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29. *Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction.*

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1. Introduction

The attenuation of seismic waves has been studied by many authors[1], and the present writer[2], too, has obtained some results in that sphere. Though the numerical precision of the information obtained is still questionable, it seems that the attenuation coefficient is almost inversely proportional to the period of the wave, and not to the square of the period, but, at the same time, it is slightly large for the waves with short period and, perhaps also, for the waves with a period in certain range. And examining various points noticed in the course of those studies, it seems necessary to find out some other mechanism of the attenuation of waves, in addition to the generally accepted ones. In this paper, an idea is presented for the elucidation of the above-mentioned relationship of the attenuation coefficient to the period from the stability of the wave through a heterogeneous medium, a preliminary reasoning of which was once explained in a previous paper[3]. Some allied problem was treated by Lord Rayleigh[4] soon after Hill studied the differential equation that is called after his name. However, besides that the main point of the problem, or in other words, manner of applying the mathematical results to the physical interpretation, is

1) M. EWING and F. PRESS, "An Investigation of Mantle Rayleigh Waves." *B.S.S.A.*, **44** (1954), 127-147; "Mantle Rayleigh Waves from the Kamchatka Earthquake of November 4, 1952." *B.S.S.A.*, **44** (1954), 471-479.

Y. SATÔ, "Attenuation, Dispersion, and the Wave Guide of the G Wave." *B.S.S.A.*, **48** (1958), 231-251.

2) R. YOSHIYAMA, "Maximum Amplitude and Epicentral Distance." *Bull. Earthq. Res. Inst.*, **37** (1959), 389-404; "Propagation of Surface Waves and Internal Friction." *Bull. Earthq. Res. Inst.*, **38** (1960), 361-368.

3) R. YOSHIYAMA, "Waves through a heterogeneous medium." (in Japanese). *Zisin*, I-**14** (1941), 363-366.

4) Lord Rayleigh, "Maintenance of vibrations by forces of double frequency, and propagation of waves through a medium with a periodic structure." *Sci. Papers*. Vol. 3. 1-14.

different, the mathematical method is somewhat improved in the writer's studies by a reformation of the differential equation.

2. Recapitulation from previous studies with some extension

For simplicity's sake in the mathematical treatment, it is assumed that the property of the medium changes in one direction x, and a plain wave is propagated in that direction: propagation of P-wave or SH-wave of normal incidence applies to this case. The equation of motion of the wave is given by,

$$\rho\frac{\partial^2 u}{\partial T^2}=\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right), \tag{1}$$

where T is time-coordinate. E, elastic constant, stands for $\lambda+2\mu$ in the case of P-wave, for μ in the case of SH-wave, and is a function of x; whether density ρ may be a function of x or a constant, it does not affect the main part of the results of the discussion in this paper. u is a component of displacement: x-component in the case of P-wave, and a transverse component perpendicular to x in the case of SH-wave. The velocity of propagation of the wave is

$$c(x)=\sqrt{\frac{E}{\rho}}\ . \tag{2}$$

Following the method of mathematical treatment developed in a previous paper, the travel time, t, of the wave is used for space co-ordinate in place of x, where t is defined by,

$$t=\int\frac{dx}{c(x)}\ . \tag{3}$$

And putting $\sqrt{\rho c}\,u=\phi(t, T)$, we obtain, from (1), a differential equation for ϕ as follows:

$$\frac{\partial^2\phi}{\partial T^2}=\frac{\partial^2\phi}{\partial t^2}-\alpha^2\phi\,, \tag{4}$$

where

$$\alpha^2=c^2\left\{\frac{1}{\sqrt{E}}\frac{d^2\sqrt{E}}{dx^2}-\frac{1}{\sqrt{c}}\frac{d^2\sqrt{c}}{dx^2}\right\},$$

or, putting $K=\rho c$,

$$=c^2\left\{\frac{\sqrt{\rho}}{K}\frac{d^2}{dx^2}\frac{K}{\sqrt{\rho}}-\sqrt{\frac{\rho}{K}}\frac{d^2}{dx^2}\sqrt{\frac{K}{\rho}}\right\}. \tag{5}$$

