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10. *An Example of Application of the General Method of Treating Water Waves Produced by a Vibrating Bottom with an Arbitrary Form.*

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1. Introduction

The method of obtaining theoretical formulae of water waves generated by a vibrating bottom with an arbitrary form has already been introduced by the author[1].

In this paper the author intends to apply this method to a particular case—a continuosly vibrating square bottom.

2. Basic Equation and Boundary Conditions

We used cylindrical coordinates (r, θ, z), taking (r, θ) at the undisturbed free surface of water and z vertically upwards.

Provided that Φ is the velocity potential, t the time, g the acceleration of gravity, H the depth of water, and η the velocity of bottom displacement, we have

$$\frac{\partial^2\Phi}{\partial t^2}+g\frac{\partial\Phi}{\partial z}=0 \qquad (z=0) \,,$$

$$\frac{\partial\Phi}{\partial z}=\eta \qquad (z=-H) \,,$$

$$\Delta\Phi=0 \,,$$

where

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2} \,.$$

1) T. MOMOI, "General Method of Treating Water Waves Produced by a Vibrating Bottom with an Arbitrary Form," *Bull. Eathq. Res.* **40** (1962), 261-271. In subsequent discussions we refer to this paper as "T. M.".

Rewriting the above conditions and equation into dimensionless forms with regard to length we get

$$\frac{\partial^2\Phi'}{\partial t^2}+g'\frac{\partial\Phi'}{\partial z'}=0 \qquad (z'=0)\ ,$$

$$\frac{\partial\Phi'}{\partial z'}=\eta' \qquad (z'=-H')\ ,$$

$$\Delta'\Phi'=0\ ,$$

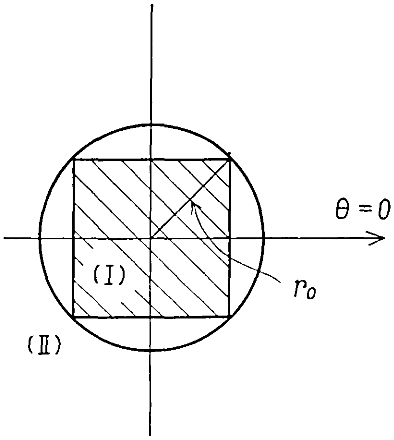

Fig. 1.

where

$$\Phi'=\Phi/r_0^2\ , \qquad g'=g/r_0\ ,$$

$$z'=z/r_0\ , \qquad \eta'=\eta/r_0\ ,$$

$$H'=H/r_0\ , \qquad r'=r/r_0\ ,$$

$$\Delta'=\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z'^2}\ ,$$

and r_0 is the radius of circumscribed circle of the square bottom (Fig. 1).

In the case of the harmonically vibrating bottom, these conditions and equation are reduced to

$$\left.\begin{aligned} &-\omega^2\Phi_0'+g'\frac{\partial\Phi_0'}{\partial z'}=0 \qquad (z'=0)\ ,\\ &\frac{\partial\Phi_0'}{\partial z'}=\eta_0' \qquad (z'=-H')\ ,\end{aligned}\right\} \tag{1}$$

$$\Delta'\Phi_0'=0\ , \tag{2}$$

where ω: angular frequency of vibration,

$$\Phi'=\Phi_0'\exp(-i\omega t)\ ,$$

$$\eta'=\eta_0'\exp(-i\omega t)\ .$$

3. Fourier Bessel Expression of the Vibrating Bottom

According to the paper T. M.[1], the amplitude of the bottom displacement in dimensionless from is written as

$$(D'_{\mathrm{bot}})_0=\sum_{n=0}^{\infty}\sum_{s=1}^{\infty}(A_{n,s}\cos n\theta+B_{n,s}\sin n\theta)\cdot J_n(\lambda_{n,s}r')\ , \tag{3}$$

where

$$\left.\begin{matrix} A_{n,s} \\ B_{n,s} \end{matrix}\right\} = \frac{\varepsilon_n}{\pi[J_{n+1}(\lambda_{n,s})]^2} \int_{-\pi}^{\pi} \int_0^1 (D'_{\mathrm{bot}})_0 \left\{\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\} \cdot J_n(\lambda_{n,s} r') \cdot d\theta r' dr' ,$$

$\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \cdots = 2$,

$\lambda_{n,s}$ $(s=1, 2, 3, \cdots)$ are the positive roots of $J_n(\lambda_{n,s})=0$, taken in order from the smallest positive value,

$D'_{\mathrm{bot}} = (D'_{\mathrm{bot}})_0 \cdot \exp(-i\omega t)$,

$D'_{\mathrm{bot}} = D_{\mathrm{bot}}/r_0$, and

D_{bot} is the bottom displacement.

Here we take the basal line of cylindrical coordinates parallel to the brim of the square bottom (Fig. 1).

