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11. *On the Directivity of Water Waves Generated by a Vibrating Elliptic Wave Source.*

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1. Introduction

We obtained theoretical formulae of the wave height produced by a harmonically vibrating elliptic area of the bottom.[1] Then an asymptotic formula of the wave height at a distant point from the elliptic wave-generating source was given, whose variation versus direction for some numerical values was also presented. In this paper more numerical examples of the wave height in asymptotic form are shown for several values of the major and the minor axes of the wave-generating ellipse.

The article entitled "On Water Waves Generated by a Vibrating Bottom with an Elliptic Form" was reported in Japanese, so we give outlines of derivation of the theoretical formulae of water waves produced by the vibrating elliptic wave source in the following section.

2. Derivation of Theoretical Formulae

Supposing the water to extend to infinity horizontally and taking the axis of x and y on the undisturbed free surface of water and that of z vertically upwards, we have, on the assumption that the motion is infinitely small and irrotational,

$$p/p=\partial\Phi/\partial t+gz\ ,$$

where p denotes the pressure, ρ the density of water, g the constant of gravity and Φ the velocity potential which satisfies the equation

$$\Delta\Phi=0\ . \tag{1}$$

If ζ denotes the elevation of water-level at the point $(x,\ y,\ 0)$ and at time t above the undisturbed surface, then the pressure condition

1) T. MOMOI, "On Water Waves Generated by a Vibrating Bottom with an Elliptic Form", *Zisin*, [ii], **14**, (1962), 9-22 (in Japanese).

to be satisfied at the free surface, supposing ζ to be small, is

$$\zeta=-\frac{1}{g}\left(\frac{\partial \Phi}{\partial t}\right)_{z=0}$$

and the kinematical condition is

$$\frac{\partial \zeta}{\partial t}=\left(\frac{\partial \Phi}{\partial z}\right)_{z=0}.$$

Hence, for $z=0$, we must have

$$\frac{\partial^2 \Phi}{\partial t^2}+g\frac{\partial \Phi}{\partial z}=0 \tag{2}$$

As the bottom condition, we have

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=-H}=\eta_0\,, \tag{3}$$

where H is the depth of water and η_0 the velocity of bottom displacement.

If the problem is confined to the case of vibration, the equation (1) and the conditions (2) and (3) can be reduced to the following forms ;–

$$\Delta \Phi'=0 \tag{1'}$$

$$-\omega^2\Phi'+g\frac{\partial \Phi'}{\partial z}=0 \quad (z=0) \tag{2'}$$

$$\frac{\partial \Phi'}{\partial z}=\eta_0' \quad (z=-H)\,, \tag{3'}$$

where the primed functions disignate those from which the time factor $\exp(-i\omega t)$ is eliminated.

Now by separating the velocity potential as $\Phi'=\Phi_{zy}'\cdot\Phi_z'$, where Φ_{xy}' and Φ_z' are respectively the functions of x, y and of z alone, the equation (1′) and the conditions (2′) and (3′) again degenerate into the next forms ;–

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_{xy}'+a^2\Phi_{xy}'=0 \tag{4}$$

$$\frac{d^2\Phi_z'}{dz^2}=a^2\Phi_z' \tag{5}$$

$$-\omega^2\Phi_z'+g\frac{d\Phi_z'}{dz}=0 \quad (z=0) \tag{6}$$

$$\frac{d\Phi'}{dz}=\eta_0' \quad (z=-H)\ , \tag{7}$$

where a is the constant of separation.

Since we are treating the problem of an elliptic wave source, it is more convenient to use the elliptic coordinates rather than the Cartesian coordinates. The equation (4) becomes as follows in the elliptic coordinates $(\xi,\ \eta)$;

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\Phi_{xy}'+2k^2(\cosh 2\xi-\cos 2\eta)\Phi_{xy}'=0 \tag{8}$$

where $2k=ha$ and $2h$ is the interfocal length of the vibrating elliptic bottom.

