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23. *Coupling Effect of Shear Vibrations of the Structure with Elastic Foundations, and the Maximum Response of Rocking Motion.*

By Yasuo SATÔ and Rinzo YAMAGUCHI,

Earthquake Research Institute.

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1. Introduction

In our previous paper[1] we studied the coupling oscillation of a building and an elastic foundation, the problem being classified into three kinds, namely the vertical, rocking and horizontal motion. In the case of the third kind of problem, we assumed a one mass system for simplicity in calculation. In the present paper, however, we assumed a uniform circular cylinder and studied the mechanical characteristics of the shear vibration of this system, the basic part of the calculation being adopted from the study of I. Toriumi[2] as before. Consequently, the following assumptions of his also hold in our study.

1) The stress is uniform within the circular base of the structure and vanishes outside this circle.

2) Purely horizontal seismic waves are incident from vertical downward.

Parameters are assumed to vary more extensively than before so that the obtained result may also be applicable to the structures such as piers. As an appendix formulas are given in the last chapter connecting the maximum amplitude and other physical quantities.

2. Notations

$A \quad = 2\mu\pi r_0/(f_{1H}+if_{2H})$.

A_H Amplitudes of the incident waves. (Horizontal)

1) Y. SATÔ and R. YAMAGUCHI, "Vibration of Buildings on the Ground," *Zisin* [ii], **9** (1957), 156. (in Japanese) Y. SATÔ and R. YAMAGUCHI, "Vibration of a Building upon the Elastic Foundation," *Bull. Earthq. Res. Inst.*, **35** (1957), 545.

2) I. TORIUMI, "Vibrations in Foundations of Machines," *Technology Reports of the Osaka Univ.*, **5** (1955), 103. I. TORIUMI "Vibrations in Foundation of Machine on the Ground," *Zisin* [ii], **7** (1955), 216.

a_0 $=pr_0\sqrt{\rho/\mu}=pr_0/V_s=2\pi r_0/$(Wave length of S-waves).
a_{0MAX} Values of a_0 that gives the maximum magnification.
E_0 $=\pi r_0^2$.
f_{1H}, f_{2H} f_1 and f_2 in Toriumi's paper. (Horizontal motion)
l_0 Height of a structure from its base to the center of gravity.
p Circular frequency of incident waves.
Q_H Horizontal force transmitted from a structure to the ground.
r_0 Radius of a structure.
U_1 Displacement at the free surface. (Horizontal motion)
U_D Additional displacement of the center of the base caused by the existence of a structure. (Horizontal motion)
U_R Horizontal displacement of the center of gravity. (Rocking motion)
V_s, V_{s0} Velocity of S-waves of the foundation and structure respectively.
$|X|$ $=|U_R/U_1|$.
$|Y'|$ $=|l_0\Gamma_R/U_1|$.
y_x Absolute displcement of the structure at the height x.
Γ_R Angle of inclination of a structure. (Rocking motion)
μ, μ_0 Rigidity of the foundation and structure respectively.
ρ, ρ_0 Density of the foundation and structure respectively.
ϑ_H Phase difference between the incident waves and the oscillation of the structure. (Horizontal motion)
$\mathfrak{V}_{HB}$ Magnification of the displacement of the base.
$\mathfrak{V}_{HT}$ Magnification of the displacement of the top.

3. Shear vibration of a continuous body

We assume a uniform circular cylinder and solve the problem of its forced oscillation excited by purely horizontal waves incident from the vertical downward with the amplitude A_H and the circular frequency p. In Fig. 1 the oscillation at the free surface is, if there is no structure upon the surface

$$U_1=2A_H\exp(ipt)\ . \tag{3.1}$$

Put additional displacement caused by the existence of a structure as

$$U_D=U\exp(ipt)\ , \tag{3.2}$$

and the displacement of the structure at the height x relative to the base as

$$y = Y \exp(ipt) , \tag{3.3}$$

then the absolute displacement of the structure at x is

$$y_x = U_1 + U_D + y . \tag{3.4}$$

The equation of motion for the structure is

$$\rho_0 \frac{\partial^2 y_x}{\partial t^2} = \mu_0 \frac{\partial^2 y}{\partial x^2} . \tag{3.5}$$

Introducing the above relation we have

$$\frac{d^2 Y}{dx^2} + k^2(2A_H + U + Y) = 0 , \tag{3.6}$$

where

$$k^2 = p^2 \rho_0 / \mu_0 = p^2 / V_{s0}^2 .$$

The solution of this equation is given in the form

$$2A_H + U + Y = \alpha \cos kx + \beta \sin kx , \tag{3.7}$$

where the constants α and β are determined by the boundary conditions.

At the base of the structure $x=0$, we have

$$Y = 0 \text{ and } E_0 \mu_0 \cdot dY/dx = Q_H , \tag{3.8}$$

where Q_H is the force related to the additional displacement U_D caused by the existence of a structure. From the theory of I. Toriumi we have

$$U_D = \frac{1}{A} Q_H \exp(ipt) , \tag{3.9}$$

or

$$AU = Q_H .$$

Using the equations in (3.8) the relations

$$2A_H + U = \alpha, \quad AU = E_0 \mu_0 k \beta , \tag{3.10}$$

are obtained.

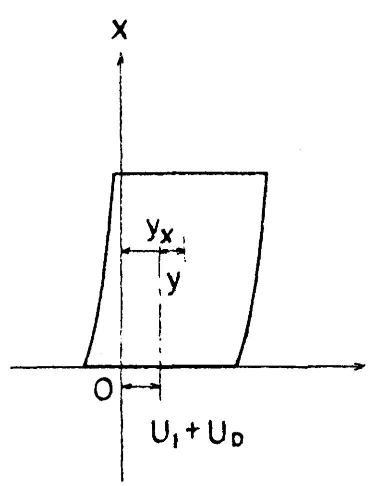

Fig. 1.