Now since we are treating the case of the square bottom, the expression (3) becomes as follows by virtue of symmetry :-

$$\begin{aligned} D_{\mathrm{bot}}(r, \theta) &= D_{\mathrm{bot}}(r, -\theta) \\ &= D_{\mathrm{bot}}(r, \pi-\theta) \\ &= D_{\mathrm{bot}}\left(r, \frac{\pi}{2}-\theta\right) , \end{aligned}$$

$$(D'_{\mathrm{bot}})_0 = \sum_{l=0}^{\infty} \sum_{s=1}^{\infty} A_{4l,s} \cos 4l\theta \cdot J_{4l}(\lambda_{4l,s} r') \tag{4}$$

where

$$A_{4l,s} = \frac{8\varepsilon_{4l}}{\pi[J_{4l+1}(\lambda_{4l,s})]^2} \int_0^{\pi/4} \int_0^1 (D'_{\mathrm{bot}})_0 \cos 4l\theta \cdot J_{4l}(\lambda_{4l,s} r') \cdot d\theta \cdot r' dr' . \tag{5}$$

Also the uniform vibration and the form of the square bottom make the expression (5) simpler as follows;

$$A_{4l,s} = \frac{8\varepsilon_{4l}(D'_{\mathrm{bct}})_0}{\pi[J_{4l+1}(\lambda_{4l,s})]^2} \int_0^{\pi/4} \cos 4l\theta \cdot d\theta \int_0^{(1/\sqrt{2})\sec\theta} J_{4l}(\lambda_{4l,s} r') \cdot r' dr' . \tag{6}$$

Here we define the domain in the range $r'<1$ as "Domain (I)" and that in the range $r'>1$ as "Domain (II)" (Fig. 1).

4. The Solution in Domain (II)

In this region $\eta_0'=0$ in the condition (1), therefore the separation of variables of Φ_0' so that $\Phi_0' = \Phi'_{r\theta} \cdot \Phi_z'$, where $\Phi'_{r\theta}$ and Φ_z' are functions respectively of only r, θ and only z, gives

$$\left.\begin{aligned} -\omega^2\Phi_z'+g'\frac{d\Phi_z'}{dz'}=0 \qquad (z'=0) \\ \frac{d\Phi_z'}{dz'}=0 \qquad (z'=-H') \\ \frac{d^2\Phi_z'}{dz'^2}=a'^2\Phi_z' \end{aligned}\right\} \tag{7}$$

$$\left(\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial \gamma^2}+\frac{1}{r'^2}\frac{\partial^2}{\partial\theta^2}+a'^2\right)\Phi'_{r\theta}=0\ , \tag{8}$$

where a' is the separation constant to be determind from the first three equations.

A group of equations and conditions in (7) give the solutions,

$$\begin{aligned} \Phi_z'=\cosh a_0'(H'+z')\ , \\ \cos a_s'(H'+z')\ , \\ (s=1,\ 2,\ 3,\cdots) \end{aligned} \tag{9}$$

where a_0' and ia_s' $(s=1,\ 2,\ 3,\cdots)$ are a real and an imaginery solution of Airy's relation $\omega^2=a'g'\tanh a'H'$.

After substituting the "eigen" values $a'=a_0'$ and ia_s' $(s=1,\ 2,\ 3,\cdots)$ obtained from (7) into (8), we have the solutions of (8), i. e.,

$$\begin{aligned} \Phi'_{r\theta}=(C_{n,0}^{(II)}\cos n\theta+D_{n,0}^{(II)}\sin n\theta)\cdot H_n^{(1)}(a_0'r')\ , \\ (C_{n,s}^{(II)}\cos n\theta+D_{n,s}^{(II)}\sin n\theta)\cdot K_n(a_s'r')\ , \\ \begin{pmatrix} n=0,\ 1,\ 2,\ 3,\cdots \\ s=1,\ 2,\ 3,\cdots \end{pmatrix}, \end{aligned} \tag{10}$$

where the Bessel functions are selected so as to fulfil the condition that only the out-going progressive waves remain at infinite point of r', and $C_{n,0}^{(II)}$, $C_{n,s}^{(II)}$, $D_{n,0}^{(II)}$, $D_{n,s}^{(II)}$ are arbitrary constants.