Expect for the conditions (6) and (7), the velocity potential must satisfy the next conditions, i. e.,

i) $\Phi\rightarrow 0$ at the infinite point of ξ,

ii) $\Phi(0,\ \eta)=\Phi(0,\ -\eta)$ (continuity of pressure),

$$\frac{\partial}{\partial\xi}[\Phi(\xi,\eta)]_{\xi\rightarrow 0}=-\frac{\partial}{\partial\xi}[\Phi(\xi,\ -\eta)]_{\xi\rightarrow 0}$$

(continuity of the velocity of water particle)

on the interfocal line of the ellipse.

Under these conditions we obtained in the previous paper[2] the velocity potentials, $\Phi^{(1)}$ in the region where the bottom vibrates and $\Phi^{(2)}$ in the region where the bottom is at rest, as follows ;-

$$\begin{aligned}\Phi^{(1)}=e^{-i\omega t}\Bigg[&\left\{\sum_{n=0}^{\infty}C_{2n,0}Ce_{2n}(\xi,\ q_0)ce_{2n}(\eta,\ q_0)\right\}\cdot\cosh a_0(H+z)\\&+\sum_{r=1}^{\infty}\left\{\sum_{n=0}^{\infty}C_{2n,r}Ce_{2n}(\xi,\ q_r)ce_{2n}(\eta,\ q_r)\right\}\cdot\cos a_r(H+z)\\&-i\omega D'_{\text{bot}}\left(z+\frac{g}{\omega^2}\right)\Bigg]\end{aligned} \tag{9}$$

$$\begin{aligned}\Phi^{(2)}=e^{-i\omega t}\Bigg[&\left\{\sum_{n=0}^{\infty}M_{2n,0}Me_{2n}^{(1)}(\xi,\ q_0)ce_{2n}(\eta,\ q_0)\right\}\cdot\cosh a_0(H+z)\\&+\sum_{r=1}^{\infty}\left\{\sum_{n=0}^{\infty}M_{2n,r}\cdot Me_{2n}^{(1)}(\xi,\ q_r)\cdot ce_{2n}(\eta,\ q_r)\right\}\cdot\cos a_r(H+z)\Bigg]\end{aligned} \tag{10}$$

where a_0 and ia_r $(r=1,\ 2,\ 3,\cdots)$ are a real and an imaginary solution

2) T. MOMOI, *loc. cit.*, 1).

of the relation $\omega^2=ag\tanh aH$; D'_{bot} is the amplitude of the bottom displacement; $q_0=(ha_0/2)^2$ and $q_r=-(ha_r/2)^2$ $(r=1,\ 2,\ 3,\cdots)$; $ce_{2n}(\eta)$, $Ce_{2n}(\xi)$ and $Me_{2n}^{(1)}(\xi)$ $(n=0,\ 1,\ 2,\ 3,\cdots)$ are Mathieu and modified Mathieu functions respectively;

$$\left.\begin{aligned}M_{2n,s}&=-E_{2n,s}^{(ce)}\cdot Ce'_{2n}(\xi_0,\ q_s)/F_{2n,s}\\ C_{2n,s}&=-E_{2n,s}^{(ce)}\cdot M_{2n}^{(1)\prime}(\xi_0,\ q_s)/F_{2n,s}\end{aligned}\right\},$$

$$\begin{pmatrix}n=0,\ 1,\ 2,\ 3,\cdots\\ s=0,\ 1,\ 2,\ 3,\cdots\end{pmatrix}$$

where

$$\left.\begin{aligned}E_{2n,0}^{(ce)}&=\frac{\dfrac{-i\omega D'_{\text{bot}}}{a_0}\left\{\dfrac{g}{\omega^2}\sinh a_0H-\dfrac{1}{a_0}(\cosh a_0H-1)\right\}}{\dfrac{1}{2}\left(\dfrac{1}{2a_0}\sinh 2a_0H+H\right)}\cdot A_0^{(2n)}\\ E_{2n,0}^{(ce)}&=\frac{\dfrac{-i\omega D'_{\text{bot}}}{a_r}\left\{\dfrac{g}{\omega^2}\sin a_rH+\dfrac{1}{a_r}(\cos a_rH-1)\right\}}{\dfrac{1}{2}\left(\dfrac{1}{2a_r}\sin 2a_rH+H\right)}\cdot(-1)^n\cdot 2A_0^{(2n)}\end{aligned}\right\}$$