In a heterogeneous medium, α^2 is originally a function of x, and is transformed into a function of t in virtue of (3), and α^2 vanishes when the medium is homogeneous. If we neglect the change of α^2 from place to place, the solution of the equation (4) is obtained,

$$\phi(t, T)=\exp i(pT\pm\sqrt{p^2-\alpha^2}t) . \tag{6}$$

So that,

$$u=\frac{1}{\sqrt{\rho c}}\exp i(pT\pm\sqrt{p^2-\alpha^2}t) , \tag{7}$$

$K=\rho c$ is called in acoustics specific acoustic impedance, and, when it is constant, α^2 vanishes, even if the medium is heterogeneous; the results from such an assumption, however, will have little importance in the application to the study of seismic bodily waves. On the other hand, in some cases, for example, when $\rho=$const. and $c=a+bx$, α^2 turns out a constant, and (7) is a rigorous solution of the equation of motion (1); since these assumptions are conventionally used and the solution is rigorous, (7) will be useful in some future studies. α^2 effects a non-dissipative attenuation of the wave, arising in nature from selective reflection: reflection of this kind was studied by Matuzawa[5].

One of the important problems related to the study of seismic waves occurs when the medium has a periodic, or, in general, fluctuating structure in a certain direction; though it is assumed in this paper to set about the mathematical study of the problem that $\rho=$const. and $c(x)=a(1+b\cos\gamma x)$, the simplest of oscillatory functions, we may be able to deduce some fundamental properties of the wave through the medium in consideration.

3. Reasoning to apparent internal friction

Putting $c(x)=a(1+b\cos\gamma x)$ in (2),

$$t=\frac{2}{a\gamma\sqrt{1-b^2}}\tan^{-1}\frac{\sqrt{1-b^2}}{1+b}\tan\frac{\gamma x}{2} ,$$

$$x=\frac{2}{\gamma}\tan^{-1}\frac{1+b}{\sqrt{1-b^2}}\tan\frac{\gamma at\sqrt{1-b^2}}{2} . \tag{8}$$

And, if we assume ρ constant, we have from (5),

5) T. MATUZAWA; "Reflexion und Refraktion der seismischen Wellen durch eine kontinuierlich verändernde Schicht," *Bull. Earthq. Res. Inst.*, **33** (1955), 543–548.

$$\alpha^2 = \frac{\sqrt{c}}{3}\frac{d^2}{dx^2}c^{\frac{3}{2}}. \tag{9}$$

Assuming $c(x)=a(1+b\cos\gamma x)$, applying $x-t$ relation given by (8), and denoting $\gamma at\sqrt{1-b^2}/2$ by z, we obtain from (9),

$$c(x)=a(1+b\cos\gamma x)=\frac{a(1-b^2)}{1-b\cos 2z},$$

$$\alpha^2=\frac{\gamma^2 a^2(1-b^2)}{4(1-b\cos 2z)^2}b\left\{\frac{5}{2}b-2\cos 2z-\frac{b}{2}\cos 4z\right\}. \tag{10}$$

So that, putting $\phi(t, T)=\phi_1(t)\exp(ipT)$ in (4), p being a circular frequency of the wave, a differential equation for $\phi_1(t)$ is obtatined as follows:

$$\frac{d^2\phi_1}{dz^2}+\left\{q^2+\frac{b(-5b+4\cos 2z+b\cos 4z)}{2(1-b\cos 2z)^2}\right\}\phi_1=0, \tag{11}$$

$$q=\frac{2p}{a\gamma\sqrt{1-b^2}}; \qquad z=\frac{\gamma at\sqrt{1-b^2}}{2},$$

which is clearly on the lines of Hill's equation, and, assuming that b is small, (11) is transformed approximately to Mathieu's equation. Since,

$$\frac{b(-5b+4\cos 2z+b\cos 4z)}{2(1-b\cos 2z)^2}=b\left\{-\frac{b}{2}+2\cos 2z+\frac{5}{2}b\cos 4z+O(b^2)\right\},$$

it is approximately

$$\frac{d^2\phi_1}{dz^2}+\left\{q^2+2b\cos 2z\right\}\phi_1=0. \tag{12}$$

Then we have a solution of $\phi_1(z)$, when z is large, in the form,

$$\phi_1(z)=\psi(z)\exp(-\mu z), \tag{13}$$

where $\psi(z)$ is a periodic function of z. Therefore, the attenuation of ϕ_1, which consequently applies to that of the wave is determined by μ in (13). According to the theory of Mathieu's equation, when b is fixed, μ is a function of q^2, and the largest real value of μ, for a range of $q^2 \geq 0$, is

$$\mu_m \doteqdot \frac{1}{2}|b|, \qquad \text{when } q^2 \doteqdot 1-\frac{1}{8}b^2. \tag{14}$$

