \quad \left(\begin{matrix} n=0,\ 1,\ 2, \cdots \\ s=1,\ 2,\ 3, \cdots \end{matrix} \right) , \\
&\left.\begin{matrix} D^{(\mathrm{I})}_{n,s} \\ D^{(\mathrm{II})}_{n,s} \end{matrix}\right\} = a_s' \cdot R^{B_n}_{n,s} \left\{ \begin{matrix} K_n(a_s') \\ I_n(a_s') \end{matrix} \right. , \quad \left(\begin{matrix} n=1,\ 2,\ 3, \cdots \\ s=1,\ 2,\ 3, \cdots \end{matrix} \right) .
\end{aligned}\right\} \quad (29)$$

10. Theoretical Formulae of Wave Height

By use of the relation $\zeta' = \frac{1}{-g'} \left(\frac{\partial \Phi'}{\partial t} \right)_{z'=0}$, the heights of waves in dimensionless form are obtained from (20), (23) and (29) as follows;

$$\zeta^{(I)'}=\frac{i\omega}{g'}e^{-i\omega t}\sum_{n=0}^{\infty}(C_{n,0}^{(I)}\cos n\theta+D_{n,0}^{(I)}\sin n\theta)\cdot J_n(a_0'r')\cdot\cosh a_0'H'$$

$$+\frac{i\omega}{g'}e^{-i\omega t}\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(I)}\cos n\theta+D_{n,s}^{(I)}\sin n\theta)\cdot I_n(a_s'r')\cdot\cos a_s'H'$$

$$+\omega^2\cdot e^{-i\omega t}\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}(A_{n,s}\cos n\theta+B_{n,s}\sin n\theta)\cdot J_n(\lambda_{n,s}r')\cdot\frac{2}{M_{n,s}}\ , \qquad (30)$$

$$\zeta^{(II)'}=\frac{i\omega}{g'}e^{-i\omega t}\sum_{n=0}^{\infty}(C_{n,0}^{(II)}\cos n\theta+D_{n,0}^{(II)}\sin n\theta)\cdot H_n^{(1)}(a_0'r')\cdot\cosh a_0'H'$$

$$+\frac{i\omega}{g'}\cdot e^{-i\omega t}\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}(C_{n,s}^{(II)}\cos n\theta+D_{n,s}^{(II)}\sin n\theta)\cdot K_n(a_s'r')\cdot\cos a_s'H'\ , \qquad (31)$$

where $\zeta^{(I)'}$ and $\zeta^{(II)'}$ stand respectively for the wave heights in dimensionless form in domains (I) and (II).

Using the relation $\zeta=r_0\cdot\zeta'$, we finally obtain the wave heights

$$\zeta^{(I)}=r_0\cdot\zeta^{(I)'}$$
$$\zeta^{(II)}=r_0\cdot\zeta^{(II)'}\ , \qquad (32)$$

where $\zeta^{(I)}$ and $\zeta^{(II)}$ are the wave heights in (I) and (II) respectively, and $\zeta^{(I)'}$ and $\zeta^{(II)'}$ are given in (30) and (31). As far as other constants included in (30) and (31) are concerned, reference should be made to the preceding sections.

11. Asymptotic Formula of the Out-going Wave

According to Watson[2], asymptotic formulae of Bessel and modified Bessel functions are given by

$$H^{(1)}(a_0'r')\approx\sqrt{\frac{2}{\pi a_0'r'}}\cdot e^{i\left(a_0'r'-\frac{\pi}{4}-\frac{n}{2}\pi\right)}\ ,$$

$$K_n(a_0'r')\approx\frac{1}{2}\pi i^{n+1}\cdot\sqrt{\frac{2}{\pi a_0'r'\cdot i}}\cdot e^{-a_0'r'-i\left(\frac{\pi}{4}+\frac{n}{2}\pi\right)}\ .$$

By virtue of the above asymptotic expressions, the out-going wave of $\zeta^{(II)}$ is found to be composed only of terms with the factor $H_n^{(1)}(a_0'r')$. Hence the formula of the out-going wave is written as follows;

2) G. N. WASTON, *Theory of Bessel Functions*, (Cambridge Press, 1922).

$$\zeta^{(II)} \approx \frac{i\omega \cdot r_0^2}{g} \cdot \sqrt{\frac{2}{\pi a_0 r}} \cdot \cosh a_0 H \cdot N(\theta) \cdot e^{i\left(-\omega t + a_0 r - \frac{\pi}{4}\right)}, \tag{33}$$

where

$$N(\theta) = \sum_{n=0}^{\infty} (C_{n,0}^{(II)} \cos n\theta + D_{n,0}^{(II)} \sin n\theta) \cdot e^{-i \cdot \frac{n}{2}\pi},$$

and the relations

$$g = g' r_0, \qquad a_0 = a_0' / r_0, \qquad H = H' r_0,$$

have been used.

Here we can see that the factor $N(\theta)$ plays the main part in causing the directivity of the out-going wave at a distant point from the wave-generating source.

In due course we shall apply this method to a particular problem in a subsequent article.

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9. 任意の形をした振動する底によつておこされた水波を取り扱う方法について

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著者は本論文において，任意の形をした水底によりおこされる波を理論的に解く方法を導いた．従来の方法と異り，著者は振動域を囲む円の内部と外部とを別々に解き，円周上で滑らかに連続するように，円の内外の解の中の任意常数を決定した．内部の解を求めるに際し，著者は振動域を囲む底を取り，これを Fourier Bessel 展開を行い，方程式を円筒函数で解くのを便ならしめた．この Fourier Bessel 展開による方法は，方程式を解くのを便ならしめたが，他面では，任意の形をした底を Fourier Bessel 展開したときの，Fourier Bessel 係数を求めることを，むずかしくしている．この Fourier Bessel 係数は解析的にも解けるけれども，著者はあえて，その形を与えなかつた．それは，Fourier Bessel 係数を直接数値積分した方が，より容易であると考えられるからである．

本論文によつて，一応“任意”の形をした底が振動するときの Water Wave の問題は解析的には解決したかのごとく見える．