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27. The Dispersion of Love Waves in Heterogeneous Half Space overlain by a Homogeneous Layer.

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1. Introduction

The author once treated the dispersion of surface waves in heterogeneous half space[1]. Formulation was made in a general form, and a scheme was presented for determining the structure of the medium from the dispersion curve of Love waves. However, the assumption of slowly varying material constants of the medium has prevented us from applying the theory to many practical problems. The numerical example at the end of [I], where the discontinuity structure was replaced by a continuous layer, has to be understood as a mere trial.

In order to get numerically precise results, it might be most adequate to divide the medium into several parts, each of which satisfies the above mentioned assumption.

As such we shall consider here the heterogeneous half space having a homogeneous upper layer, and shall treat the dispersion of Love waves.

Such a problem has already been solved by several authors[2] assuming special functions for density and rigidity. Our treatment will be nothing but the generalization of their results. We shall treat the case when density and shear velocity in the lower layer are given only numerically, and show that the scheme for the inverse problem still holds under certain restrictions.

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1) T. TAKAHASHI, *Bull. Earthq. Res. Inst.*, **33** (1955) 287; **35** (1957) 297. Cited as I and II respectively in this paper.

2) T. MATSUZAWA, *Bull. Earthq. Res. Inst.*, **6** (1929) 213.
H. JEFFREYS, M. N. R. S. *Geophys. Suppl.*, **2** (1928) 101.
Y. SATÔ, *Bull. Earthq. Res. Inst.*, **30** (1952) 1.
J. T. WILSON, *Bull. Seism. Soc. Amer.*, **30** (1940) 273.

Notations are as follows:

T period
L wave-length
$p=2\pi/T$
$f=2\pi/L$
$v_Q=p/f=L/T$ phase velocity
$U=dp/df$ group velocity
$V(z)$ amplitude
$\rho(z)$ density
$\mu(z)$ rigidity
$v_s(z)=\sqrt{\mu/\rho}$ shear velocity
$\kappa(z)=(v_Q/v_s(z))^2=p^2\rho/f^2\mu$
H zero point of $\kappa-1$. Then $\kappa(H)=1$,
or $v_Q=v_s(H)$.
$\mu(0)$, $\kappa(0)$ etc. are written merely as μ_0, κ_0 etc.

2. Recapitulation of the formulas of the previous papers

Consider the differential equation

$$\frac{1}{\mu}\frac{d}{dz}\left(\mu\frac{dV}{dz}\right)+f^2(\kappa-1)V=0\,, \tag{1}$$

where μ and κ are slowly varying functions of z, and $d\kappa/dz$ is assumed to be negative.

Then the approximate solution of (1) which is valid for large f and converges to zero at $z=\infty$, is expressed as

$$V(z)=\left(\frac{-\varphi}{\mu\sqrt{\kappa-1}}\right)^{1/2}K_{1/3}(-i\varphi)\,, \tag{2}$$

where $K_{1/3}$ means modified Bessel function of order 1/3, and φ is defined by

$$\varphi(z)=f\int_z^H\sqrt{\kappa-1}\,dz\,. \tag{3}$$

Next we introduce a function $H(\varphi)$:

$$H(\varphi)=\frac{d}{d\varphi}\log\{\varphi^{1/2}K_{1/3}(-i\varphi)\}\,. \tag{4}$$

The asymptotic expression of $H(\varphi)$ for large φ is

$$H(\varphi) \sim \tan\left(\frac{\pi}{4}-\varphi\right), \tag{5}$$

which is useful for calculating higher modes.

From (2) and (4) we obtain

$$\frac{V'(0)}{V(0)}=-\frac{\mu_0'}{2\mu_0}-\frac{\kappa_0'}{4(\kappa_0-1)}-f\sqrt{\kappa_0-1}\ H(\varphi_0)\ , \tag{6}$$

where the letters with prime are derivatives with respect to z.