Under these conditions, the attenuation of the wave is given by

$\exp(-|b|z/2)$. So that the apparent or mean attenuation constant of the wave is $|b|z/2x$, which is equal to $|b|\gamma at\sqrt{1-b^2}/4x$, or approximately to $|b\gamma|/4$.

Suppose a wave of a certain period, then p is specified. And γ-value which satisfies (14) for the specified p is approximately given by,

$$q \simeq 1\,; \qquad \gamma = 2p/a\,. \tag{15}$$

If the wave length of the periodic structure of the medium is L, $\gamma=2\pi/L$. So that, from (15), the L-value that effects the largest attenuation to the wave whose period is $\tau=2\pi/p$ is given by,

$$L = a\tau/2\,. \tag{16}$$

It is a debatable point whether we have the L-value represented by (16) in the actual structure of the earth for the wave of an assigned period τ. If we assume that there is always, that means for any period of the wave, or, at least, for a certain range of period of the wave, such L-value that satisfies (16) in the structure, the attenuation constant of the wave of the period of that range is $|b\gamma|/4$, or, putting $\gamma=2p/a$, $|b|\pi/a\tau$. Consequently, apparent, or say, equivalent, internal friction on account of heterogeneity, $1/Q'$ is

$$1/Q' = |b|\,. \tag{17}$$

$1/Q'$ diminishes the wave amplitude in the course of its propagation, and the attenuation effect is clearly additive to the genuine internal friction, if any. The mechanism of the attenuation caused by $1/Q'$ is not, however, dissipative; it means that some part of the energy of the wave is transformed into harmonic waves. Those circumstances will be understood, also, from the results obtained in a previous paper[6] which proves that there is a generation of a characteristic stationary wave and diminution of the amplitude of the first shock. In this case, the harmonic waves generated are given by each term in the infinite harmonic series $\psi(z)$ of (13). Those harmonic waves will be studied in the near future, and until the study is finalised some doubtful points will remain unsolved, from a theoretical point of view, in the acceptance of formula (17), because the solution of Mathieu's equation is too complicated to be easily adapted to the mathematical description of the propagation of waves. In spite of some uncertainties, we can assume that the order of magni-

6) R. YOSHIYAMA, "Elastic Waves from a Point in an Isotropic Heterogeneous Sphere. Part 2," *Bull. Earthq. Res. Inst.*, **18** (1940), 41-56.

tude of $1/Q'$ can not be much different at any rate from that set down in (17), and the effect of the heterogeneity of the medium in the apparent attenuation of the waves will not be negligible, since the internal friction computed from the observations of seismic waves is also a small quantity and does not exceed 9.10^{-3}. Of course, the above stated results are worked out to apply to special bodily waves, but will be applicable to some extent even to surface waves; at least, if it is said generally, the attenuation effect will not be less than for bodily waves. And the magnitude of b, that is a measure of the irregularities of structure in the horizontal direction, as large as 10^{-2} will be reasonably presumable in any district, if it is expected to cover some hundred kilometres, and if it is expected, also, in some range to have an arbitrary wave length, to effect the largest attenuation to the wave with a certain period. On the other hand, the present writer expects the effect of the heterogeneity of the earth's crust of the order of magnitude $3-5\times10^{-3}$, at the largest, in $1/Q'$, to fill up some part of the difference between the two internal frictions, one being obtained from the computation of a surface wave with a long period which ranges from fifty seconds to some hundreds of seconds, and the other from the study of "Maximum amplitude and the Epicentral distance relation" of large earthquakes in Japan, in which the period of the seismic waves ranges from a few seconds to several seconds.

For a given small b-value, real μ decreases very rapidly either for $q^2>1$ or for $q^2<1$, and, for a certain range of q^2, μ is a pure imaginary. So that, the attenuation of the wave, whose period is τ, by the periodic structure of a wave length other than that which exerts the largest effect and is obtained above in (16), is negligible.