By combination of (9) and (10), we have the velocity potential $\Phi_0^{(II)'}$ in Domain (II) as follows;

$$\begin{aligned} \Phi_0^{(II)'}=\sum_{n=0}^{\infty}(C_{n,0}^{(II)}\cos n\theta+D_{n,0}^{(II)}\sin n\theta)\cdot H_n^{(1)}(a_0'r')\cdot\cosh a_0'(H'+z') \\ +\sum_{s=1}^{\infty}\sum_{n=0}^{\infty}(C_{n,s}^{(II)}\cos n\theta+D_{n,s}^{(II)}\sin n\theta)\cdot K_n(a_s'r')\cdot\cos a_s'(H'+z') \end{aligned} \tag{11}$$

5. The Solution in Domain (I)

In this region $\eta_0' \neq 0$ in the condition (1), so we consider the general solution $\Phi_0^{(\mathrm{I})\prime}$ in domain (I) as the sum of $\Phi'_{(h)}$ and $\Phi'_{(p)}$, where $\Phi'_{(h)}$ is the general solution under the "homogeneous" boundary conditions, in which we assume $\eta_0' \equiv 0$, and $\Phi'_{(p)}$ the particular solution under the "*imhomogeneous*" conditions.

Then we have, as the particular solution, according to the paper T.M.,

$$\Phi'_{(p)} = -i\omega \sum_{l=0}^{\infty} \sum_{s=1}^{\infty} A_{4l,s} \cdot \cos 4l\theta \cdot J_{4l}(\lambda_{4l,s} r') \cdot (E_{4l,s} e^{\lambda_{4l,s} z'} + F_{4l,s} e^{-\lambda_{4l,s} z'}) , \tag{12}$$

where

$$E_{4l,s} = \frac{1}{M_{4l,s}} \cdot \frac{\omega^2 + g'\lambda_{4l,s}}{\lambda_{4l,s}} ,$$

$$F_{4l,s} = \frac{1}{M_{4l,s}} \cdot \frac{-\omega^2 + g'\lambda_{4l,s}}{\lambda_{4l,s}} ,$$

$$M_{4l,s} = \begin{vmatrix} (-\omega^2 + g'\lambda_{4l,s}) , & (-\omega^2 + g'\lambda_{4l,s}) \\ e^{-\lambda_{4l,s} H'} , & -e^{\lambda_{4l,s} H'} \end{vmatrix} ,$$

and, as the general solution satisfying the "homogeneous" conditions,

$$\begin{aligned} \Phi'_{(h)} = & \sum_{n=0}^{\infty} (C_{n,0}^{(\mathrm{I})} \cos n\theta + D_{n,0}^{(\mathrm{I})} \sin n\theta) \cdot J_n(a_0' r') \cdot \cosh a_0'(H' + z') \\ & + \sum_{s=1}^{\infty} \sum_{n=0}^{\infty} (C_{n,s}^{(\mathrm{I})} \cos n\theta + D_{n,s}^{(\mathrm{I})} \sin n\theta) \cdot I_n(a_s' r') \cdot \cos a_s'(H' + z') , \end{aligned} \tag{13}$$

where $C_{n,0}^{(\mathrm{I})}$, $C_{n,s}^{(\mathrm{I})}$, $D_{n,0}^{(\mathrm{I})}$, $D_{n,s}^{(\mathrm{I})}$ are arbitrary constants.

Thus we finally obtain the general solution $\Phi_0^{(\mathrm{I})\prime}$ in Domain (I),

$$\Phi_0^{(\mathrm{I})} = \Phi'_{(h)} + \Phi'_{(p)} , \tag{14}$$

where $\Phi'_{(h)}$ and $\Phi'_{(p)}$ are given by (12) and (13).

6. The Determination of Arbitrary Constants $C_{n,u}^{(l)}$ and $D_{n,u}^{(l)}$ (l=I, II; n=0, 1, 2, 3,$\cdots$; u=0, 1, 2, 3,$\cdots$)

The following two conditions are enough to determine the arbitrary constants $C_{n,u}^{(l)}$, $D_{n,u}^{(l)}$, that is to say,

(1) continuity of pressure, at $r'=1$,

$$\Phi_0^{(\mathrm{I})\prime} = \Phi_0^{(\mathrm{II})} ,$$

(2) continuity of velocity of water particle, at $r'=1$,

$$\frac{\partial \Phi_0^{(\mathrm{I})'}}{\partial r}=\frac{\partial \Phi_0^{(\mathrm{II})'}}{\partial r}\,, \tag{16}$$

where the condition $\dfrac{\partial \Phi_0^{(\mathrm{I})'}}{\partial \theta}=\dfrac{\partial \Phi_0^{(\mathrm{II})'}}{\partial \theta}$ (at $r'=1$) is found to be equivalent to (15), so the above two conditions are used.