When κ_0 becomes very near to 1, equation (6) is transformed into

$$\frac{V'(0)}{V(0)}=-\frac{\mu_0'}{2\mu_0}-0.72904\,f^{2/3}(-\kappa_0')^{1/3}+1.10554\,\frac{f^{4/3}(\kappa_0-1)}{(-\kappa_0')^{1/3}}$$
$$+\frac{\kappa_0''+5f^2(\kappa_0-1)^2}{10(-\kappa_0')}+\ldots, \tag{7}$$

where

$$0.72904=9^{-1/3}\Gamma(2/3)/\Gamma(4/3)\ ,\ 1.10554=9^{-1/3}\{\Gamma(2/3)/\Gamma(4/3)\}^2\ .$$

Usually κ is defined only for $z\geqq 0$. Accordingly, if $\kappa_0<1$, we cannot determine H, and $\varphi(z)$ must have some expression other than (3). (See Appendix.) The expression (7) is proved to hold both for $\kappa_0>1$ and $\kappa_0<1$.

Lastly, if κ_0 is fairly small as compared with 1, we can use the asymptotic expansion of $K_{1/3}(|\varphi|)$ and obtain

$$\frac{V'(0)}{V(0)}=-\frac{\mu_0'}{2\mu_0}-f\sqrt{1-\kappa_0}\left(1-\frac{\kappa_0'}{4(1-\kappa_0)^{3/2}}-\frac{5}{32f^2}\frac{\kappa_0'^2}{(1-\kappa_0)^3}\right.$$
$$\left.-\frac{1}{8f^2}\frac{\kappa_0''}{(1-\kappa_0)^2}-\ldots\right). \tag{8}$$

3. Equation of dispersion

We shall discuss the theory of Love wave propagation in media consisting of a homogeneous upper layer of thickness h and a heterogeneous lower layer, where rigidity and density are slowly varying functions of the depth. The shear velocity is assumed to be increasing with the depth.

The z-axis is taken vertically downward, and the origin at the interface of two layers. Then the free surface is indicated by the equation $z=-h$.

The letters with suffix 1, such as ρ_1, μ_1, κ_1 and $V_1(z)$ are used to denote that they belong to the upper layer. Shear velocity in the upper layer is expressed merely as v_1. ρ_1, μ_1, κ_1 and v_1 are all constants, and $v_1 < v_s(0)$.

The amplitude $V(z)$ of the Love wave satisfies (1). Then, in the lower layer we can employ the expression (2).

The boundary conditions are such that the stress shall vanish at the free surface, and at the interface displacement and stress shall be continuous with those given by (2). That is

$$V_1'(-h)=0\ , \tag{9}$$

and

$$\frac{V'(0)}{V(0)}=\frac{\mu_1 V_1'(0)}{\mu_0 V_1(0)}\ . \tag{10}$$

It is easily verified that in the homogeneous layer,

$$V_1(z)=A\cos\{f\sqrt{\kappa_1-1}(z+h)\} \tag{11}$$

is the solution of (1) which satisfies (9).

Substituting (11) in (10), we obtain

$$\frac{V'(0)}{V(0)}=-\frac{\mu_1}{\mu_0}f\sqrt{\kappa_1-1}\tan(fh\sqrt{\kappa_1-1})\ . \tag{12}$$

This is the equation of dispersion, and has three different expressions corresponding to the various values of κ_0.

i) $\kappa_0>1$ $(v_Q>v_s(0))$.

From (6) and (12) we have

$$H(\varphi_0)=\frac{\mu_1\sqrt{\kappa_1-1}}{\mu_0\sqrt{\kappa_0-1}}\tan(fh\sqrt{\kappa_1-1})+\frac{1}{4f(\kappa_0-1)^{3/2}}\left\{(2-\kappa_0)\frac{\mu_0'}{\mu_0}-\kappa_0\frac{\rho_0'}{\rho_0}\right\}\ . \tag{13}$$

To solve (13), we transform (13) into

$$H(\varphi_0)=\frac{\mu_1\sqrt{\kappa_1-1}}{\mu_0\sqrt{\kappa_0-1}}\tan\left(\varphi_0\frac{h\sqrt{\kappa_1-1}}{\bar{\varphi}}\right)$$
$$+\frac{1}{\varphi_0}\frac{\varphi}{4(\kappa_0-1)^{3/2}}\left\{(2-\kappa_0)\frac{\mu_0'}{\mu_0}-\kappa_0\frac{\rho_0'}{\rho_0}\right\}\ , \tag{14}$$

where

$$\bar{\varphi}=\varphi_0/f=\int_0^H \sqrt{\kappa-1}\, dz\ .$$

When $v_s(z)$ is given, fix a value of v_Q and calculate $\bar{\varphi}$, κ_0 and κ_1. Assuming that μ_1/μ_0, h, μ_0'/μ_0 and ρ_0'/ρ_0 are all known, we can determine φ_0 from (14), then $f=\varphi_0/\varphi$ is the wave number corresponding to v_Q.