At the top of the structure $x=2l_0$ we have

$$E_0\mu_0\cdot dY/dx=0\ . \tag{3.11}$$

Consequently

$$-\alpha\sin(2kl_0)+\beta\cos(2kl_0)=0\ . \tag{3.12}$$

From (3.10) and (3.12) α, β and U are obtained as follows:

$$\begin{aligned}\alpha&=2A_H\cdot A\cos(2kl_0)/\Delta\ ,\\ \beta&=2A_H\cdot A\sin(2kl_0)/\Delta\ ,\\ U&=2A_H\cdot E_0\mu_0k\sin(2kl_0)/\Delta\ ,\\ &\Delta=A\cos(2kl_0)-E_0\mu_0k\sin(2kl_0)\ .\end{aligned} \tag{3.13}$$

Introducing the above expressions into (3.7), we have

$$2A_H+U+Y=2A_H\cdot A\cos\{k(2l_0-x)\}/\Delta\ , \tag{3.14}$$

or

$$Y=2A_H\cdot A[-\cos(2kl_0)+\cos\{k(2l_0-x)\}]/\Delta\ . \tag{3.15}$$

From the last expression the displacement of a structure at any height can be obtained.

The first expression of (3.10) is the absolute displacement of the base of the structure, therefore the magnification of the displacement is given in the following form:

$$\begin{aligned}\mathfrak{B}_{HB}\exp(-i\vartheta_H)&=(2A_H+U)/(2A_H)\\ &=1\Big/\left\{1-\frac{E_0\mu_0k}{A}\tan(2kl_0)\right\}\\ &=1\Big/\left\{1-\frac{1}{2}\frac{\rho_0V_{s0}}{\rho V_s}a_0(f_{1H}+if_{2H})\tan\left(2\frac{V_s}{V_{s0}}\frac{l_0}{r_0}a_0\right)\right\}\ .\end{aligned} \tag{3.16}$$

With regard to the top of the structure a similar expression is obtained by introducing $x=2l_0$ into equation (3.14). Namely

$$\begin{aligned}\mathfrak{B}_{HT}\exp(-i\vartheta_H)&=(2A_H+U+Y)_{x=2l_0}/(2A_H)\\ &=1\Big/\cos(2kl_0)\left\{1-\frac{E_0\mu_0k}{A}\tan(2kl_0)\right\}\ .\end{aligned} \tag{3.17}$$

Combining the above two expressions, we have

$$\mathfrak{B}_{HT}=\mathfrak{B}_{HB}\Big/\cos\left(2\frac{V_s}{V_{s0}}\frac{l_0}{r_0}a_0\right). \tag{3.18}$$

3.1 Numerical computation of $\mathfrak{B}_{HB}$ and $\mathfrak{B}_{HT}$.

In this section the result of numerical computation is given assuming various values for the quantities in (3.16) (3.17) and (3.18).

At first we assume that $\rho_0/\rho=1/4$.

Fig. 2 gives the relation $\mathfrak{B}_{HB}$ versus $a_0=pr_0/V_s$, the parameter being the (l_0/r_0), and V_s and V_{s0} are assumed to be equal. A similar relation with regard to $\mathfrak{B}_{HT}$ is given in Fig. 3.

In Fig. 2 the zero points of curves correspond to the resonance of the structure. General tendencies of these curves are similar to those of our previous paper, but we can find second resonances in the present case.

Fig. 4 gives $\mathfrak{B}_{HB}$. In this figure, the parameter is the ratio (V_s/V_{s0}), and $(l_0/r_0)=1.0$ is assumed. Fig. 5 is a similar relation with regard to $\mathfrak{B}_{HT}$. Figs. 6 9 are the similar graphs, in which $(l_0/r_0)=1.5$ and 2.0.

In Figs. 10-13 the angle of phase difference (ϑ_H) is given corresponding to the above cases. As is seen in the equations (3.16) and (3.17) there is no difference in the phase angles for the base and the top of the structure.

If the vibrating structure is a building we may safely assume that $\rho_0/\rho=1/4$. However, for the case such as piers we must adopt a larger value of (ρ_0/ρ). The cases of that ρ_0/ρ being assumed to be 1.0 and 2.0 are shown in § 4.

4. Relation between the maximum amplitude and the period

Summalizing the results obtained above we can conclude the following characteristics which hold between the maximum amplitude and the period.

1) Maximum amplitude of the base of the structure can be approximately expressed as a function of a_0 only. We can find this nature in Fig. 14, but it is also proved analytically by the following process.

From (3,16)

$$\begin{aligned}\mathfrak{B}_{HB}(-i\vartheta_H)&=1\Big/\left\{1-\frac{1}{2}\frac{\rho_0}{\rho}\frac{V_{s0}}{V_s}a_0(f_{1H}+if_{2H})\tan\left(2\frac{V_0}{V_{s0}}\frac{l_0}{r_0}a_0\right)\right\}\\&\equiv1/\{1-c_0(f_{1H}+if_{2H})\}\\&=\exp(-i\vartheta_H)/\sqrt{(1-c_0f_{1H})^2+c_0^2f_{2H}^2}\end{aligned} \tag{4.1}$$