Substituting (11) and (14) into (15) and (16), and applying the operators,

$$\int_{-\pi}^{\pi}\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}d\theta\int_{-H}^{0}\cosh a_0'(H'+z')dz'\,,\quad \int_{-\pi}^{\pi}\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}d\theta\int_{-H'}^{0}\cos a_s'(H'+z')dz'\,,$$

$$(n=0,\ 1,\ 2,\cdots;\ s=1,\ 2,\ 3,\cdots)\,,$$

to the relations (15) and (16), we obtain the following compatible equations, i. e.,

$$C_{n,0}^{(\mathrm{I})}J_n(a_0')-C_{n,0}^{(\mathrm{II})}H_n^{(1)}(a_0')=0 \quad (n=0,\ 1,\ 2,\cdots)\,,$$

$$D_{n,0}^{(\mathrm{I})}J_n(a_0')-D_{n,0}^{(\mathrm{II})}H_n^{(1)}(a_0')=0 \quad (n=1,\ 2,\ 3,\cdots)\,,$$

$$C_{n,s}^{(\mathrm{I})}I_n(a_s')-C_{n,s}^{(\mathrm{II})}K_n(a_s')=0 \quad \begin{pmatrix}n=0,\ 1,\ 2,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix},$$

$$D_{n,s}^{(\mathrm{I})}I_n(a_s')-D_{n,s}^{(\mathrm{II})}K_n(a_s')=0 \quad \begin{pmatrix}n=1,\ 2,\ 3,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix},$$

$$C_{4l,0}^{(\mathrm{I})}J_{4l}'(a_0')-C_{4l,0}^{(\mathrm{II})}H_{4l}^{(1)\prime}(a_0')=R_{4l,0}$$
$$(l=0,\ 1,\ 2,\ 3,\cdots)\,,$$

$$C_{u,0}^{(\mathrm{I})}J_u'(a_0')-C_{s,0}^{(\mathrm{II})}H_u^{(1)\prime}(a_0')=0$$
$$\begin{Bmatrix}u:\ \text{positive integers except}\\ \text{for}\ \ u=4l\ \ (l=0,\ 1,\ 2,\cdots)\end{Bmatrix},$$

$$D_{n,0}^{(\mathrm{I})}J_n'(a_0')-D_{n,0}^{(\mathrm{II})}H_n^{(1)\prime}(a_0')=0$$
$$(n=1,\ 2,\ 3,\cdots)$$

$$C_{4l,s}^{(\mathrm{I})}I_{4l}'(a_s')-C_{4l,s}^{(\mathrm{II})}K_{4l}'(a_s')=R_{4l,s}$$
$$\begin{pmatrix}l=0,\ 1,\ 2,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix},$$

$$C_{u,s}^{(\mathrm{I})}I_u'(a_s')-C_{u,s}^{(\mathrm{II})}K_u'(a_s')=0$$
$$\begin{Bmatrix}u:\ \text{positive integers except for}\\ u=4l\ \ (l=0,\ 1,\ 2,\ 3,\cdots)\\ s=1,\ 2,\ 3,\cdots\end{Bmatrix},$$

$$D_{n,s}^{(\mathrm{I})} I_n'(a_s') - D_{u,s}^{(\mathrm{II})} K_n'(a_s') = 0$$
$$\begin{pmatrix} n=1,\ 2,\ 3,\cdots \\ s=1,\ 2,\ 3,\cdots \end{pmatrix},$$

where

$$R_{4l,0} = \frac{i\omega \cdot 4}{\sinh 2a_0'H' + 2a_0'H'} \cdot \sum_{u=1}^{\infty} A_{4l,u} \cdot \lambda_{4l,u} J_{4l}'(\lambda_{4l,u}) \cdot L_{4l,u}^{(0)}, \tag{17}$$

$$L_{4l,u}^{(0)} = \frac{1}{M_{4l,u}} \cdot \frac{2}{\lambda_{4l,u}^2 - a_0'^2} \cdot \{\omega^2(\cosh a_0'H' - \cosh \lambda_{4l,u}H')$$
$$+ g'(-a_0' \sinh a_0'H' + \lambda_{4l,u} \sinh \lambda_{4l,u}H')\},$$

$$R_{4l,s} = \frac{i\omega \cdot 4}{\sin 2a_s'H' + 2a_s'H'} \cdot \sum_{u=0}^{\infty} A_{4l,u} \cdot \lambda_{4l,u} J_{4l}'(\lambda_{4l,u}) \cdot L_{4l,u}^{(s)}, \tag{18}$$

$$L_{4l,u}^{(s)} = \frac{1}{M_{4l,u}} \cdot \frac{2}{\lambda_{4l,u}^2 + a_s'^2} \cdot \{\omega^2(\cos a_s'H' - \cosh \lambda_{4l,u}H')$$
$$+ g'(a_s' \sin a_s'H' + \lambda_{4l,u} \sinh \lambda_{4l,u}H')\},$$

$$M_{4l,u} = \begin{vmatrix} (-\omega^2 + g'\lambda_{4l,u}), & -(\omega^2 + g'\lambda_{4l,u}) \\ e^{-\lambda_{4l,u}H'}, & -e^{\lambda_{4l,u}H'} \end{vmatrix}.$$