A brief table of $H(\varphi)$ for real φ is given in [II]. The more precise one (Table 1) is calculated, using the table of Airy functions.[3] (See Appendix (A3))

Table 1

φ	$H(\varphi)$	$\varphi H(\varphi)$	φ	$H(\varphi)$	$\varphi H(\varphi)$
.00	∞	.16667	.50	.53195	.26598
.02	10.51098	.21022	.52	.49598	.25791
.04	5.81366	.23255	.54	.46151	.24922
.06	4.15550	.24933	.56	.42836	.23988
.08	3.32878	.26630	.58	.39641	.22992
.10	2.73762	.27376	.60	.36553	.21932
.12	2.35715	.28286	.62	.33632	.20852
.14	2.07403	.29036	.64	.30653	.19618
.16	1.85316	.29651	.66	.27954	.18450
.18	1.67467	.30144	.68	.25065	.17044
.20	1.52635	.30527	.70	.22370	.15659
.22	1.40048	.30811	.72	.19729	.14205
.24	1.29166	.31000	.74	.17141	.12684
.26	1.19622	.31102	.76	.14596	.11093
.28	1.11148	.31121	.78	.12093	.09433
.30	1.03538	.31061	.80	.09623	.07698
.32	.96642	.30925	.82	.07185	.05892
.34	.90342	.30716	.84	.04772	.04008
.36	.84543	.30435	.86	.02385	.02051
.38	.79174	.30086	.88	.00018	.00016
.40	.74171	.29668	.90	−.02332	−.02099
.42	.69486	.29184	.92	−.04670	−.04296
.44	.65077	.28634	.94	−.07996	−.07516
.46	.60911	.28019	.96	−.09315	−.08942
.48	.56958	.27340	.98	−.11633	−.11400
			1.00	−.13951	−.13951

ii) $\kappa_0 \doteqdot 1$ $(v_Q \doteqdot v_s(0))$.

From (7) and (12) we have

$$\tan\,(fh\sqrt{\kappa_1-1})=\frac{\mu_0}{\mu_1\sqrt{\kappa_1-1}}\left\{\frac{\mu_0'}{2f\mu_0}+0.72904\left(\frac{-\kappa_0'}{f}\right)^{1/3}\right.$$

3) A. D. SMIRNOV, *Tables of Airy Functions* (Pergamon Press, 1960).

$$-1.10554\frac{f^{1/3}(\kappa_0-1)}{(-\kappa_0')^{1/3}}-\frac{\kappa_0''+5f^2(\kappa_0-1)^2}{10f(-\kappa_0')}+\dots\Big\}\,. \tag{15}$$

When $\kappa_0=1$ ($v_Q=v_s(0)$), (15) becomes

$$\xi\tan\xi=\frac{\mu_0}{\mu_1}\frac{h}{2}\left(\frac{\mu_0'}{\mu_0}-\frac{\kappa_0''}{5(-\kappa_0')}\right)+0.72904\frac{\mu_0}{\mu_1}\left(\frac{h(-\kappa_0')}{\kappa_1-1}\right)^{1/3}\xi^{2/3}\,. \tag{16}$$

where

$$\xi=fh\sqrt{\kappa_1-1}\,.$$

iii) $\kappa_0<1$ $(v_s(0)>v_Q>v_1)$.