4. Theoretical calculation and some remarks

As already mentioned, it seems to the writer that the solution of Mathieu's equation, hitherto obtained, is not satisfactory for a straightforward study of the propagation of waves; the solution implies inseparable two sets of waves, primary waves and reflected waves. Though the co-existence of the two sets of waves will be an important property, utterly natural in itself, of waves through the medium with a periodic structure, it gives rise to some theoretical difficulties, when we try to interpret physically the solution for the elucidation of a progressive wave. To remove those difficulties completely, analytical study of the solution, or reformation of the infinite series of the solution will be necessary. However, it is not a problem to be solved easily. And, in this paper, to remove a part of those difficulties and to arrive at formula (17), the following example is studied.

A three layered medium is assumed as shown in Fig. 1; two of the three, I and III, are homogeneous, and the intermediate medium, II, is the one that is considered with a special interest in this paper, having a periodic structure. A sinusoidal wave with a period $\tau=2\pi/p$ is propagated from medium I to medium III through medium II; $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ are the displacement vectors in each medium caused by the wave propagated in x-direction, x-direction being assumed from medium I to medium III; $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are those from the wave in the opposite direction; clearly $\boldsymbol{B}_3$ is zero. Then it follows;

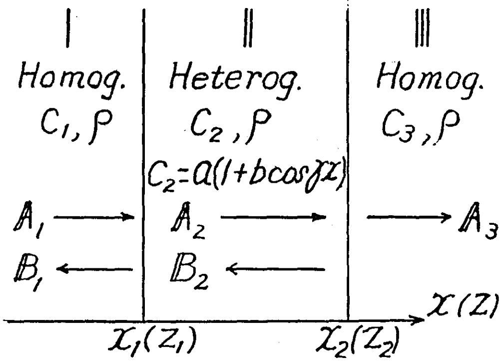

Fig. 1. Schematic illustration of the assumption and the notations.

$$\text{I.}\quad \boldsymbol{A}_1=\frac{A_1}{\sqrt{\rho_1 c_1}}\exp(ipT)\exp\{-ik_1(x-x_1)\}\,, \tag{18}$$

$$\boldsymbol{B}_1=\frac{B_1}{\sqrt{\rho_1 c_1}}\exp(ipT)\exp\{ik_1(x-x_1)\}\,,$$

$$\text{II.}\quad \boldsymbol{A}_2+\boldsymbol{B}_2=\frac{1}{\sqrt{\rho_2 c_2}}\exp(ipT)\{A_2\psi(-z)\exp(-\mu z)+B_2\psi(z)\exp(\mu z)\}\,, \tag{19}$$

$$\text{III.}\quad \boldsymbol{A}_3=\frac{A_3}{\sqrt{\rho_3 c_3}}\exp(ipT)\exp\{-ik_3(x-x_2)\}\,, \tag{20}$$

c_1 and c_3 are constant respectively, while $c_2=a(1+b\cos\gamma x)$. There will be no necessity to explain the notations in (18) and (20). The formula in the bracket of (19) is the complete solution of Mathieu's equation (12); in (13), one of the two terms, which diverges at $x=\infty$, was discarded and the study of theoretical basis of such discarding is one of the subjects in this section.

When $q^2\fallingdotseq 1$ and b is small, the function in the complete solution of Mathieu's equation is given by,

$$\psi(z)\fallingdotseq\sin(z-\sigma)+s_3\sin(3z-\sigma) \tag{21}$$

where $\sigma=\pi/4$, $s_3=b/8$ and $\mu=b/2$, regardless of the sign of b. $\psi(-z)$ is obtained by putting $\sigma=-\pi/4$. Boundary conditions at the two interfaces, that imply continuity of displacement and stress, assign four simultaneous equations to determine B_1, A_2, B_2 and A_3 in A_1. Those equations are obtained in the following, being only assumed that density ρ is constant throughout the whole medium:

$$\begin{array}{ccccc}
B_1 & +A_2 & +B_2 & +A_3 & =A_1 \\
-1\,, & \sqrt{c_{12}}\psi(-z_1)\exp(-\mu z_1)\,, & \sqrt{c_{12}}\psi(z_1)\exp(\mu z_1)\,, & 0\,, & 1\,. \\
1\,, & ic_{21}^{3/2}\gamma_{12}F(-z_1)\exp(-\mu z_1)\,, & ic_{21}^{3/2}\gamma_{12}F(z_1)\exp(\mu z_1)\,, & 0\,, & 1\,. \\
0\,, & \psi(-z_2)\exp(-\mu z_2)\,, & \psi(z_2)\exp(\mu z_2)\,, & -\sqrt{c_{23}}\,, & 0\,. \\
0\,, & \gamma_{32}F(-z_2)\exp(-\mu z_2)\,, & \gamma_{32}F(z_2)\exp(\mu z_2)\,, & ic_{32}^{3/2}\,, & 0\,.
\end{array} \tag{22}$$

where x_1, or z_1, and x_2, or z_2, are the x- or z-co-ordinates of the two interfaces as shown in Fig. 1; $c_{12}=1/c_{21}=c_1/c_2$ at $x_1(z_1)$, $c_{23}=1/c_{32}=c_2/c_3$ at $x_2(z_2)$; $\gamma_{12}=\gamma a\sqrt{1-b^2}/2k_1c_2(x_1)$, $\gamma_{32}=\gamma a\sqrt{1-b^2}/2k_3c_2(x_2)$;

$$F(z)=\left\{\frac{2c_2}{\gamma a\sqrt{1-b^2}}\cdot\sqrt{c_2}\frac{d}{dx}\frac{1}{\sqrt{c_2}}+\mu\right\}\psi(z)+\frac{d\psi(z)}{dz}$$

$$F(-z)=\left\{\frac{2c_2}{\gamma a\sqrt{1-b^2}}\cdot\sqrt{c_2}\frac{d}{dx}\frac{1}{\sqrt{c_2}}-\mu\right\}\psi(-z)+\frac{d\psi(-z)}{dz}\,. \tag{23}$$

Considering the limited purposes of our study at present, we may assume for the sake of mathematical simplicity the following: 1) origins of x- and z-co-ordinates are at the interface of the I and the II medium, i.e., $x_1=z_1=0$, then we may put $z_2=z_0$; 2) $c_1=c_3$; 3) velocity of propagation and its gradient are continuous at the interfaces. After all, our assumption on the velocity distribution throughout the whole medium is such as shown in Fig. 2; then the co-ordinates of the interface between II and III is $x_2=2n\pi/\gamma$, or $z_2(z_0)=n\pi$. From these assumptions, it follows, putting $z_1=0$, $z_2(z_0)=n\pi$:

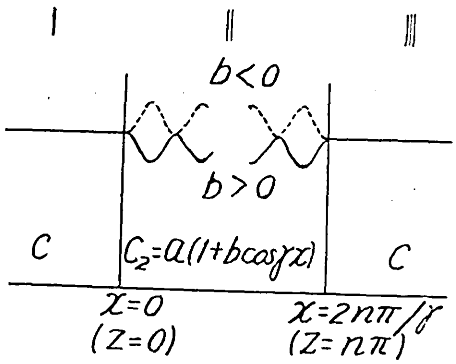

Fig. 2. Velocity distribution, assumed.

$$\begin{aligned}
&\psi(z_1)=-(1+s_3)/\sqrt{2}=-\psi(-z_1)\ ,\\
&\psi(z_2)=(-1)^{n+1}(1+s_3)/\sqrt{2}=-\psi(-z_2)\ ,\\
&F(z_1)=(1-s_3)/\sqrt{2}=F(-z_1)\ ,\\
&F(z_2)=(-1)^n(1-s_3)/\sqrt{2}=F(-z_2)\ , \qquad s_3=b/8\\
&\quad c_{12}=c_{21}=c_{23}=c_{32}=1\ ,\\
&\quad \gamma_{12}=\gamma_{32}=\gamma a\sqrt{1-b^2}/2p\ .
\end{aligned} \tag{24}$$