Solving the above simultaneous equations we get the arbitrary constants :-

$$\left.\begin{array}{l} C_{4l,0}^{(\mathrm{I})} = \dfrac{i\pi a_0'}{2} \cdot H_{4l}^{(1)}(a_0') \cdot R_{4l,0}, \\[2ex] C_{4l,0}^{(\mathrm{II})} = \dfrac{i\pi a_0'}{2} \cdot H_{4l}^{(1)}(a_0') \cdot R_{4l,0}, \\[1ex] \quad (l=0,\ 1,\ 2,\ 3,\cdots) \\ C_{4l,s}^{(\mathrm{I})} = i a_s' \cdot K_{4l}(a_s') \cdot R_{4l,s}; \\ C_{4l,s}^{(\mathrm{II})} = i a_s' \cdot I_{4l}(a_s') \cdot R_{4l,s}, \\ \quad \begin{pmatrix} l=0,\ 1,\ 2,\ 3,\ \cdots \\ s=1,\ 2,\ 3,\ \cdots \end{pmatrix} \end{array}\right\} \tag{19}$$

where $R_{4l,u}$ ($l=0,\ 1,\ 2,\ 3,\cdots;\ u=0,\ 1,\ 2,\ 3,\cdots$) are given in (17) and (18). Other constants are zero.

7. Theoretical Formulae of Wave Height

By virtue of (19) and the relation, $\zeta' = \dfrac{1}{-g'}\left(\dfrac{\partial \Phi'}{\partial t}\right)_{z'=0}$ ($\zeta' = \zeta/r_0$, ζ :

wave height), we obtain from (11) and (14) the wave heights in dimensionless form :-

$$\zeta^{(\mathrm{I})\prime}=\frac{i\omega}{g'}\cdot e^{-i\omega t}\cdot[\Phi'_{(h)}+\Phi'_{(p)}]_{z'=0} \quad \text{in Domain (I)} ,$$

$$[\Phi'_{(h)}]_{z'=0}=\sum_{l=0}^{\infty} J_{4l}(a_0'r')\cdot C_{4l,0}^{(\mathrm{I})}\cdot\cos 4l\theta\cdot\cosh a_0'H'$$

$$+\sum_{l=0}^{\infty}\sum_{s=1}^{\infty} I_{4l}(a_s'r')\cdot C_{4l,s}^{(\mathrm{I})}\cdot\cos 4l\theta\cdot\cos a_s'H' ,$$

$$[\Phi'_{(p)}]_{z'=0}=-i\omega\sum_{l=0}^{\infty}\sum_{s=1}^{\infty} A_{4l,s}\cdot\cos 4l\theta\cdot J_{4l}(\lambda_{4l,s}r')\cdot\frac{2g'}{M_{4l,s}} ,$$

$$\zeta^{(\mathrm{II})\prime}=\frac{i\omega}{g'}\cdot e^{-i\omega t}\cdot[\Phi^{(\mathrm{II})\prime}]_{z'=0} \quad \text{in Domain (II)} ,$$

$$[\Phi^{(\mathrm{II})\prime}]_{z'}=\sum_{l=0}^{\infty} H_{4l}^{(1)}(a_0'r')\cdot C_{4l,0}^{(\mathrm{II})}\cdot\cos 4l\theta\cdot\cosh a_0'H'$$

$$+\sum_{l=0}^{\infty}\sum_{s=1}^{\infty} K_{4l}(a_s'r')\cdot C_{4l,s}^{(\mathrm{II})}\cdot\cos 4l\theta\cdot\cos a_s'H' ,$$

where $C_{4l,u}^{(j)}$ ($j=\mathrm{I}, \mathrm{II}$; $l=0, 1, 2, 3,\cdots$; $u=0, 1, 2, 3,\cdots$) are given in (19) and other constants are presented in the preceding sections.

8. Asymptotic Formula of the Out-going Wave

In this section we consider the asymptotic formula of the out-going wave.

As mentioned in the paper T. M., the asymptotic formula of the out-going wave is given as follows;

$$\zeta^{(\mathrm{II})}\approx\frac{i\omega\cdot r_0^2}{g}\sqrt{\frac{2}{\pi a_0 r}}\cdot\cosh a_0 H\cdot N(\theta)\cdot\exp\left\{i\left(-\omega t+a_0 r-\frac{\pi}{4}\right)\right\}, \tag{20}$$

where

$$N(\theta)=\sum_{l=0}^{\infty} C_{4l,0}^{(\mathrm{II})}\cos 4l\theta ,$$

$\zeta^{(\mathrm{II})}$: wave height of the out-going wave and the constants included in the equation are given in the preceding sections.