From (8) and (12) we have

$$\tan(fh\sqrt{\kappa_1-1})=\frac{\mu_0\sqrt{1-\kappa_0}}{\mu_1\sqrt{\kappa_1-1}}\Big\{1+\frac{\mu_0'}{2f\mu_0\sqrt{1-\kappa_0}}-\frac{\kappa_0'}{4f(1-\kappa_0)^{3/2}}$$
$$-\frac{5}{32f^2}\frac{\kappa_0'^2}{(1-\kappa_0)^3}-\frac{\kappa_0''}{8f^2(1-\kappa_0)^2}-\dots\Big\}\,. \tag{17}$$

(15) and (17) are solved by iteration.

4. Explicit expression of $\bar{\varphi}$

Examples of the explicit expression of $\bar{\varphi}$ are given in this section.

i) When ρ/μ is a linear function of z, we put

$$\rho/\mu=(\rho_0/\mu_0)(1-\beta z)\,,$$

or

$$\kappa=\kappa_0(1-\beta z)\,. \tag{18}$$

From $\kappa(H)=1$, β is expressed by H as

$$\beta=(\kappa_0-1)\kappa_0^{-1}H^{-1}\,. \tag{19}$$

The calculation of $\bar{\varphi}$ is easily performed:

$$\bar{\varphi}=\int_0^H\sqrt{\kappa-1}\,dz=\frac{2}{3}H\sqrt{\kappa_0-1}\,. \tag{20}$$

We add expressions for $d\bar{\varphi}/d\kappa_0$, κ_0' and κ_0''.

$$\frac{1}{\bar{\varphi}}\frac{d\bar{\varphi}}{d\kappa_0}=\frac{3}{2(\kappa_0-1)}-\frac{1}{\kappa_0}\,, \tag{21}$$

$$\frac{-\kappa_0'}{\kappa_0}=\beta=\frac{\kappa_0-1}{H\kappa_0}\,, \qquad \kappa_0''=0\,. \tag{22}$$

ii) When $\sqrt{\mu/\rho}$ is a linear function of z, we put

$$\sqrt{\mu/\rho}=\sqrt{\mu_0/\rho_0}(1+\gamma z)\,,$$

or

$$\kappa=\kappa_0(1+\gamma z)^{-2}\,. \tag{23}$$

From $\kappa(H)=1$, γ is expressed by H as

$$\gamma=(\sqrt{\kappa_0}-1)H^{-1}\,. \tag{24}$$

If we put

$$\sqrt{\kappa}=\cosh\theta\,, \tag{25}$$

the calculation of $\bar{\varphi}$ is easily carried out:

$$\bar{\varphi}=H(\theta_0\cosh\theta_0-\sinh\theta_0)(\cosh\theta_0-1)^{-1}\,. \tag{26}$$

We add expressions for $d\bar{\varphi}/d\kappa_0$, κ_0' and κ_0''.

$$\frac{d\bar{\varphi}}{d\kappa_0}=\frac{H}{2}\frac{\theta_0}{\cosh\theta_0(\cosh\theta_0-1)}\,, \tag{27}$$

$$-\frac{\kappa_0'}{\kappa_0}=\frac{2}{H}(\sqrt{\kappa_0}-1)\,, \qquad \frac{\kappa_0''}{\kappa_0}=\frac{6}{H^2}(\kappa_0-1)^2\,. \tag{28}$$

iii) When μ/ρ is a linear function of z, we put

$$\mu/\rho=(\mu_0/\rho_0)(1+\delta z)\,,$$

or

$$\kappa=\kappa_0(1+\delta z)^{-1}\,. \tag{29}$$

From $\kappa(H)=1$, δ is expressed by H as

$$\delta=(\kappa_0-1)H^{-1}\,. \tag{30}$$

If we put

$$\frac{2}{\kappa}-1=\cos\omega\,, \tag{31}$$

the calculation of $\bar{\varphi}$ is easily performed:

$$\bar{\varphi}=H(\omega_0-\sin\omega_0)(1-\cos\omega_0)^{-1}\ . \tag{32}$$

We add the expressions for $d\bar{\varphi}/d\kappa_0$, κ_0' and κ_0''.

$$\frac{d\bar{\varphi}}{d\kappa_0}=\frac{H}{2}\frac{\sin\omega_0(1+\cos\omega_0)}{1-\cos\omega_0}\ , \tag{33}$$

$$\frac{-\kappa_0'}{\kappa_0}=\frac{\kappa_0-1}{H}\ , \qquad \frac{\kappa_0''}{\kappa_0}=2\left(\frac{\kappa_0-1}{H}\right)^2 . \tag{34}$$

5. Group velocity

We have only to differentiate the period equation with f in order to obtain the equation for the group velocity U.