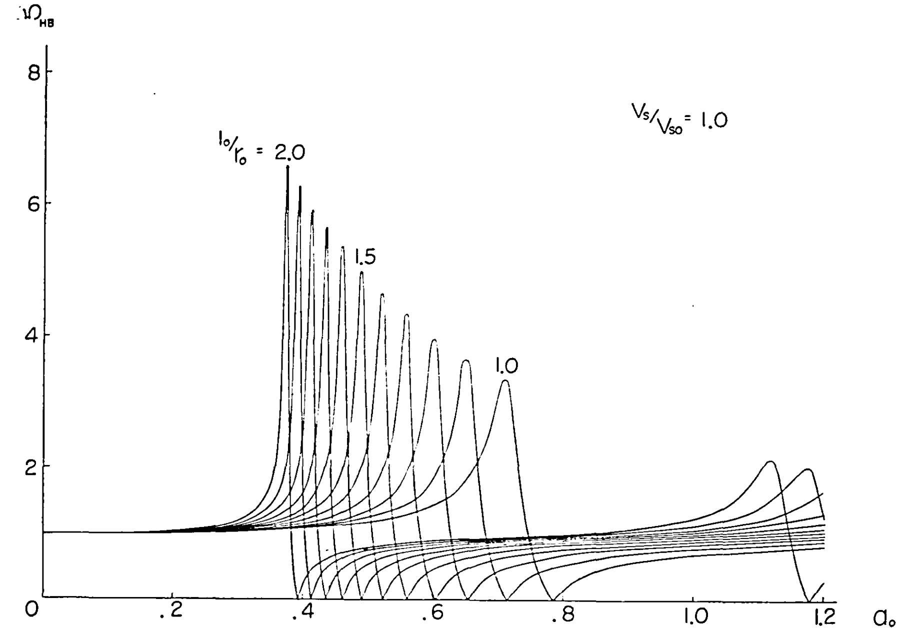

Fig. 2. Amplitude of the base of structure for the case $V_s = V_{s0}$.

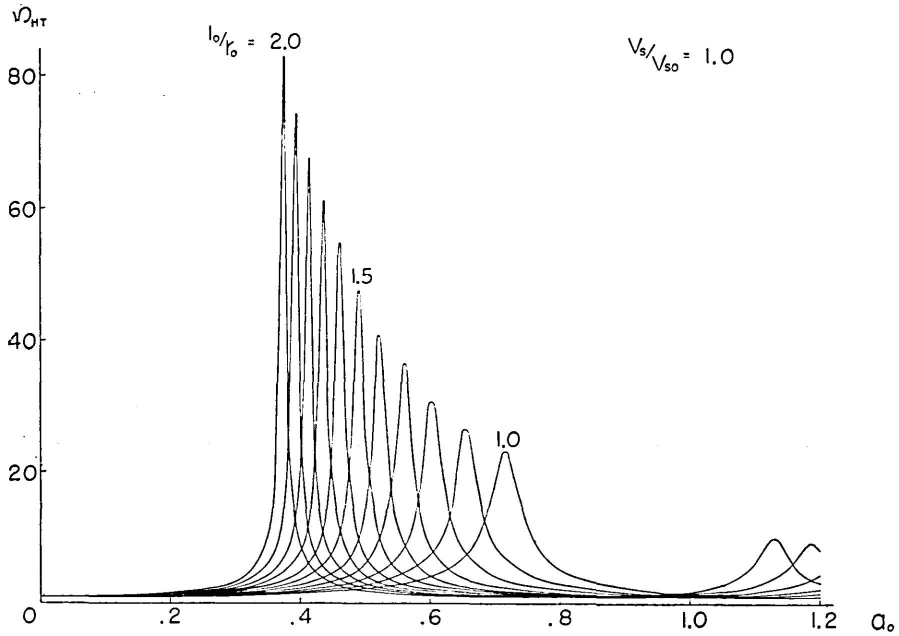

Fig. 3. Amplitude of the top of structure for the case $V_s = V_{s0}$.

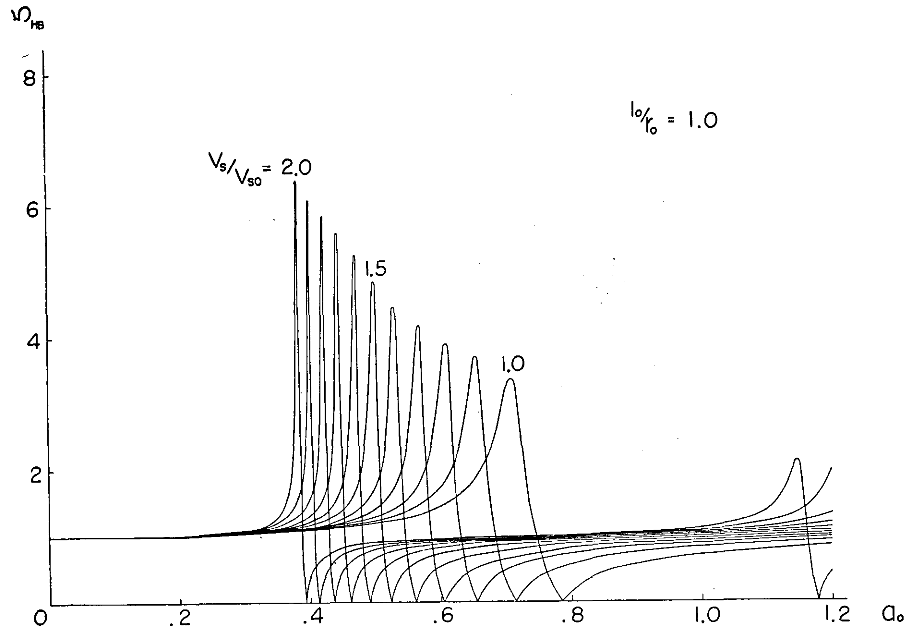

Fig. 4. Amplitude of the base of structure for the case $l_0/r_0=1.0$.