9. Numerical Consideration of the Out-going Wave

In numerical consideration we adopt the following numerical value, i. e.,

amplitude of bottom displacement (D_{bot})=1 cm,
depth of water (H)=31.2 cm,
period of vibration (T)=1 sec.,
acceleration of gravity (g)=980.5 cm/sec^2.

Substituting these values into Airy's relation, $\omega^2=ag\tanh aH$, we ascertain the value of wave number a_0 to be 0.045323.

For convenience of numerical consideration, let us rewrite the form of (20) as follows;

$$\zeta^{(\mathrm{II})}\approx\frac{1}{\sqrt{r}}\sin\left(\omega t-a_0r+\frac{\pi}{4}\right)\zeta_r^{(\mathrm{II})}\ , \tag{21}$$

where

$$\zeta_r^{(\mathrm{II})}=\frac{\omega\cdot r_0^{\,2}}{g}\cdot\sqrt{\frac{2}{\pi a_0}}\cdot\cosh a_0H\cdot N(\theta)\ ,$$

$$N(\theta)=\sum_{l=0}^{\infty}C_{4l,0}^{(\mathrm{II})}\cdot\cos 4l\theta\ ,$$

$$C_{4l,0}^{(\mathrm{II})}=\frac{i\pi a_0'}{2}\cdot J_{4l}(a_0')\cdot R_{4l,0}\quad(\text{from (19)})\ ,$$

$$R_{4l,0}=\frac{i\omega\cdot 4}{\sinh 2a_0'H'+2a_0'H'}\cdot\sum_{u=0}^{\infty}A_{4l,u}\cdot\lambda_{4l,u}\cdot J_{4l}'(\lambda_{4l,u})\cdot L_{4l,u}^{(0)}\quad(\text{from (17)}\ ,$$

$$L_{4l,u}^{(0)}=\frac{1}{M_{4l,u}}\cdot\frac{2}{\lambda_{4l,u}^2-a_0'^2}\cdot\{\omega^2(\cosh a_0'H'-\cosh\lambda_{4l,u}H')$$
$$+g'(-a_0'\sinh a_0'H'+\lambda_{4l,u}\sinh\lambda_{4l,u}H')\}\ ,$$

$$M_{4l,u}=\begin{vmatrix}(-\omega^2+g'\lambda_{4l,u})\ , & -(\omega^2+g'\lambda_{4l,u})\\ e^{-\lambda_{4l,u}H'}\ , & -e^{\lambda_{4l,u}H'}\end{vmatrix}\ ,$$

$$A_{4l,u}=\frac{8\varepsilon_{4l}\cdot(D_{\text{bot}}')_0}{\pi[J_{4l+1}(\lambda_{4l,u})]^2}\int_0^{\pi/4}\cos 4l\theta\cdot d\theta\cdot\int_0^{(1/\sqrt{2})\sec\theta}J_{4l}(\lambda_{4l,u}r')\cdot r'dr'\quad(\text{from (6)})$$

$$=\frac{8\varepsilon_{4l}\cdot(D_{\text{bot}}')_0}{\pi[J_{4l+1}(\lambda_{4l,u})]^2}\cdot I_{4l,u}\ , \tag{22}$$

$$I_{4l,u}=\sum_{t=0}^{\infty}(-1)^t\cdot U(4l,u,t)\cdot V(4l,t)\ ,$$

$$U(4l,u,t)=\frac{\left(\frac{\lambda_{4l,u}}{2}\right)^{4l+2t}}{t\,!\cdot(4l+t)\,!}\cdot\frac{\left(\frac{1}{2}\right)^{2l+t+1}}{2(2l+t+1)}\ ,$$

$$V(4l,\, t)=L_{2(2l+t+1)}-\frac{(4l)^2}{2\,!}\cdot J_{2(2l+t+1),\,2}+\frac{(4l)^2\{(4l)^2-2^2\}}{4\,!}\cdot J_{2(2l+t+1),\,4}$$
$$-\frac{(4l)^2\{(4l)^2-2^2\}\cdot\{(4l)^2-4^2\}}{6\,!}\cdot J_{2(2l+t+1),\,6}+\cdots,$$

$$L_{2(2l+t+1)}=\int_0^{\pi/4}\cos^{-2(2l+t+1)}\theta\cdot d\theta$$
$$=\frac{1}{2(2l+t+1)-1}\cdot 2^{(2l+t+1)-1}+\frac{2(2l+t+1)-2}{2(2l+t+1)-1}\cdot L_{2(2l+t)}\ ,$$
$$\cdots\cdots$$
$$L_2=1\,,$$
$$J_{2(2l+t+1),2p}=\int_0^{\pi/4}\cos^{-2(2l+t+1)}\theta\cdot\sin^{2p}\theta\cdot d\theta$$
$$=\frac{1}{2p-2(2l+t+1)}\left(-2^{-p+(2l+t+1)}+(2p-1)\cdot J_{2(2l+t+1),\,2(p-1)}\right),$$
$$\cdots\cdots$$
$$J_{2(2l+t+1),\,0}=L_{2(2l+t+1)}\ .$$

Here we difine $\zeta_r^{(\text{II})}$ as "relative wave height".