$$\begin{pmatrix}n=0,\ 1,\ 2,\ 3,\cdots\\ s=1,\ 2,\ 3,\cdots\end{pmatrix}$$

and

$$F_{2n,s}=\begin{vmatrix}Me_{2n}^{(1)}(\xi_0,\ q_s), & -Ce_{2n}(\xi_0,\ q_s)\\ Me_{2n}^{(1)\prime}(\xi_0,\ q_s), & -Ce'_{2n}(\xi_0,\ q_s)\end{vmatrix},$$

provided ξ_0 is the value of ξ, which discriminates the region where the bottom vibrates and that where the bottom is at rest, and $A_0^{(2n)}$ is the coefficient of the first term in Fourier expansion of $ce_{2n}(\eta)$ with regard to η in the range $0\leq\eta<2\pi$.

From (9), (10) and the relation $\zeta=\dfrac{1}{-g}\left(\dfrac{\partial\Phi}{\partial t}\right)_{z=0}$ we finally obtained the wave heights $\zeta^{(1)}$ in the region where the bottom vibrates and $\zeta^{(2)}$ in the region where the bottom is at rest as follows;

$$\zeta^{(1)}=\frac{i\omega}{g}e^{-i\omega t}\left[\left\{\sum_{n=0}^{\infty}C_{2n,0}Ce_{2n}(\xi,\ q_0)ce_{2n}(\eta,\ q_0)\right\}\cdot\cosh a_0H\right.$$

$$\left.+\sum_{r=1}^{\infty}\left\{\sum_{n=0}^{\infty}C_{2n,r}Ce_{2n}(\xi,\ q_r)ce_{2n}(\eta,\ q_r)\right\}\cdot\cos a_rH-iD'_{\text{bot}}\cdot\frac{g}{\omega}\right]$$

$$\zeta^{(2)}=\frac{i\omega}{g}e^{-i\omega t}\left[\left\{\sum_{n=0}^{\infty}M_{2n,0}Me_{2n}^{(1)}(\xi,\,q_0)ce_{2n}(\eta,\,q_0)\right\}\cdot\cosh a_0H\right.$$

$$\left.+\sum_{r=1}^{\infty}\left\{\sum_{n=0}^{\infty}M_{2n,r}Me_{2n}^{(1)}(\xi,\,q_r)ce_{2n}(\eta,\,q_r)\right\}\cdot\cos a_rH\right]. \tag{11}$$

Next we consider an asymptotic formula for the out-going wave. According to McLachlan[3], an asymptotic form of the modified Mathieu function $Me_{2n}^{(1)}$ is given by

$$Me_{2n}^{(1)}(\xi,\,q_0)\approx\frac{ce_{2n}(0,\,q_0)ce_{2n}(\pi/2,\,q_0)}{A_0^{(2n)}}\sqrt{\frac{2}{\pi a_0R}}\cdot e^{i(a_0R-\pi/4)}\,. \tag{12}$$

$$Me_{2n}^{(1)}(\xi,\,q_r)\approx-\frac{2i}{\pi}\cdot\frac{(-1)^n ce_{2n}(0,\,|q_r|)ce_{2n}(\pi/2,\,|q_r|)}{A_0^{(2n)}}\cdot K_{2n}(a_rR)\,, \tag{13}$$

where $K_{2n}(a_rR)$ is modified Bessel function.

