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# 15. *Studies of the Thermal State of the Earth. The Second Paper: Heat Flow associated with Magma Intrusion.*

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## Summary

On the assumption that hot magma intrudes suddenly in a spherical cavity in the earth's crust, the conduction of heat inside and outside the magma is discussed. The changes in the geothermal gradient and consequently the heat flow at the earth's surface right up the magma are then calculated. It is assumed 1) that the initial temperatures are constant and zero respectively inside and outside the sphere and 2) that the temperature at the earth's surface is always kept at zero.

It turns out that the geothermal gradient at the surface gradually increases reaching a maximum and then slowly diminishes. The maximum of the gradient is largely dependent on both the size of the sphere and the depth. If the radius and the depth of the centre are assumed respectively as 2 and 5 *km*, the gradient corresponding to an initial temperature, which is assumed as 1300°*C*, takes the maximum around 40°*C*/*km* for a usual thermal diffusivity of rocks some $6\times10^4$ *years* past after the intrusion. This value amounts to only double the gradient due to the general heat flow of crustal origin, so that it would not be easy to detect the magma mass of the size by measuring the geothermal gradient anomaly provided the intrusion has occurred at the depth considered or deeper. Large geothermal gradients observed at some volcanic areas are likely to be caused by a high temperature mass permeated by hot gases at a depth of 1 *km* or less.

## 1. Introduction

One of the aims of the geothermal studies[1] which we have recently started is to bring out geothermal anomalies which may be associated

1) S. UYEDA, T. YUKUTAKE and I. TANAOKA, *Bull. Earthq. Res. Inst.*, **36** (1958), 251.

with geological and geophysical activities. On Volcano Sirane, an active volcano, we actually found a geothermal gradient as large as $242°C/km$. The heat flow is estimated at $10.8 \times 10^{-6}\, cal/sec\, cm^2$ there. These values would be one of the most straightforward examples of a large thermal anomaly which may be caused by underground high temperatures.

We then come to a question. How deep and how large will the hot region be? Since it is not practicable to conduct measurements of geothermal gradient at many points over the area, however, we can answer practically nothing from observation. In order to have a rough idea, we had better investigate theoretically a few typical models relevant to possible geothermal anomaly at the present state of investigation.

In this paper, account will be taken of heat conduction in the earth's crust when an underground spherical region is heated up quickly. When a volcano becomes active, the interior of the volcano would become hot as was inferred from the geomagnetic studies[2)] on Volcano Mihara at the time of its 1950 eruption. Although some parts of the heat thus stored in the volcano are sent out from the crater together with lava, ashes, water-vapour and other gases, there should be some heat diffusing through the crust. It is intended here to examine quantitatively the geothermal gradient at the earth's surface or heat flow associated with such heat conduction by assuming depth, size and initial temperature of the magma mass.

This sort of study will be also useful for examining the possibility of detecting underground magma intrusion which is supposed to be the cause of earthquake swarms. Since a long time is required in order to observe an appreciable heat flow at the earth's surface, however, the underground activity for this case must occur continually over a long period.

Heat conduction within a volcano has been a topic of geophysics[3)]. Cooling of a laccolith or intrusive magma mass is also a good exercise of mathematical theory of heat conduction in solids[4)]. To the writer's knowledge, however, no study on cooling of magma has been published with adequate considerations of the boundary condition at the earth's surface. Since the geothermal gradient near the surface is the main interest in order to discuss the possibility of detecting magma mass by geothermal method, the boundary condition, on which the geothermal

2) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **29** (1951), 161.
3) Y. KODAIRA, *Geophys. Mag.*, **1** (1926), 14.
4) For example, H. S. CARSLAW and J. G. JAEGER, *Conduction of heat in solids*, Oxford (1948), 200.

gradient is largely dependent, should be taken into account.

In the light of the above, the cooling of a hot spherical mass, of which the initial temperature is assumed as constant, embedded in the cool earth having a plane boundary, where the temperature is kept zero, will be solved in this paper.

