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28. *Dispersion Curves for the Higher Modes of Surface Waves in Heterogeneous Media.*

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1. Higher modes of surface waves

In the previous papers[1)] concerning the dispersion of surface waves in heterogeneous media, we treated only fundamental mode. Turning to higher modes, we readily see that the dispersion formula becomes rather simple, owing to the fact that the parameter φ_0 is now very large and we have only to use the first term in the asymptotic expansion of the solution for large argument.

For Love waves, we can use I (11):

$$\tan\left(\frac{\pi}{4}-\varphi_0\right)=\frac{1}{4f(\kappa_0-1)^{3/2}}\left\{(2-\kappa_0)\frac{\mu_0'}{\mu_0}-\kappa_0\frac{\rho_0'}{\rho_0}\right\}, \tag{1}$$

instead of I (17).

In the dispersion equation for Rayleigh wave II (52), the left side is written merely as $\tan\left(\frac{\pi}{4}-\varphi_0\right)$:

$$\tan\left(\frac{\pi}{4}-\varphi_0\right)=\frac{\left(1-\frac{\kappa_0}{2}\right)^2}{\sqrt{(\kappa_0-1)(1-\eta_0)}}-\frac{\kappa_0'}{f\kappa_0}\Delta_\sigma(\kappa_0)-\frac{\rho_0'}{f\rho_0}\frac{\Delta_\rho(\kappa_0)}{\sqrt{1-\kappa_0}}. \tag{2}$$

Using (1) and (2) we have drawn the dispersion curves for the second mode of both Love and Rayleigh waves, assuming that ρ be constant and μ be linear, and have confirmed the coincidence with the results achieved by Y. Satô.[2)]

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1) T. TAKAHASHI, *Bull. Earthq. Res. Inst.*, **33** (1955) 287; **35** (1957) 297. Cited as I and II respectively in this paper.

2) Y. SATÔ, *Bull. Seism. Soc. Amer.*, **49** (1959) 57.

In the following we shall treat the case where a low velocity layer exists. Wave number f is assumed to be very large, so that the first approximation is obtained by putting $1/f=0$. For (1) we have

$$\varphi_0 \doteqdot \left(n+\frac{1}{4}\right)\pi . \qquad (n=1, 2, \ldots) \tag{3}$$

2. Higher modes of Love waves in media having a low velocity zone

We use the following notations:

v_Q phase velocity
$v_s(z)$ shear velocity
$\kappa(z)=(v_Q/v_s(z))^2$, $\kappa(H)=1$.

When $\kappa>1$ $(v_Q>v_s(z))$, we define $\varphi_{\alpha\beta}$ by

$$\varphi_{\alpha\beta}=f\int_\alpha^\beta \sqrt{\kappa-1}\,dz . \tag{4}$$

Similarly for $\kappa<1$ $(v_Q<v_s(z))$, we define $\psi_{\alpha\beta}$ by

$$\psi_{\alpha\beta}=f\int_\alpha^\beta \sqrt{1-\kappa}\,dz . \tag{5}$$

Then the asymptotic expression of the amplitude $V(z)$ of the Love wave is

$$\mu^{-1/2}(1-\kappa)^{-1/4}e^{\pm\psi_{\alpha z}} \qquad (\kappa<1), \tag{6}$$

or

$$\mu^{-1/2}(\kappa-1)^{-1/4}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\pi}{4}-\varphi_{\alpha z}\right) \qquad (\kappa>1). \tag{7}$$

In case $(\kappa-1)$ changes sign at $z=\alpha$, there exist the so-called connection formulas of H. Jeffreys[3] which are expressed as

$$2(\kappa-1)^{-1/4}\cos\left(\frac{\pi}{4}-|\varphi_{z\alpha}|\right) \longleftrightarrow (1-\kappa)^{-1/4}e^{-|\psi_{\alpha z}|} , \tag{8}$$

$$(\kappa-1)^{-1/4}\sin\left(\frac{\pi}{4}-|\varphi_{z\alpha}|\right) \longleftrightarrow (1-\kappa)^{-1/4}e^{|\psi_{\alpha z}|} . \tag{9}$$

$V(z)$ must converge to zero at infinity. Then, near the surface,

3) H. JEFFREYS, *Proc. Cam. Phil. Soc.*, **52** (1956) 61.