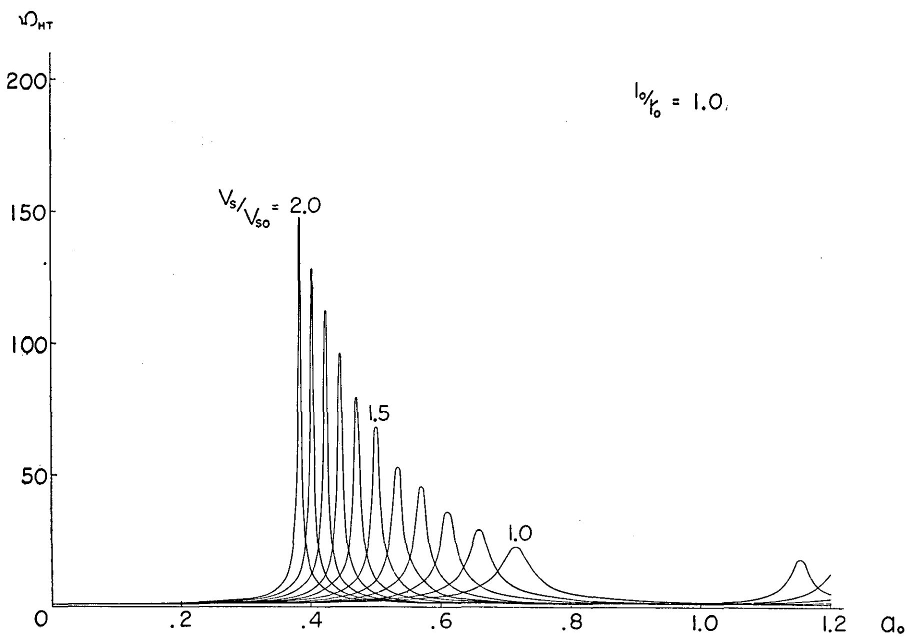

Fig. 5. Amplitude of the top of structure for the case $l_0/r_0=1.0$.

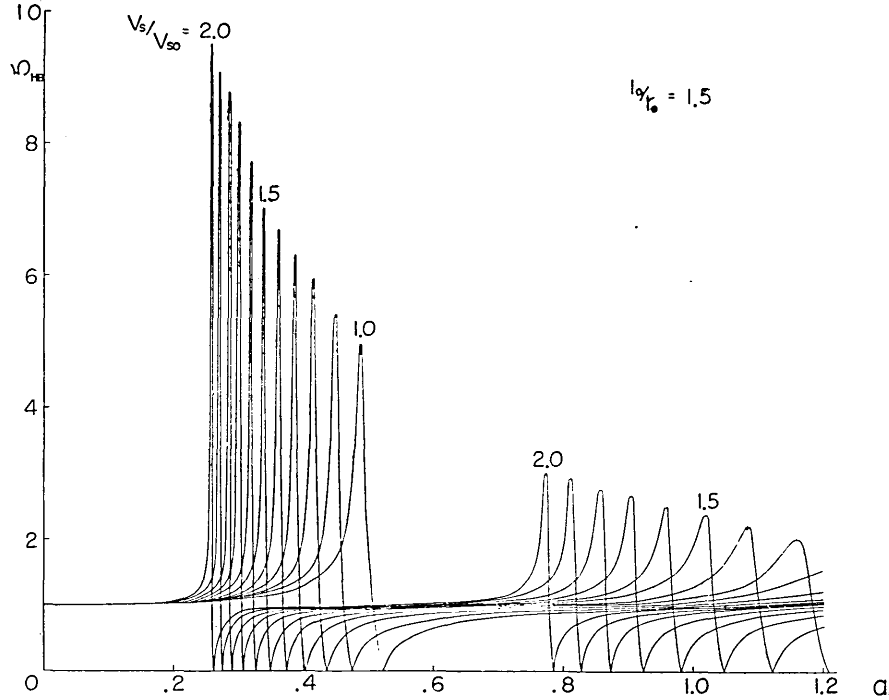

Fig. 6. Amplitude of the base of structure for the case $l_0/r_0=1.5$.

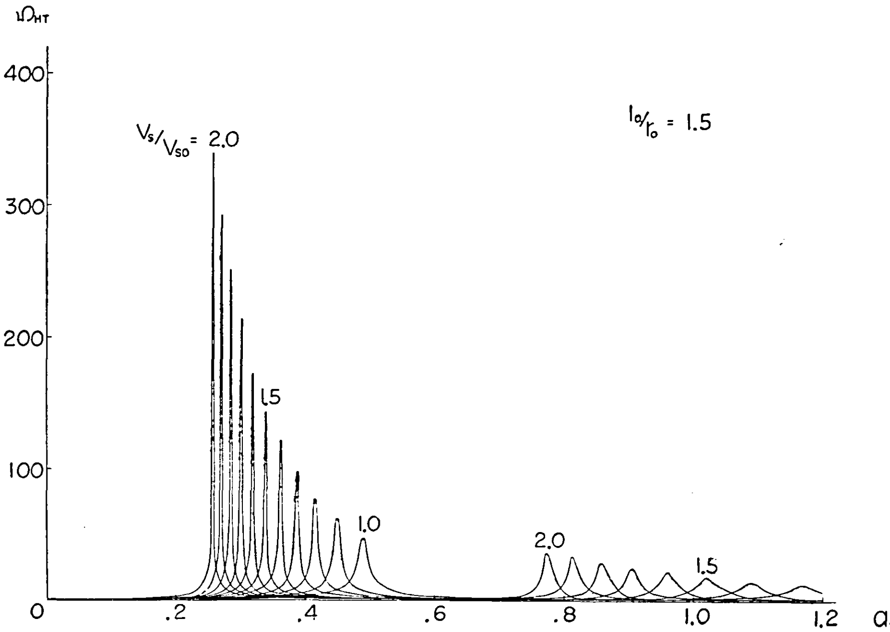

Fig. 7. Amplitude of the top of structure for the case $l_0/r_0=1.5$.