Making use of (12) and (13) we get the asymptotic formula of the out-going wave from (11) as follows;

$$\zeta^{(2)}\approx\frac{\omega}{g}\cdot 2\cdot\sqrt{\frac{2}{\pi a_0R}}\cdot\cosh a_0H$$

$$\cdot\frac{\dfrac{\omega D'_{\text{bot}}}{a_0}\left\{\dfrac{g}{\omega^2}\sinh a_0H-\dfrac{1}{a_0}(\cosh a_0H-1)\right\}}{\dfrac{1}{2}\left(\dfrac{1}{2a_0}\sinh a_0H+H\right)}\cdot e^{-i(\omega t-a_0R+\pi/4)}$$

$$\cdot\sum_{n=0}^{\infty}\frac{-Ce'_{2n}(\xi_0,\,q_0)ce_{2n}(0,\,q_0)ce_{2n}(\pi/2,\,q_0)}{F_{2n,0}}\cdot ce_{2n}(\eta,\,q_0)\,, \tag{14}$$

where

$$F_{2n,0}=\begin{vmatrix} Me_{2n}^{(1)}(\xi_0,\,q_0)\,, & -Ce_{2n}(\xi_0,\,q_0) \\ Me_{2n}^{(1)\prime}(\xi_0,\,q_0)\,, & -Ce'_{2n}(\xi_0,\,q_0)\end{vmatrix}\,, \tag{15}$$

R: polar coordinate which is related with elliptic coordinates $(\xi,\,\eta)$ as follows;

$$R=\frac{h}{2^{1/2}}(\cosh 2\xi+\cos 2\eta)^{1/2}\,.$$

Now let us treat the height of the out-going wave in asymptotic form in numerical consideration.

3) N. W. McLachlan, *Theory and Application of Mathieu Functions*, (1947), Oxford Press.

3. Outlines of Numerical Analysis

Modified Mathieu function $Me_{2n}^{(1)}$ is related with Ce_{2n} and Fey_{2n} as follows ;[4]

$$Me_{2n}^{(1)}=Ce_{2n}+iFey_{2n} .$$

Consequently, the determinant (15) is reduced to

$$F_{2n,0}=-i\begin{vmatrix} Fey_{2n} & Ce_{2n} \\ Fey'_{2n} & Ce'_{2n} \end{vmatrix} \tag{16}$$

Next, Mathieu function $ce_{2n}(\eta, q_0)$ is expressed in Fourier form as follows[5] ;

$$ce_{2n}(\eta, q_0)=\sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2r\eta \tag{17}$$

where $A_{2r}^{(2n)}$ ($n=0, 1, 2, 3,\cdots$) stand for Fourier coefficients, of which a table for values of q_0 from 0 to 40 up to $2n=4$ has been given by Ince.[6]

Making use of the above-mentioned Fourier coefficients $A_{2r}^{(2n)}$ ($r=0, 1, 2, 3,\cdots, n=0, 1, 2, 3,\cdots$), modified Mathieu functions $Ce_{2n}(\xi, q_0)$ and $Fey_{2n}(\xi, q_0)$ are represented in Bessel series,[7] that is to say,

$$Ce_{2n}(\xi, q_0)=\frac{ce_{2n}(1/2\cdot\pi, q_0)}{A_0^{(2n)}} \sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)} J_{2r}(2k_0 \cosh \xi) , \tag{18}$$

$$Fey_{2n}(\xi, q_0)=\frac{ce_{2n}(\pi/2, q_0)}{A_0^{(2n)}} \sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)} Y_{2r}(2k_0 \cosh \xi) \tag{19}$$

$$(\cosh \xi>1) ,$$

where

$$k_0=\frac{ha_0}{2} .$$

Although modified Mathieu functions $Ce_{2n}(\xi, q_0)$ and $Fey_{2n}(\xi, q_0)$ are represented by hyperbolic series, the above expressions (18) and (19) in Bessel series are recommended for more rapid convergence.

4) N. W. MCLACHLAN, *loc. cit.*, 3).

5) N. W. MCLACHLAN, *loc. cit.*, 3).

6) E. L. INCE, "Tables of the Elliptic-cylinder Functions", *Proc. Roy. Soc. Edinburgh* (1931-32), LII, p. 355.

7) N. W. MCLACHLAN, *loc. cit.*, 3).