$V(z)$ takes the form

$$\mu^{-1/2}(\kappa-1)^{-1/4}\cos\left(\frac{\pi}{4}-\varphi_{z\alpha}\right), \tag{10}$$

when $(\kappa-1)$ has only one zero point. (1) is the immediate result of applying the boundary condition

$$\frac{d}{dz}V(0)=0 \tag{11}$$

to (10).

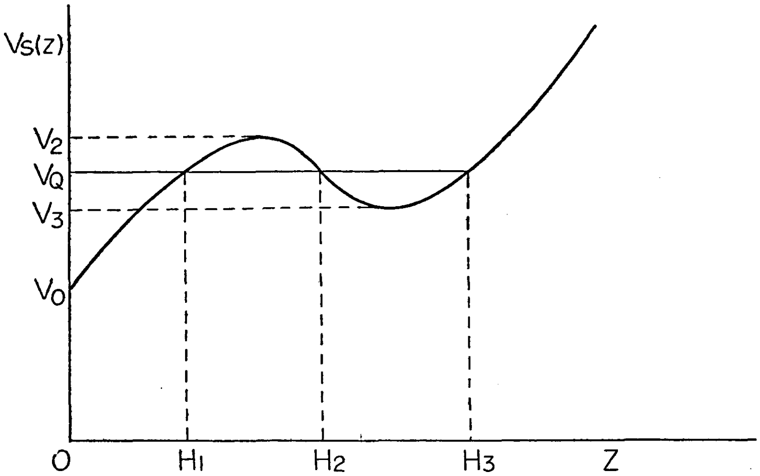

Fig. 1

Now we assume that there exists a low velocity internal layer as shown in Fig. 1. For $v_3<v_Q<v_2$, $(\kappa-1)$ has three zero-points H_1, H_2 and H_3, v_2 or v_3 being the maximum or minimum value of $v_s(z)$.

We observe

$$\begin{aligned}
\kappa-1&>0 \quad \text{for} \quad H_1>z\geqq 0\,,\\
\kappa-1&<0 \quad \text{for} \quad H_2>z>H_1\,,\\
\kappa-1&>0 \quad \text{for} \quad H_3>z>H_2\,,\\
\kappa-1&<0 \quad \text{for} \quad z>H_3\,.
\end{aligned}$$

We further assume that in the above four regions, the asymptotic expression as (6) or (7) holds, which may be allowed for fairly large f and for v_Q not too near to v_2 or v_3.

Following the treatment of H. Jeffreys,[4] we proceed as follows: When $z>H_3$, we have

$$V(z)\sim\mu^{-1/2}(1-\kappa)^{-1/4}e^{-\psi_{3z}}\,, \tag{12}$$

where ψ_{3z} means ψ_{H_3z} .

By the connection formula, we have for $H_3>z>H_2$

$$\begin{aligned} V(z)&\sim 2\mu^{-1/2}(\kappa-1)^{-1/4}\cos\left(\frac{\pi}{4}-\varphi_{z3}\right)\\ &=2\mu^{-1/2}(\kappa-1)^{-1/4}\cos\left(\frac{\pi}{4}+\varphi_{2z}-\varphi_{23}\right)\\ &=2\mu^{-1/2}(\kappa-1)^{-1/4}\left\{\sin\varphi_{23}\cos\left(\frac{\pi}{4}-\varphi_{2z}\right)+\cos\varphi_{23}\sin\left(\frac{\pi}{4}-\varphi_{2z}\right)\right\}, \end{aligned} \tag{13}$$

where $\varphi_{23}\equiv\varphi_{H_2H_3}=\varphi_{2z}+\varphi_{z3}$.