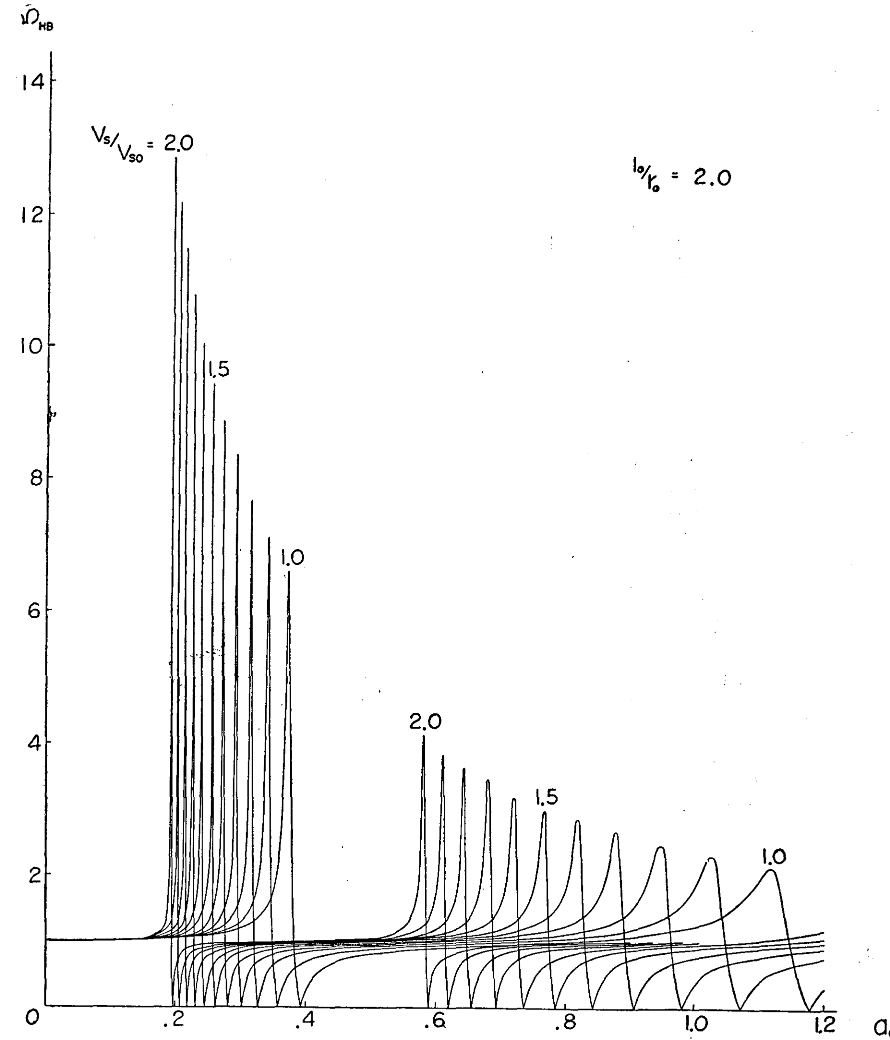

Fig. 8. Amplitude of the base of structure for the case $l_0/r_0=2.0$.

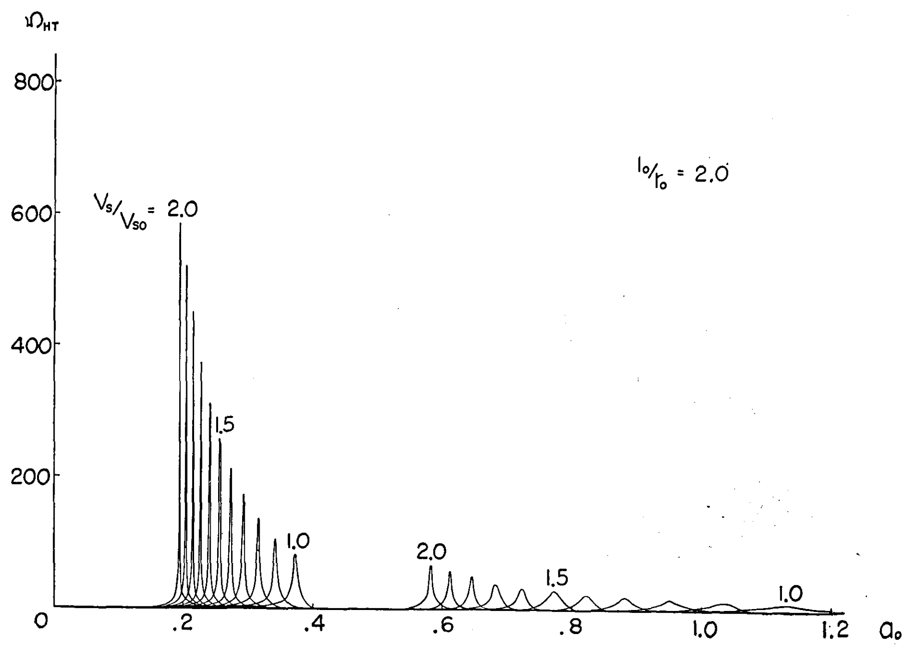

Fig. 9. Amplitude of the top of structure for the case $l_0/r_0=2.0$.