As to the integration of (22), reference should be made to the Appendix, and $\lambda_{4l,u}$ ($l=0,\ 1,\ 2,\ 3,\cdots;\ u=1,\ 2,\ 3,\cdots$) may be obtained from the table of Bessel functions.

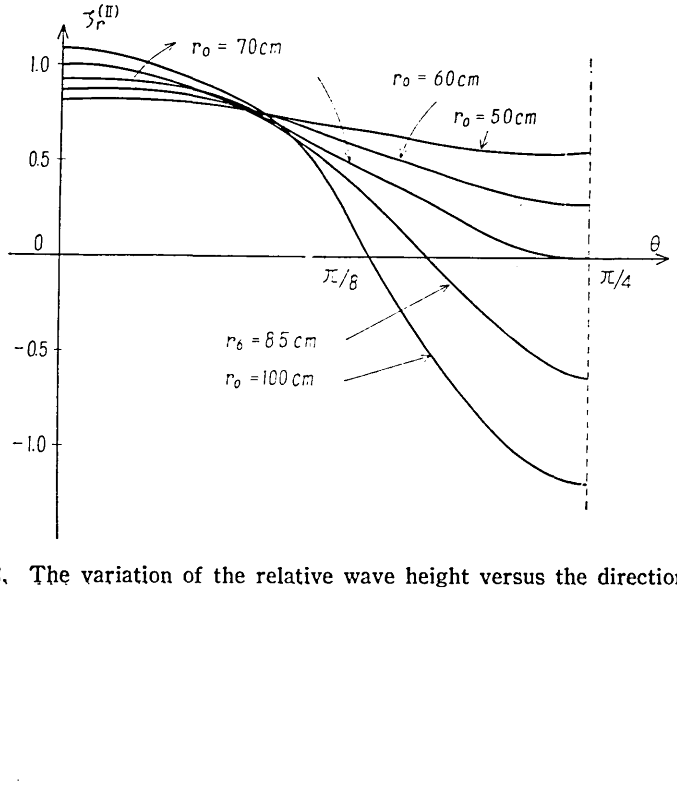

Fig. 2. The variation of the relative wave height versus the direction θ.

Now by the use of the above mentioned values and relations, we finally obtained variations of the relative wave height $\zeta_r^{(II)}$ versus direction θ for each value of r_0, i.e., 50 cm, 60 cm, 70 cm, 85 cm and 100 cm (Fig. 2). From this Figure we find that

(1) the relative wave height in the direction $\theta=0$ is generally larger than that in the direction $\theta=\pi/4$,

(2) the phase difference of the waves in the direction $\theta=0$ and $\theta=\pi/4$ becomes more remarkable, as the diagonal length of the square bottom increases. And they are finally in inverse phase.

10. Acknowledgments

The author is indebted to Professor R. Takahasi and Assistant Professor K. Kajiura of this Institute and also to Dr. K. Takano of the Geophysical Institute of Tokyo University for their kind discussions.

Appendix

Here we will consider the method of integration of (22). Firstly, Bessel function is expressed as an ascending series[2], i.e.,

$$J_n(z)=\sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{1}{2}z\right)^{n+2r}}{r!\cdot(n+r)!}\ .$$

By virtue of this expression, we have

$$\int_0^{(1/\sqrt{2})\sec\theta} J_{4l}(\lambda_{4l,s}r')\cdot r'dr'=\sum_{t=0}^{\infty}(-1)^t\cdot\frac{\left(\frac{\lambda_{4l,s}}{2}\right)^{4l+2t}}{t!\cdot(4l+t)!}\cdot\int_0^{(1/\sqrt{2})\sec\theta} r'^{4l+2t+1}dr'\ , \qquad (A,1)$$

where $J_{4l}(z)$ is integrable term by term, so $J_{4l}(\lambda_{4l,s}r')r'$ is also integrable term by term.

After integration, (A, 1) becomes

$$\int_0^{(1/\sqrt{2})\sec\theta} J_{4l}(\lambda_{4l,s}r')r'dr'=\sum_{t=0}^{\infty}(-1)^t\cdot U(4l,\ s,\ t)\cdot\sec^{(4l+2t+2)}\theta\ , \qquad (A,2)$$

$$U(4l,\ s,\ t)=\frac{\left(\frac{\lambda_{4l,s}}{2}\right)^{4l+2t}}{t!\cdot(4l+t)!}\cdot\frac{2^{-(2l+t+1)}}{2(2l+t+1)}\ . \qquad (A,3)$$

2) G. N. WATSON, *Theory of Bessel Functions*, (Cambridge press, 1922).