In the region $H_2>z>H_1$, (12) is connected to

$$\begin{aligned} V(z)&\sim\mu^{-1/2}(1-\kappa)^{-1/4}\{\sin\varphi_{23}e^{-\psi_{z2}}+2\cos\varphi_{23}e^{\psi_{z2}}\}\\ &=\mu^{-1/2}(1-\kappa)^{-1/4}\{\sin\varphi_{23}e^{-\psi_{12}+\psi_{1z}}+2\cos\varphi_{23}e^{\psi_{12}-\psi_{1z}}\}\ . \end{aligned} \tag{14}$$

Similarly we have for $H_1>z>0$

$$\begin{aligned} V(z)\sim\mu^{-1/2}(\kappa-1)^{-1/4}&\left\{\sin\varphi_{23}e^{-\psi_{12}}\sin\left(\frac{\pi}{4}-\varphi_{1z}\right)\right.\\ &\left.+4\cos\varphi_{23}e^{\psi_{12}}\cos\left(\frac{\pi}{4}-\varphi_{1z}\right)\right\}. \end{aligned} \tag{15}$$

3. Appearance of C branch

Putting (15) in the boundary condition (11), we obtain the dispersion equation

$$\frac{-\frac{1}{4}\tan\varphi_{23}e^{-2\psi_{12}}+\tan\left(\frac{\pi}{4}-\varphi_{01}\right)}{\frac{1}{4}\tan\varphi_{23}\tan\left(\frac{\pi}{4}-\varphi_{01}\right)e^{-2\psi_{12}}+1}=\frac{-1}{2f\sqrt{\kappa_0-1}}\left(\frac{\mu_0'}{\mu_0}+\frac{\kappa_0'}{2(\kappa_0-1)}\right). \tag{16}$$

As $1/f$ is assumed to be very small, we have approximately

4) H. JEFFREYS, M.N.R.A.S. *Geophys. Suppl.*, **7** (1957) 332.

$$\tan \varphi_{23} \cot \left(\frac{\pi}{4} - \varphi_{01}\right) \doteqdot 4e^{2\Phi_{12}} \ . \tag{17}$$

For large f, $e^{2\Phi_{12}}$ becomes very large, then at least one of the factors of the left side, $\tan \varphi_{23}$ or $\cot\left(\frac{\pi}{4} - \varphi_{01}\right)$, must be very large. And we have as a rough estimation

$$\varphi_{23} \doteqdot \left(n + \frac{1}{2}\right)\pi \qquad (n=1,\ 2,\ \ldots)\ , \tag{18}$$

or

$$\varphi_{01} \doteqdot \left(m + \frac{1}{4}\right)\pi \qquad (m=1,\ 2,\ \ldots)\ . \tag{19}$$

(18) and (19) show that there exist two sorts of branches of dispersion curves.

From the definition of φ we see easily that when v_Q approaches v_3 in (18), f becomes indefinitely large, which means that these branches start from the point with coordinate v_3 on the velocity axis. The existence of such a branch has already been pointed out by R. Yamaguchi[5], who treated the model of four homogeneous layers, of which the third is the low velocity layer and the last is the half-space, and obtained the results as shown in Fig. 2. We follow his lead in adapting the name C branch.

The branches (19) are the continuations of the ordinary higher modes which start from v_0.

To see the nature of these branches, we write

$$\varphi_{01} = f\int_0^{H_1} \sqrt{\kappa - 1}\, dz = fH_1 \sqrt{\bar{\kappa}_0 - 1} \ , \tag{20}$$

$$\varphi_{23} = f\int_{H_2}^{H_3} \sqrt{\kappa - 1}\, dz = fH_{23} \sqrt{\bar{\kappa}_3 - 1} \ , \tag{21}$$

where $\bar{\kappa}_0$ and $\bar{\kappa}_3$ are some mean values of κ in the corresponding region and $H_{23} = H_3 - H_2$.