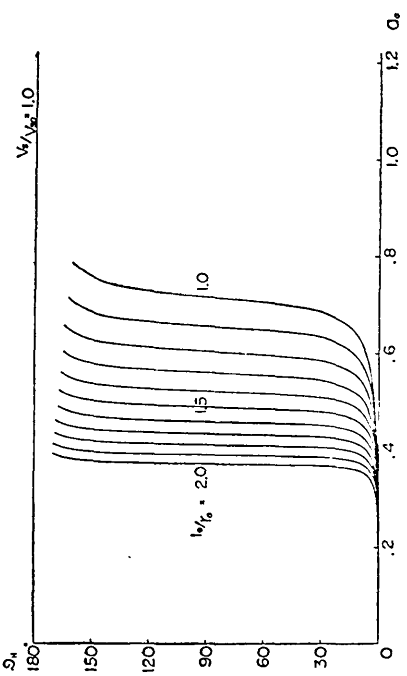

Fig. 10.

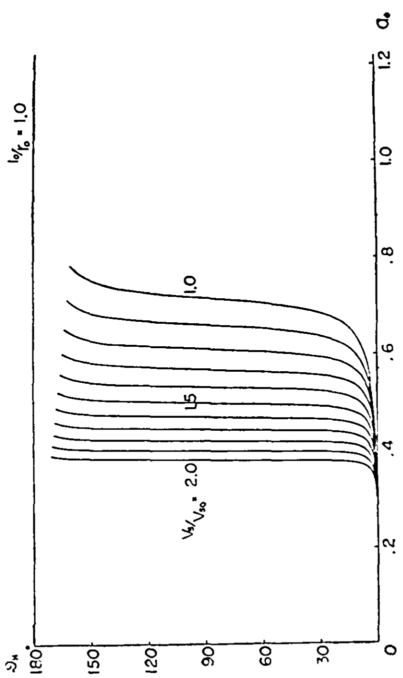

Fig. 11.

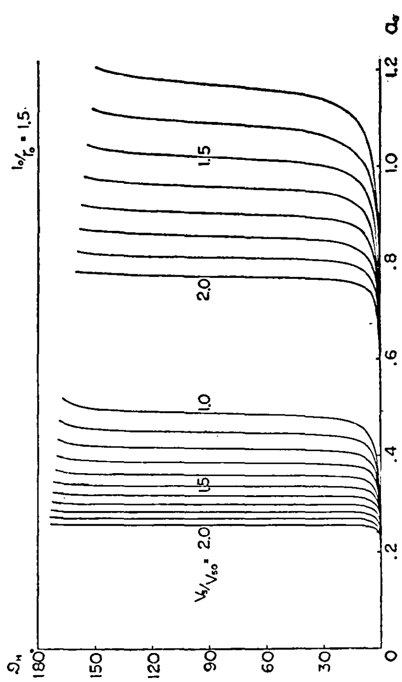

Fig. 12.

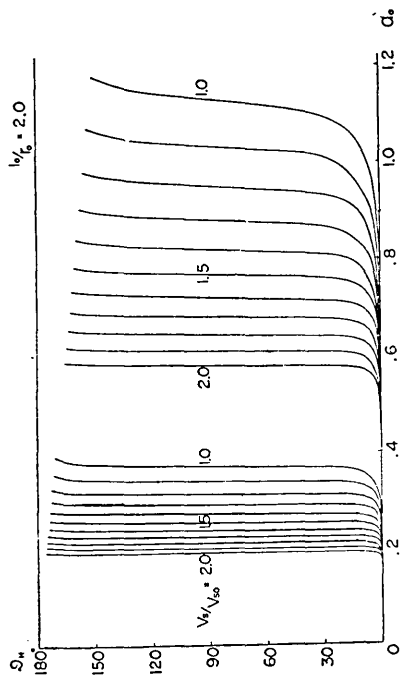

Fig. 13.

Phase difference between the incident waves and the oscillation of structure.

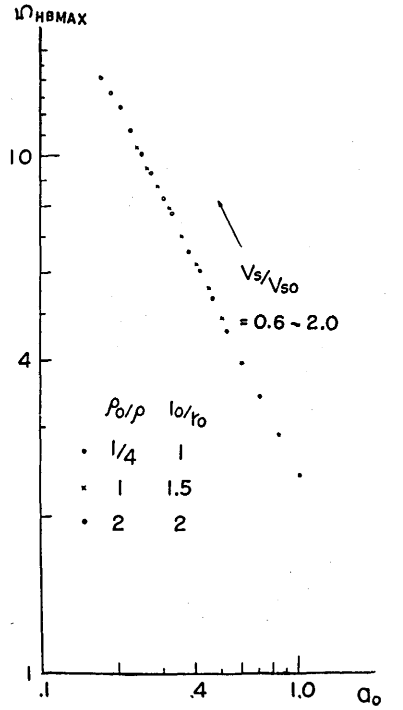

Fig. 14. Frequency which gives the maximum amplitude of the base of structure.

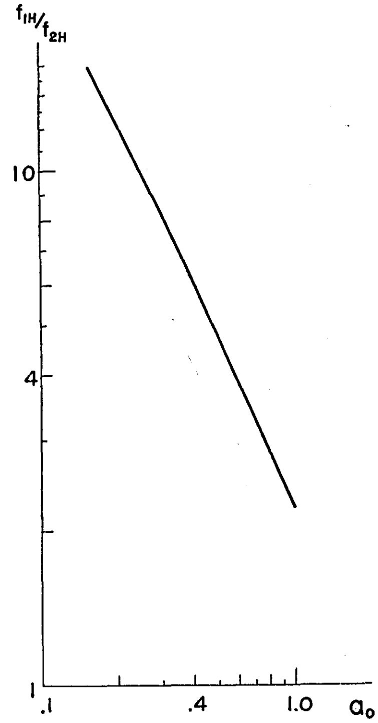

Fig. 15. Relation between a_0 and f_{1H}/f_{2H} which give the maximum amplitude of the base of structure.