From the relation

$$\kappa_0 \propto p^2/f^2$$

we obtain

$$\frac{1}{\kappa}\frac{d\kappa}{df}=\frac{1}{\kappa'}\frac{d\kappa'}{df}=\frac{1}{\kappa''}\frac{d\kappa''}{df}=\frac{2}{f}\left(\frac{U}{v_Q}-1\right).$$

Then the differentiation can be carried out easily, and only the results are listed.

i) When $\kappa_0>1$, we have from (13)

$$\begin{aligned}\frac{U}{v_Q}=1-&\left\{-\sqrt{\kappa_0-1}\,\frac{dH}{d\varphi_0}\varphi_0-\sqrt{\kappa_0-1}\,H(\varphi_0)+\frac{\mu_1}{\mu_0}\sqrt{\kappa_1-1}\,(\tan\xi+\xi\sec^2\xi)\right\}\\ &\div\left\{-2f\kappa_0\sqrt{\kappa_0-1}\,\frac{d\bar{\varphi}}{d\kappa_0}\frac{dH}{d\varphi_0}-\frac{\kappa_0}{\sqrt{\kappa_0-1}}H(\varphi_0)\right.\\ &\left.+\frac{\mu_1}{\mu_0}\frac{\kappa_1}{\sqrt{\kappa_1-1}}(\tan\xi+\xi\sec^2\xi)-\frac{(-\kappa_0')}{2f(\kappa_0-1)^2}\right\} .\end{aligned} \tag{35}$$

When $\bar{\varphi}$ has explicit expression, $d\bar{\varphi}/d\kappa_0$ can be computed easily. However, in case $\bar{\varphi}$ is obtained by numerical integration, it would be better to give up the use of (35) and return to the usual method of numerical (or graphical) differentiation of v_Q.

ii) When $\kappa_0=1$, we have from (15)

$$\frac{v_Q}{U}=1-\left\{0.72904\frac{2h}{3}\frac{\mu_0}{\mu_1}\left(\frac{f}{-\kappa_0'}\right)^{-1/3}-\frac{1}{f}(\tan\xi+\xi\sec^2\xi)\right\}$$

$$\div\left\{1.10554\cdot 2h\frac{\mu_0}{\mu_1}\left(\frac{f}{-\kappa_0'}\right)^{1/3}+\frac{1}{f(\kappa_1-1)}(\tan\xi+\xi\sec^2\xi)\right\}. \quad (36)$$

iii) When $\kappa_0<1$, we have from (17)

$$\frac{U}{v_Q}=1-\left\{\xi(\tan\xi+\xi\sec^2\xi)+\frac{5}{32}\frac{\kappa_0'}{f(1-\kappa_0)^{5/2}}+\frac{\kappa_0''}{8f}\frac{1}{(1-\kappa_0)^{3/2}}-\mu_0\sqrt{1-\kappa_0}\right\}$$

$$\div\left\{\frac{\kappa_1}{\kappa_1-1}\xi(\tan\xi+\xi\sec^2\xi)+\frac{\mu_0\kappa_0}{\sqrt{1-\kappa_0}}+\frac{\kappa_0'}{2(1-\kappa_0)^2}\right.$$

$$\left.+\frac{5\kappa_0'^2}{32f^2}\frac{(4+\kappa_0)}{(1-\kappa_0)^{7/2}}+\frac{\kappa_0''}{8f}\frac{(2+\kappa_0)}{(1-\kappa_0)^{5/2}}\right\}. \quad (37)$$

6. Numerical results

As an example of calculation, it will be desirable to take up the case when the distribution of shear velocity is given only numerically. And a knowledge of the accurate results, if we could have them, would be useful for checking our calculation.

N. Kobayashi and H. Takeuchi[4] treated the dispersion of mantle Love waves. Their computation was based on the method of variation which is considered to have fairly good accuracy, therefore their results would be able to be taken as our criterion.