For the wave, to which this structure, assigned by a, b and γ, effects the largest attenuation, $\gamma a\sqrt{1-b^2}/2p \doteqdot 1$. Substituting (24) into (22) and solving the equations, we obtain A_3, A_2, B_2 and B_1:

$$\frac{A_3}{A_1}=\frac{(-1)^n}{\cosh \mu z_0-2is_3 \sinh \mu z_0} \tag{25}$$

$$\frac{A_2}{A_1}=\frac{(1-s_3)-i(1+s_3)}{\cosh \mu z_0-2is_3 \sinh \mu z_0}\cdot\frac{\exp(\mu z_0)}{\sqrt{2}} \tag{26}$$

$$\frac{B_2}{A_1}=\frac{-(1-s_3)-i(1+s_3)}{\cosh \mu z_0-2is_3 \sinh \mu z_0}\cdot\frac{\exp(-\mu z_0)}{\sqrt{2}} \tag{27}$$

$$\frac{B_1}{A_1}=\frac{-i \sinh \mu z_0}{\cosh \mu z_0-2is_3 \sinh \mu z_0}\ , \tag{28}$$

$$(\mu=b/2,\ \ s_3=b/8)$$

z_0 is proportional to the thickness of the medium II and, accordingly, to the distance over which the wave is propagated through the medium of a periodic structure. From (25), when $z_0=0$, $A_3/A_1=1$; and when z_0 is large,

$$\frac{A_3}{A_1}=\frac{2\exp(-|\mu|z_0)}{1-2is_3}\ . \tag{29}$$

Therefore, at large distances, $|A_3|$ decreases as $\exp(-|\mu|z_0)$; this result may be interpreted to support our reasoning based on (13). At the same time, however, we must notice, that if the amplitude at the origin is estimated by (13) from observations at large distances, it will be over-estimated almost twofold, chiefly because of the difference between $\exp(-|\mu|z_0)$ and $1/\cosh \mu z_0$; the non-dissipative attenuation concerned contrasts in this point with the ordinary dissipative one, in which we have two linearly independent particular solutions with $\exp(-|\mu|z)$ and none with $\exp(|\mu|z)$.

Concerning $\boldsymbol{A}_2+\boldsymbol{B}_2$, according as $b>0$ or $b<0$, there is a slight change of the notations in the following description, which clearly never affects the main point of the discussion. We may assume, for the present, $b>0$. The wave motion in medium II, when z_0-z, which is always positive in medium II, is large, contribution to $\boldsymbol{A}_2+\boldsymbol{B}_2$ from B_2 vanishes, and the wave motion is expressed by A_2 only. So that, when z_0 is large, while z is fixed,

$$\boldsymbol{A}_2+\boldsymbol{B}_2=A_1\frac{\exp(ipT)}{\sqrt{\rho_2c_2}}\cdot\frac{\sqrt{2}\{(1-s_3)-i(1+s_3)\}}{1-2is_3}\cdot\psi(-z)\exp(-\mu z), \tag{30}$$

$\boldsymbol{A}_2+\boldsymbol{B}_2$ in (30) expresses clearly a stationary wave, whose amplitude decreases as $\exp(-\mu z)$; it was already pointed out by Rayleigh that the wave propagated through the medium of a periodic structure is ultimately totally reflected back to the origin, and here obtained result also indicates the same conclusion. The behaviour of the primary wave apart from the reflected wave must be studied by the solution where z_0-z is not large, the intense effect of the reflected wave being avoided there. And, since $z_0>z_0-z>0$ in the medium, exponential factor of B_2 is transformed,

$$\exp(-\mu z_0)\cdot\exp(\mu z)=(1-\delta)\exp\mu(z_0-z)\,, \tag{31}$$

where $0<\delta<1$; $\delta=0$ when $\mu(z_0-z)=0$, $\delta=1$ when $\mu(z_0-z)=\infty$; and the problem to be studied is the case when $\delta\ll 1$. So that,

$$\boldsymbol{A}_2+\boldsymbol{B}_2=A_1\frac{\exp(ipT)}{\sqrt{\rho_2c_2}}\cdot\frac{\exp\mu(z_0-z)}{\cosh\mu z_0-2is_3\sinh\mu z_0}$$
$$\times\left[\exp(-iz)-s_3\exp(iz)+s_3\exp(-i3z)+\frac{\delta}{\sqrt{2}}\{(1-s_3)+i(1+s_3)\}\psi(z)\right]. \tag{32}$$

We can see in (32) the primary wave progressing in $+x$ direction with a small quantity of its reflected wave, the 3rd harmonic wave in the same direction with the primary wave and stationary wave whose amplitude is proportional to δ; phase velocity of the 3rd harmonic wave is one third of the primary waves.