Consequently, applying the operator, $\int_0^{\pi/4} \cos 4l\theta \cdot d\theta$ to (A, 2), it produces

$$\int_0^{\pi/4} \cos 4l\theta \cdot d\theta \int_0^{(1/\sqrt{2})\sec\theta} J_{4l}(\lambda_{4l,s} r') r' dr' = \sum_{t=0}^{\infty} (-1)^t \cdot U(4l, s, t) \cdot V(4l, t) \; , \quad \text{(A, 4)}$$

$$V(4l, t) = \int_0^{\pi/4} \cos 4l\theta \cdot \sec^{(4l+2t+2)} \theta \cdot d\theta \; .$$

Here $\cos(4l\theta)$ can be expanded to the following series,

$$\cos 4l\theta = 1 - \frac{(4l)^2}{2\,!} \cdot \sin^2\theta + \frac{(4l)^2 \cdot \{(4l)^2 - 2^2\}}{4\,!} \sin^4\theta$$
$$- \frac{(4l)^2 \cdot \{(4l)^2 - 2^2\}\,\{(4l)^2 - 4^2\}}{6\,!} \sin^6\theta + \cdots \; ,$$

or

$$\cos 4l\theta \cdot \sec^{(4l+2t+2)}\theta = \frac{1}{\cos^{(4l+2t+2)}\theta} - \frac{(4l)^2}{2\,!} \cdot \frac{\sin^2\theta}{\cos^{(4l+2t+2)}\theta}$$
$$+ \frac{(4l)^2\{(4l)^2 - 2^2\}}{4\,!} \cdot \frac{\sin^4\theta}{\cos^{(4l+2t+2)}\theta}$$
$$- \frac{(4l)^2\{(4l)^2 - 2^2\}\,\{(4l)^2 - 4^2\}}{6!} \cdot \frac{\sin^6\theta}{\cos^{(4l+2t+2)}\theta}$$
$$+ \cdots \; .$$

Integrating the last equation from 0 to $\pi/4$ with regard to θ, we have $V(4l, t)$.

$$V(4l, t) = L(4l + 2t + 2) - \frac{(4l)^2}{2\,!} \cdot J_{(4l+2t+2),\,2}$$
$$+ \frac{(4l)^2\{(4l)^2 - 2^2\}}{4\,!} \cdot J_{(4l+2t+2),\,4}$$
$$+ \frac{(4l)^2\{(4l)^2 - 2^2\} \cdot \{(4l)^2 - 4^2\}}{6\,!} \cdot J_{(4l+2t+2),\,6}$$
$$+ \cdots, \quad \text{(A, 5)}$$

where

$$L_{(4l+2t+2)} = \int_0^{\pi/4} \cos^{-(4l+2t+2)}\theta \cdot d\theta \; ,$$

$$J_{(4l+2t+2),\,2p} = \int_0^{\pi/4} \cos^{-(4l+2t+2)}\theta \cdot \sin^{2p}\theta \cdot d\theta \; .$$

The latter expression can be reduced to the former by the next recurrence formula,

$$J_{(4l+2t+2),\,2p}=\frac{1}{2p-(4l+2t+2)}\cdot\{-2^{(2l+2t+1)-p}+(2p-1)\cdot J_{(4l+2t+2),\,2p-2}\},$$

$$\cdots\cdots,$$

$$J_{(4l+2t+2),\,0}=\int_0^{\pi/4}\cos^{-(4l+2t+2)}\theta\cdot d\theta$$

$$=L(4l+2t+2)\,.$$

Accordingly if we can evaluate $L(4l+2t+2)$, (A, 5) or (A, 4) can be integrated analytically. The evaluation of $L(4l+2t+2)$ will easily be done by recurrence formulae as follows:

$$L(4l+2t+2)=\frac{1}{(4l+2t+2)-1}\cdot 2^{(2l+t)}+\frac{(4l+2t)}{(4l+2t+2)-1}\cdot L_{(4l+t)}\,,$$

$$\cdots\cdots$$

$$L_2=1\,.$$

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10. 任意の形をした振動する底によつておこされた Water Wave を扱う一般的方法の一応用例

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筆者は前に任意の形をした底によつておこされる水波を取り扱う一般的方法を導入した.

本論文では，筆者はこの方法を正方形の底が，一様な振巾をもつて振動する場合に適用して見た. その結果次のような結論を得た. すなわち,

(1) 発生する Water Wave は，正方形の辺に垂直な方向で一般に波高は大きく，正方形の対角線方向では，波高は小さい.

(2) そして，また，正方形の辺に垂直な方向と対角線方向の Wave の phase 差は，正方形の対角線の長さが大きくなるにつれて大となつてくる.