Table 2

case	T	v_Q	v_Q*	U	U*
	65.5	4.35	4.36	3.95	4.01
1	174.0	4.76	4.80		
2	181.2	4.76	4.82	4.29	4.24
3	176.7	4.76	4.81	(4.33)	4.23
4	175.5	4.76	4.81	4.26	4.23
1	346.6	5.32	5.41		
2	330.6	5.32	5.36	4.37	4.36
3	312.8	5.32	5.30	(4.46)	4.34
4	306.6	5.32	5.28	4.29	4.33

v_Q* (or U*) means phase (or group) velocity read from the results of Kobayashi and Takeuchi.

case 1. $\bar{\varphi}$ is calculated numerically.

case 2. $\bar{\varphi}=\frac{2}{3}H\sqrt{\kappa_0-1}$

case 3. $\bar{\varphi}=H(\theta_0\cosh\theta_0-\sinh\theta_0)/(\cosh\theta_0-1)$, $\cosh\theta_0=\sqrt{\kappa_0}$

case 4. $\bar{\varphi}=H(\omega_0-\sin\omega_0)/(1-\cos\omega_0)$, $\cos\omega_0=\frac{2}{\kappa_0}-1$

4) N. KOBAYASHI and T. TAKEUCHI, *Zisin* (*Journ. Seism. Soc. Japan*), **13** (1960) 232.

We performed the calculation using the same data as theirs for the case of the Jeffreys-Bullen's model, with the homogeneous upper layer of $h=35$ km, $v_1=3.5$ km/sec and $\rho=2.7$ gm/cm^3. The results are shown in Table 2.

We choose the values of v_Q equal to those of shear velocity at depths as 35 km (interface), 300 km and 500 km. Besides the values of $\bar{\varphi}$ obtained by numerical integration (case 1), we tentatively made use of the explicit expressions given in section 4, using the value of H gained from the given data (case 2, 3 and 4).

Group velocity was calculated from (35) or (36). However, the use of (36) can not always be appropriate for the present treatment, where the structure is not precisely the same as assumed in (i), (ii) or (iii) of section 4.

We see from Table 2 that, for $v_Q=5.32$ km/sec, the error of case 1 is greatest and reaches nearly 2%. As we understand, cases 2, 3 and 4 are only approximation of case 1, so that the above result seems to be somewhat queer.

Presumably, it might be that we have reached the limit of our theory, which clearly does not hold for extremely long waves.

7. Analysis of the dispersion curve

Assuming that ρ_1, μ_1, ρ_0, μ_0, ρ_0' μ_0' and h are already known, we can determine the function $v_s(z)$ from the dispersion curve. The process is essentially the same as the one discussed in [I].

For an assigned value of v_Q, f is determined from the dispersion curve. And, after calculating the right side of (13), we obtain φ_0 from the table of $H(\varphi)$. After all, we get $\bar{\varphi}=\varphi_0/f$ as a function of v_Q:

$$\bar{\varphi}\equiv\int_0^H \left\{\left(\frac{v_Q}{v_s(z)}\right)^2-1\right\}^{1/2} dz=I(v_Q) ,$$

which is the same integral equation as I(26). Its solution was given by I(30)-(32).

For relatively short wave-lengths, assuming that we can use the conventional calculation of $\bar{\varphi}$ as in the case 2 to 4 of the former section, the approximate value of H is immediately determined.

For example, using the formula of case 2, we have

$$H = \frac{3}{2}(\kappa_0 - 1)^{-1/2}\bar{\varphi} .$$

As $v_Q(H) = v_q$, the function $v_s(z)$ is fully determined.

The present research is carried out as a continuation or completion of the previous work.[5] We hope it might include practically useful results.

We acknowledge kind assistance of Mr. Rinzo Yamaguchi in the numerical calculations.

Appendix

We shall give the expression of our dispersion equation making use of the Airy function.