Concerning the reflected wave in medium I, as z_0 tends from zero to infinity, $|B_1/A_1|$ increases from zero to $1/\sqrt{1+4s_3^2}$, almost equal to unity, but that it is less than unity altogether implies that some energies are kept as a stationary wave in medium II. If we take simply $b=0$, $\sinh\mu z_0=0$ then $|B_1/A_1|=0$, but, if we take $z_0=1/b^2$ and $b\to 0$, then $|B_1/A_1|=1$. Here we have a special type of total reflection, the phase angle being shifted by $\pi/2$.

It is to be noted especially that we have harmonic waves neither in the refracted wave through medium III, nor in the reflected wave through medium I. Therefore, considering the complicated structure of the earth, it will be understood that those harmonic waves are important not so much in themselves as in their absorbing effect on the energy of the primary wave. The energy absorbing mechanism is clearly one of those by resonance effect of oscillators, the wave-mechanical elucidation of which being given by the formulas (30) and (32). Examination of the transient aspects of the wave motion from $\boldsymbol{A}_2+\boldsymbol{B}_2$ to $\boldsymbol{A}_3$ by (30) and (32), also, seems important at this stage of the studies, in which a fundamental problem of the apparent internal friction is taken up. However, for the study of the principal part of the seismic waves, the formula of A_3 (25) will be more seismologically significant than (30) or (32).

From the results above obtained, we are lead to the conclusion, for the present, that the effect of the periodic structure on the amplitude A of a progressive wave is represented by $A=1/\cosh \mu z$ in stead of the simple exponential function which is used in (13). Therefore, apparent internal friction $1/Q'$ caused by the periodic structure should be estimated somewhat smaller than $|b|$ that is obtained by (17).

Approximate attenuation formula of the seismic waves with the period $\tau=4\pi/a\gamma$ through a medium with a periodic structure whose constant of internal friction is $1/Q$ will be represented by the expression,

$$A=\frac{1}{\cosh(\pi bx/a\tau)}\cdot\exp(-\pi x/Qa\tau)\,. \tag{33}$$

Estimation of b, or, generally b_n in the earth's crust when we put $c(x)=a(1+\sum_n b_n\cos\gamma_n x)$ is a problem to be studied from observation in future. If $|b_n|$ is equal to a constant $|b|$ for any γ_n-value in a certain range such that $\gamma_1<\gamma_n<\gamma_2$, then the apparent internal friction of the medium for the seismic waves with a period τ, such that $4\pi/a\gamma_1>\tau>4\pi/a\gamma_2$ is, at a large distance, approximately $|b|+\dfrac{1}{Q}$.

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29. 不均質な媒質内を伝る弾性波の安定性について

—見掛けの内部摩擦係数—

地震研究所　吉　山　良　一

弾性波の速度を $c(x)=a(1+b\cos\gamma x)$ として Mathieu の微分方程式によつてかかる媒質内を伝播する弾性波の安定性をしらべた．考えとしては古い問題であるが波動伝播の理論とむすびつけて具体的に取扱つた例はない様である．地震波の観測に現われるいわゆる内部摩擦の見掛値との関係をしらべると b の値がほとんど直接摩擦係数 $1/Q$ に相当することになるのでその影響は見逃し得ないものがある．媒質の internal friction を $1/Q$ とすればある特定の周期 $\tau=4\pi/a\gamma$ の波，波長で比較すれば媒質の構造上の波長 $2\pi/\gamma$ の 2 倍の波長の波に対する振巾減衰の式は結局

$$A=\frac{1}{\cosh(\pi bx/a\tau)}\cdot\exp(-\pi x/Qa\tau)$$

で表わされる．したがつてその様な構造がつづく限り遠いところでの傾向を見ればそのような波に対する媒質の見掛けの internal friction は $|b|+\dfrac{1}{Q}$ となる．今後調査すべきことは地殻内の弾性波の速度は場所によつてある平均値を上下しているであろうが，それを $c(x)=a(1+\sum_n b_n\cos\gamma_n x)$ の形においた時 $|b_n|$ の大いさが種々の γ_n に対してどのような値を取るかである．