Now substituting (16), (17), (18) and (19) into (14), we finally get, after some computations,

$$\zeta^{(2)}=\frac{1}{\sqrt{R}}\sin\left(-\omega t+a_0R-\frac{\pi}{4}\right)\cdot\zeta_r^{(2)}$$

where only the real part was taken and

$$\zeta_r^{(2)}=\frac{\omega}{g}\cdot 2\sqrt{\frac{2}{\pi a_0}}\cdot\cosh a_0H\cdot\frac{\dfrac{\omega D'_{\text{bot}}}{a_0}\left\{\dfrac{g}{\omega^2}\sinh a_0H-\dfrac{1}{a_0}(\cosh a_0H-1)\right\}}{\dfrac{1}{2}\left(\dfrac{1}{2a_0}\sinh a_0H+H\right)}$$

$$\cdot\sum_{n=0}^{\infty}A_0^{(2n)}\cdot ce_{2n}(0,q_0)\cdot\sum_{r=0}^{\infty}(-1)^rA_{2r}^{(2)}\cdot J'_{2n}(a_0h\cosh\xi_0)\cdot ce_{2n}(\eta,q_0)$$

$$\Bigg/\begin{vmatrix}\sum\limits_{r=0}^{\infty}(-1)^rA_{2r}^{(2n)}Y_{2r}(a_0h\cosh\xi_0)\ , & \sum\limits_{r=0}^{\infty}(-1)^rA_{2r}^{(2n)}J_{2r}(a_0h\cosh\xi_0)\\ \sum\limits_{r=0}^{\infty}(-1)^rA_{2r}^{(2n)}Y'_{2r}(a_0h\cosh\xi_0)\ , & \sum\limits_{r=0}^{\infty}(-1)^rA_{2r}^{(2n)}J'_{2r}(a_0h\cosh\xi_0)\end{vmatrix}$$

Here, we define $\zeta_r^{(2)}$ as "relative wave height".

Now in the next section, let us see variations of the wave height $\zeta_r^{(2)}$ versus a direction for several values of the major and the minor axes.

4. Computation and Discussion

We used Ince's table[8] for the coefficient $A_{2r}^{(2n)}$ ($r=0, 1, 2, 3,\cdots$, $n=0, 1, 2,\cdots$) and Bessel's table[9] for Bessel functions. Other used values are given as follows;

depth of water $(H)=31.2$ cm,
acceleration of gravity $(g)=980.5$ cm/sec^2,
period of vibration of the bottom $(T)=1$ sec.

By the use of these values and the relation $\omega^2=a_0g\cdot\tanh(a_0H)$, the wave number of the out-going wave (a_0) was obtained as

$$a_0=0.045323\ .$$

For six pairs of values of half lengths of the major and the minor axes, which are denoted by x_0 and y_0 respectively, the relative wave height $\zeta_r^{(2)}$ versus a direction, which is denoted either by θ or η, is shown

8) E. L. INCE, *loc. cit.*, 4).

9) G. N. WATSON, *Theory of Bessel Functions*, (1922), Cambridge Press.

in Fig. 1. The angle of polar coordinate (θ) is nearly equal to the elliptic coordinate (η) for large ξ by virtue of the relation $\tan\theta = \tanh\xi \cdot \tan\eta \approx \tan\eta$.

Now let us consider features of the wave heights given in Fig. 1.