So long as the quantity f_{2H} does not vary rapidly, the maximum of $\mathfrak{V}_{HB}$ is approximately given by

$$1-c_0 f_{1H}=0 \; . \tag{4.2}$$

Consequently

$$\mathfrak{V}_{HB,\mathrm{MAX}} \fallingdotseq 1/c_0 f_{2H}=f_{1H}/f_{2H} \; . \tag{4.3}$$

The graph of f_{1H}/f_{2H} versus a_0 is given in Fig. 15, from which an approximate formula is obtained.

$$\mathfrak{V}_{HB,\mathrm{MAX}} \fallingdotseq 2.45/a_0 \; . \tag{4.4}$$

Next, the maximum amplitude of the top of the structure is given in Fig. 16 when $\rho_0/\rho=1/4$. (Cf. Figs. 3, 5, 7 and 9.) We can get the following approximation formulas.

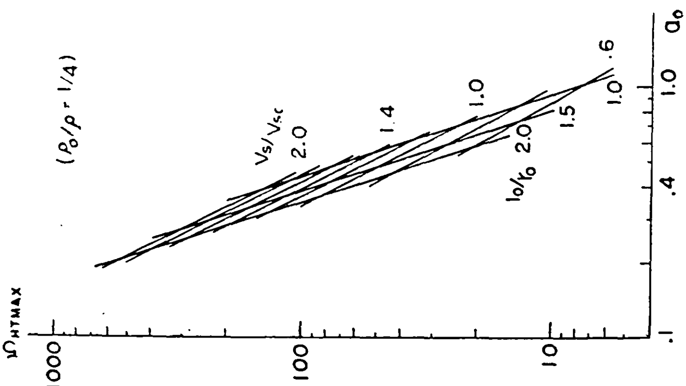

Fig. 16.

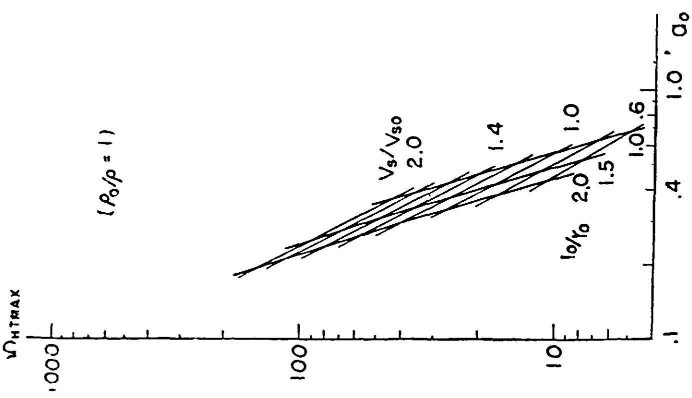

Fig. 17.

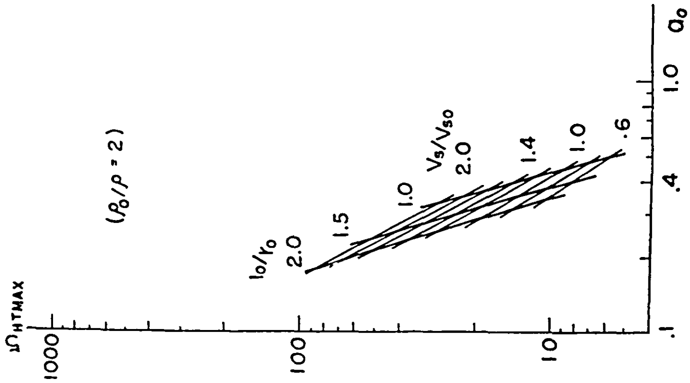

Fig. 18.

Maximum amplitudes of the top of structure for various density ratio.

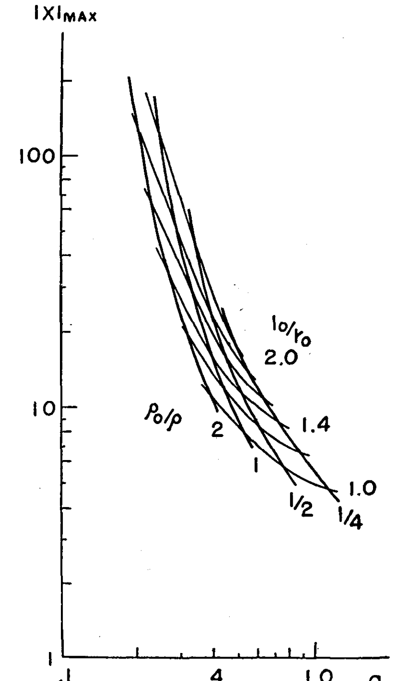

Fig. 19. Maximum amplitude of the center of gravity in horizontal motion.

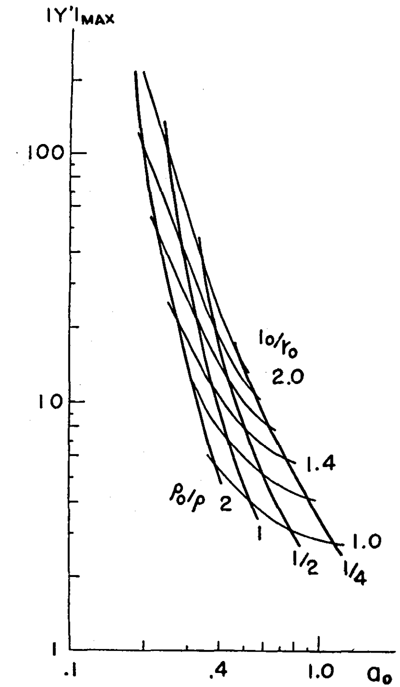

Fig. 20. Maximum displacement of the center of gravity caused by angular motion.