If we adopt the approximate values of φ_{01} and φ_{23} given by (18) and (19), we have

$$\sqrt{\bar{\kappa}_0 - 1} = \frac{\pi}{fH_1}\left(m + \frac{1}{4}\right), \qquad \sqrt{\bar{\kappa}_3 - 1} = \frac{\pi}{fH_{23}}\left(n + \frac{1}{2}\right), \tag{22}$$

5) R. YAMAGUCHI, *Bull. Earthq. Res. Inst.*, **39** (1961), 653.

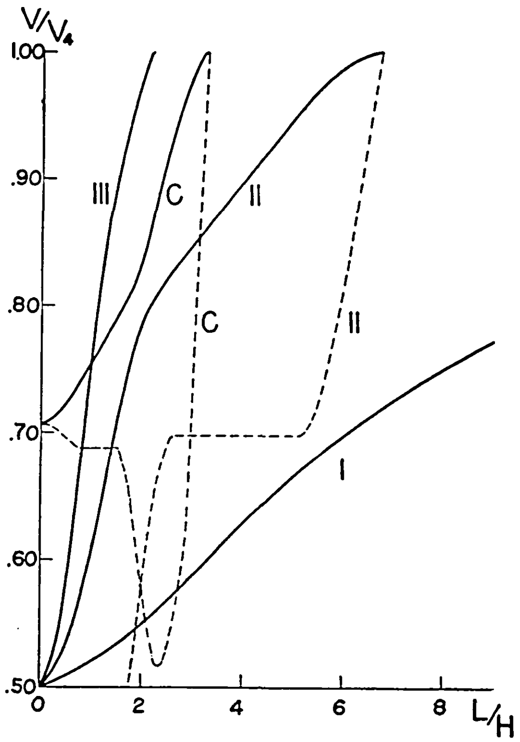

Fig. 2. Phase (solid line) and group (broken line) velocity dispersion curves of Love waves due to R. Yamaguchi. Rigidity, density and thickness of the respective layers are taken as $\mu_1 : \mu_2 : \mu_3 : \mu_4 = 1 : 3 : 2 : 4$, $\rho_1 = \rho_2 = \rho_3 = \rho_4$, $H_1 = H_2 = H_3 = H$.

or

$$v_Q = \bar{v}_0 \sqrt{1 + \frac{L^2}{4H_1^2}\left(m + \frac{1}{4}\right)^2}\,, \qquad v_Q = \bar{v}_3 \sqrt{1 + \frac{L^2}{4H_{23}^2}\left(n + \frac{1}{2}\right)^2}\,. \tag{23}$$

These are the equations of dispersion curves. Notice that $\bar{v}_0$, $\bar{v}_3$, H_1 and H_{23} are all functions of v_Q.

The condition that these branches should intersect is expressed as

$$\sqrt{\frac{\bar{\kappa}_0 - 1}{\bar{\kappa}_3 - 1}} = \frac{H_{23}}{H_1} \frac{m + \frac{1}{4}}{n + \frac{1}{2}}\,. \tag{24}$$

Putting $n=1$, $H_{23}=H_1$, (24) is written as

$$\frac{\left(\frac{4m+1}{6}\right)^2-1}{\left(\frac{4m+1}{6}\right)-(\bar{v}_3/\bar{v}_0)^2}=\bar{\kappa}_3>1\ . \tag{25}$$

As we assume $v_3/v_0>1$, it is easy to see that (25) is not satisfied when $m=1$. That is, the first C branch ($n=1$) does not intersect the ordinary second mode ($m=1$).

For $m\geqq 2$, the numerator of (25) being positive, the denominator must be positive and smaller than the numerator. Then we have

$$\bar{v}_3/\bar{v}_0<\frac{4m+1}{6}\qquad (m=2,\ 3,\ \ldots)\ . \tag{26}$$

This is the condition required for the first C branch to intersect the ordinary $(m+1)$-th mode.