The Airy function may be defined as

$$A_i(-\zeta) = \frac{1}{\sqrt{3}\,\pi}\zeta^{1/2}K_{1/3}\left(\frac{2}{3}\zeta^{3/2}\right) . \tag{A1}$$

For real arguement we have the expression

$$A_i(x) = \frac{1}{\pi}\int_0^\infty \cos\left(tx - \frac{1}{3}t^3\right)dt . \tag{A2}$$

From (A1) we easily obtain

$$\begin{aligned} H(\varphi) &= \frac{d}{d\varphi}\log\{\varphi^{1/2}K_{1/3}(-i\varphi)\} \\ &= \frac{1}{6\varphi} + \left(\frac{2}{3\varphi}\right)^{1/3}\frac{A_i'\left\{\left(\frac{3}{2}\varphi\right)^{2/3}\right\}}{A_i\left\{\left(\frac{3}{2}\varphi\right)^{2/3}\right\}} . \end{aligned} \tag{A3}$$

Putting (A3) in (6), we get

$$\frac{V'(0)}{V(0)} = -\frac{\mu_0'}{\mu_0} - \frac{\kappa_0'}{4(\kappa_0 - 1)} - \frac{f\sqrt{\kappa_0 - 1}}{6\varphi_0} - f\sqrt{\kappa_0 - 1}\left(\frac{2}{3\varphi_0}\right)^{1/3}\frac{A_i'\left\{\left(\frac{3}{2}\varphi_0\right)^{2/3}\right\}}{A_i\left\{\left(\frac{3}{2}\varphi_0\right)^{2/3}\right\}} . \tag{A4}$$

5) An interesting paper of L. Knopoff appeared recently which treated the perturbation in the dispersion curves of Love waves due to an additional inhomogeneity of density or modulus:
L. KNOPOFF, *Geophys. J.*, **4** (1961) 161.

The left side of (A4) being replaced by the right side of (12), we have the dispersion equation.

Now we can write

$$\kappa=\kappa_0+z\kappa_0'+\frac{z^2}{2}\kappa_0''+\ldots\ldots,$$

and

$$1=\kappa_0+H\kappa_0'+\frac{H^2}{2}\kappa_0''+\ldots\ldots.$$

By the iteration method, we solve the last equation for H

$$H=\frac{\kappa_0-1}{(-\kappa_0')}+\frac{(\kappa_0-1)\kappa_0''}{2(-\kappa_0')^3}+\ldots.$$

From these expressions, we obtain after some calculation

$$\bar{\varphi}=\int_0^H\sqrt{\kappa-1}\,dz=\frac{2}{3}\frac{(\kappa_0-1)^{3/2}}{(-\kappa_0')}\left\{1+\frac{2}{5}\frac{(\kappa_0-1)\kappa_0''}{(-\kappa_0')^2}+\ldots\right\}. \qquad \text{(A5)}$$

Then we can deduce the following expression of (A4), valid when κ_0 is near 1:

$$\frac{V'(0)}{V(0)}=-\frac{\mu_0'}{2\mu_0}+\frac{\kappa_0''}{10(-\kappa_0')}-(-f^2\kappa_0')^{1/3}\left\{1-\frac{2}{15}\frac{\kappa_0''(\kappa_0-1)}{(-\kappa_0')^2}\right\}\frac{A_i'\left\{\left(\frac{3}{2}\varphi_0\right)^{2/3}\right\}}{A_i\left\{\left(\frac{3}{2}\varphi_0\right)^{2/3}\right\}}. \qquad \text{(A6)}$$

In [II] we defined φ_0 for $\kappa_0<1$, as

$$\varphi_0=\frac{2i}{3}\frac{f(1-\kappa_0)^{3/2}}{(-\kappa_0')}\left\{1-\frac{2}{5}\frac{(1-\kappa_0)\kappa_0''}{(-\kappa_0')^2}\right\}. \qquad \text{(A7)}$$

If the use of (A7) is understood, (A6) can be applied for $\kappa_0>1$, $\kappa_0<1$ and $\kappa_0=1$. We observe that (A6) is the refinement of (7).

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27. 均質な表面層のある不均質な半平面における Love 波の分散

地震研究所　高　橋　健　人

不均質な部分の ρ と μ の変化がゆるやかで，μ/ρ が深さとともに増加する場合の一般的な取扱いを行ない，例として，Jeffreys-Bullen の模型に対して計算し，変分法による結果と比べた．