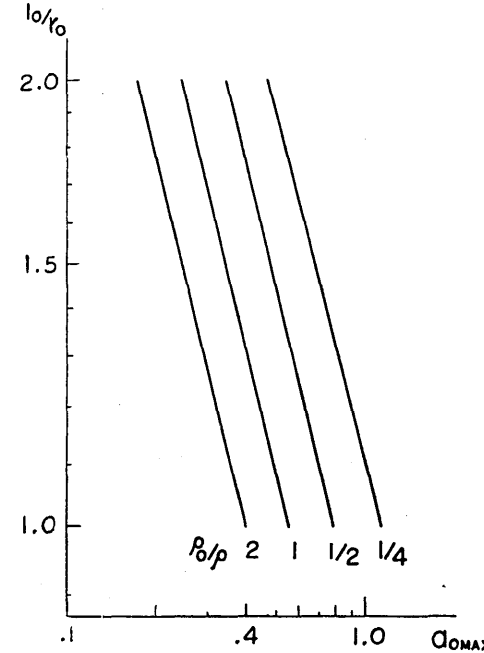

Fig. 21. Frequency which gives the maximum amplitude in rocking motion.

$$\mathfrak{V}_{HT.\mathrm{MAX}} \doteqdot 11.5(V_s/V_{s0})/a_0^2\ , \tag{4.5}$$

$$\mathfrak{V}_{HT.\mathrm{MAX}} \doteqdot 8.5/\{(l_0/r_0)a_0^3\}\ . \tag{4.6}$$

Combining the above two equations the following one is deduced.

$$a_{0,\mathrm{MAX}} \doteqdot 0.74/\{(V_s/V_{s0})\cdot(l_0/r_0)\}\ . \tag{4.7}$$

Consequently the period which gives the maximum $\mathfrak{V}_{HT}$ is

$$T_{\mathrm{MAX}} \doteqdot 8.5(l_0/V_{s0})\ . \tag{4.8}$$

A similar formula was once obtained by K. Suyehiro[3]. His calculation was based on the assumption of no coupling between the structure and the foundation. The coefficient of the right hand member of (4.8) was then 8.0 instead of 8.5.

When the structure has a larger density (e.g. $\rho_0/\rho=1, 2$), no simple formula like (4.8) can be obtained, although the tendency of the curves are similar. (See Figs. 17 and 18.)

Appendix. Rocking motion

In our previous study of rocking motion[4] we assumed ρ_0/ρ to be always 1/4. In the present paper, however, we carried out the computation for other cases, namely $\rho_0/\rho=1/2, 1, 2$, so that we may apply the theory to structures such as piers.

The result is given in Fig. 19 (maximum displacement of the center of gravity) and Fig. 20 (maximum inclination of the structure). From these figures we can find the relation between ρ_0/ρ and the maximum movement.

However, the relation for this case is not so simple as the previous one. The only simple formula obtained from Fig. 21 is the one giving the period corresponding to the maximum movement, that is

$$T_{\mathrm{MAX}} \doteqdot 12\sqrt{\rho_0/\rho}\,(l_0/V_s)\ . \tag{A.1}$$

Acknowledgement

The authors' thanks are due to Prof. H. Kawasumi, who encouraged them and gave useful suggestions throughout this work.

3) K. SUYEHIRO, *Jour. Soc. Arch., Japan,* **40** (1926), 531. (in Japanese)
4) *loc. cit.*, 1).

23. 構造物の剪断振動と弾性地盤の相互作用

附: ロッキングの最大倍率

地震研究所 {佐藤泰夫 / 山口林造}

1. 地盤と構造物とを一体の振動系と考えたときの上下動，ロッキングおよび水平動についての計算はすでに報告してある．ここではそれに引続いて連続体の剪断振動についての振動性能を求めた．そして橋脚等の振動にも適用しうるよう，前に行つた計算のパラメーターの範囲を拡張した．

2. 記号を§2に示す．

3. 剪断振動に関して構造物内任意の高さにおける変位は (3.15) の式で与えられる．先づ構造物と地盤の密度比 ρ_0/ρ を 1/4，両方の S 波の速度を等しいとおいて，構造物の縦横の比 l_0/r_0 をパラメーターにとりながら，$a_0=pr_0/V_s$ に対する底面における変位倍率 $\mathfrak{V}_{HB}$ を示したのが Fig. 2 である．同様にして頂部に関する $\mathfrak{V}_{HT}$ は Fig. 3 に示されている．Fig. 4 は $l_0/r_0=1$ とおき，S 波の速度比をパラメーターとした $\mathfrak{V}_{HB}$ を，Fig. 5 は $\mathfrak{V}_{HT}$ に関するものを示している．Fig. 6～Fig. 9 には $l_0/r_0=1.5$ および 2.0 とした場合の同様な関係を図示した．位相差に関してはそれぞれ Fig. 10～Fig. 13 に見られる通りで，(3.18) から分るごとく底面と頂部とでは相違がない．

4. 構造物底面の最大変位倍率およびそれを与える周期に関しては Fig. 14 から近似的に (4.4) のような簡単な式で表わされる．また (4.3) の関係を使つた Fig. 15 にも明らかにそのことが見られる．

次に構造物頂部に関する最大変位倍率を Fig. 16～Fig. 18 に示した．その中 $\rho_0/\rho=1/4$ の場合には簡単な近似式 (4.5)～(4.8) で表わすことができた．

附．ロッキングに関しては重心の水平あるいは回転による，それぞれの最大変位倍率を Fig. 19 および Fig. 20 に示した．いずれも密度比を 1/4, 1/2, 1, 2 として密度の変化に伴う様子を調べたが，簡単な関係式は得られなかつた．ただ最大変位倍率を与える周期に関しては，(A.1) のような関係が求められた．実用上便利な式と思われる．