For example, if

$$\frac{3}{2}>\bar{v}_3/\bar{v}_0>1\ ,$$

we can expect that the first C branch intersects the ordinary branches of $m\geqq 2$.

The above is a rough explanation of Yamaguchi's curves of Fig. 2.

When (18) holds, from (15) and (17) we obtain

$$\mu_0^{1/2}(\kappa_0-1)^{1/4}V(0)\doteqdot\sin\varphi_{23}e^{-\psi_{12}}\left\{\sin\left(\frac{\pi}{4}-\varphi_{01}\right)+\cos\left(\frac{\pi}{4}-\varphi_1\right)\cot\left(\frac{\pi}{4}-\varphi_1\right)\right\}$$

$$=\frac{\sin\varphi_{23}}{\sin\left(\frac{\pi}{4}-\varphi_{01}\right)}e^{-\psi_{12}}\ . \tag{27}$$

For the ordinary $(m+1)$-th mode, $\sin\left(\frac{\pi}{4}-\varphi_{01}\right)$ being nearly zero, we transform (27) into

$$\mu_0^{1/2}(\kappa_0-1)^{1/4}V(0)\doteqdot\frac{4\cos\varphi_{23}}{\cos\left(\frac{\pi}{4}-\varphi_{01}\right)}e^{\psi_{12}}\ . \tag{28}$$

These relations show that, even if the amplitudes in the lower layer are the same, at free surface the amplitude of the C branch becomes very small compared with that of the ordinary higher modes.

4. The case of Rayleigh waves

If we want to treat Rayleigh waves under the same assumptions as those in the former section, we have only to replace $H(\varphi_0)$ of II(52) by the left side of (16):

$$\frac{-\dfrac{1}{4}\tan\varphi_{23}e^{-\varphi_{12}}+\tan\left(\dfrac{\pi}{4}-\varphi_{01}\right)}{1+\dfrac{1}{4}\tan\varphi_{23}\tan\left(\dfrac{\pi}{4}-\varphi_{01}\right)e^{-2\varphi_{12}}}$$

$$=\frac{\left(1-\dfrac{\kappa_0}{2}\right)^2}{\sqrt{(\kappa_0-1)(1-\gamma_0)}}-\frac{\kappa_0'}{f\kappa_0}\Delta_\sigma(\kappa_0)-\frac{\rho_0'}{f\rho_0}\frac{\Delta_\rho(\kappa_0)}{\sqrt{1-\kappa_0}}\,, \tag{29}$$

which is the dispersion equation for $v_2>v_Q>v_3$.

In (29) we observe that, if $\tan\varphi_{23}$ is not great $\tan\varphi_{23}e^{-2\varphi_{12}}$ becomes very small and we can write

$$\tan\left(\frac{\pi}{4}-\varphi_{01}\right)\doteqdot\frac{\left(1-\dfrac{\kappa_0}{2}\right)^2}{\sqrt{(\kappa_0-1)(1-\gamma_0)}}-\frac{\kappa_0'}{f\kappa_0}\Delta_\sigma(\kappa_0)-\frac{\rho_0'}{f\rho_0}\frac{\Delta_\rho(\kappa_0)}{\sqrt{1-\kappa_0}}\,, \tag{30}$$

which is similar to (2).

Another alternative is that $\tan\varphi_{23}$ takes a large value and $\tan\varphi_{23}{}^{-2\varphi_{12}}$ becomes comparable with the right side. In this case we have

$$\varphi_{23}\doteqdot\left(n+\frac{1}{2}\right)\pi\,. \tag{31}$$

From (30) and (31) we can expect branches similar to those of Love waves to appear.

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28. 不均質な媒質における表面波の Higher Modes の分散曲線

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特に低速層がある場合を極めて近似的に取り扱い，山口林造が見出した分散曲線の新しい分枝について説明した。