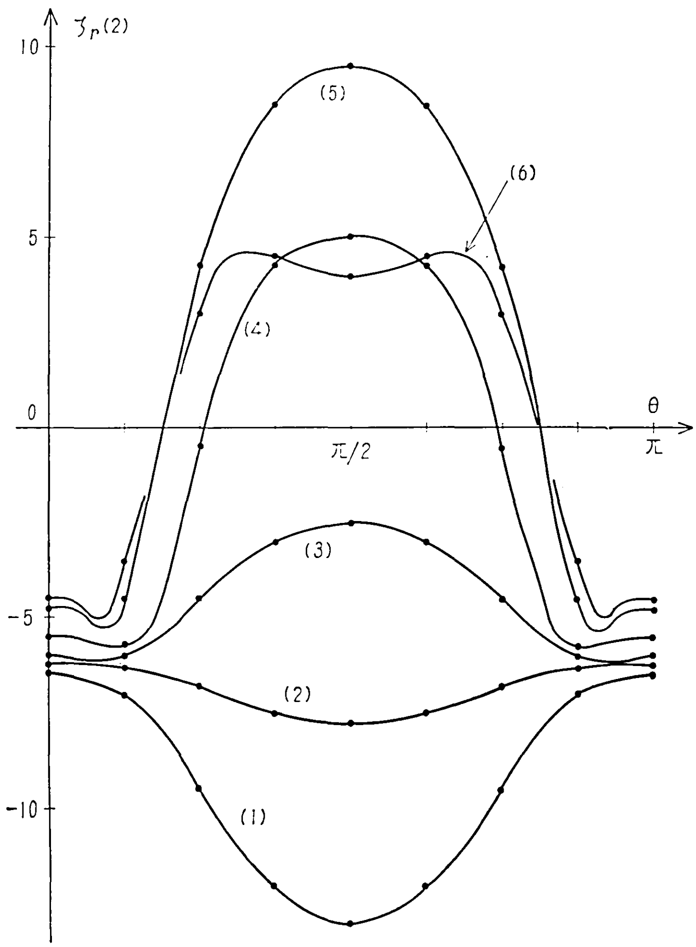

Fig. 1. The variation of the relative wave height versus the direction θ.
(1) The case where the half lengths of the major axis (x_0) and the minor axis (y_0) are 264.76 cm and 261.06 cm respectively;
(2) The case where $x_0 = 264.76$ cm and $y_0 = 225.01$ cm;
(3) The case where $x_0 = 264.76$ cm and $y_0 = 210.02$ cm;
(4) The case where $x_0 = 264.76$ cm and $y_0 = 195.40$ cm;
(5) The case where $x_0 = 264.76$ cm and $y_0 = 176.49$ cm;
(6) The case where $x_0 = 264.76$ cm and $y_0 = 150.00$ cm.

In this computation the half length of the major axis is fixed at $x_0=264.76$ cm. As the minor axis gradually decreases in length from a circle, the relative wave height in the direction of $\theta=1/2\cdot\pi$ (minor axis) decreases once, then gradually increases until it comes up to the maximum on the positive side of the relative wave height. Then it decreases again, being superposed by a small undulation.

If we start at $\theta=0$ and move counter-clockwise round a circle (R=const.) at a distant point from the elliptic source, the wave height undulates in such a way that it takes a negative value in some cases (e. g., the case where $y_0=216.06$, 225.01, 210.02 cm) and in other cases negative and positive values occur alternately. (e. g., $y=195.40$, 176.49, 150.00 cm).

These undulations seem to be caused by the interference of wavelets generated at each elementary portion of the wave-generating ellipse.

5. Acknowledgments

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11. 振動する楕円源によつて発生する Water Wave の指向性について

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著者は，一様に振動する楕円浪源によつて発生する Water Wave についての波高の理論式と，その漸近公式とを導いた (地震，第2輯，第14巻，印刷中)．そこで，若干の数値に対する波高の方向性を調べた．本論文において，著者はさらに多くの長軸，短軸の数値に対し，波高の方向性を調べた．Fig. 1 に示されるごとく，半径 264.76 cm の円から短軸が減少して行くと，まず，短軸方向の波高は減少（絶対値は増加）し，ある最小値に達するとふたたび増加し，その増加は，ついに波高の正の側にまでおよび，ここである最高に達して，ふたたび，波高は減少して行く．すなわち，浪源を中心とする，非常に大きな半径の円を考え，この円の上をわれわれがたどるとき，あるときは，波高は負の側でのみ波打つて変化し，あるときは負の側より正の側へと符号を変えて波打つ．前者の例は Fig. 1 で短軸の半分の長さ $(y_0)=261.06$, 225.01, 210.02 cm のとき，後者の例は短軸の半分の長さ $(y_0)=195.40$, 176.49, 150.00 cm のときである．これらの undulation の原因は，振動する楕円源の各部分で発生する Water Wave の干渉に基くものである．