## 2. Theory

Let us suppose a semi-infinite earth in which a sphere is embedded as can be seen in Fig. 1. If we denote the temperature and thermal diffusivity outside the sphere respectively by $u_1$ and $\kappa_1$, the equation of heat conduction, which satisfies the initial condition that $u_1=0$ at $t=0$, can be written as

$$pu_1=\kappa_1\nabla^2 u_1\ , \qquad (1)$$

where $p$ denotes the time-operator $\partial/\partial t$.

Inside the sphere, the equation is given by

$$pu_2=\kappa_2\nabla^2 u_2+p\Theta\ , \qquad (2)$$

where $u_2$ and $\kappa_2$ denote respectively the temperature and thermal diffusivity in the region.

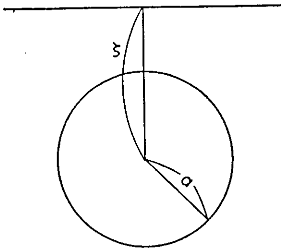


Fig. 1. The high-temperature spherical mass embedded in the earth.

The second term of the righthand-side is added to the equation in order to satisfy the initial condition that $u_2=\Theta$ throughout the sphere at $t=0$, $\Theta$ being assumed to be a constant here.

If we put

$$k_1=\kappa_1^{-1/2}p^{1/2},\quad k_2=\kappa_2^{-1/2}p^{1/2}, \qquad (3)$$

the solutions of (1) and (2), which do not become infinitely large respectively at $r\to\infty$ and $r\to 0$, where $r$ denotes the radial distance from the origin taken at the centre of the sphere, can be given as

$$u_1=A(k_1r)^{-1/2}K_{1/2}(k_1r)\ , \qquad (4)$$

$$u_2=\Theta+B(k_2r)^{-1/2}I_{1/2}(k_2r)\ , \qquad (5)$$

in which $A$ and $B$ are functions of $p$, while $I_{1/2}$ and $K_{1/2}$ are modified Bessel functions of degree 1/2.

In general, the conditions to be satisfied at the boundary of the sphere or $r=a$ are 1) the continuity of heat flow and 2) the proportionality of the heat flow to the temperature difference; they can be written as

$$K_1\frac{\partial u_1}{\partial r}=K_2\frac{\partial u_2}{\partial r}=g(u_1-u_2)\ , \tag{6}$$

where $K_1$ and $K_2$ denote the thermal conductivity outside and inside the sphere.

If we put (4) and (5) into (6) and solve the equations with respect to $A$ and $B$, we obtain

$$A=\Phi^{-1}g\Theta K_2k_2(k_1a)^{1/2}I_{3/2}(k_2a)\ , \tag{7}$$

$$B=-\Phi^{-1}g\Theta K_1k_1(k_2a)^{1/2}K_{3/2}(k_1a)\ , \tag{8}$$

where

$$\Phi=\{K_1k_1K_{3/2}(k_1a)+gK_{1/2}(k_1a)\}K_2k_2I_{3/2}(k_2a)+gK_1k_1I_{1/2}(k_2a)K_{3/2}(k_1a)\ . \tag{9}$$

In order to satisfy the boundary condition at the earth's surface, we have to add another solution $u'$ to $u_1$. Since $u'$ would become small in great depths, the thermal property inside the sphere may be assumed as the same as that outside it, for the first approximation, in obtaining $u'$. If we assume, for the sake of simplicity, that $K_1=K_2$ and $\kappa_1=\kappa_2$, the solution, which will be obtained in the above way, will be exact.

It is well known that

$$\left.\begin{aligned} K_{1/2}(x)&=\left(\frac{\pi}{2x}\right)^{1/2}e^{-x}\ , \\ K_{3/2}(x)&=\left(\frac{\pi}{2x}\right)^{1/2}(1+x^{-1})e^{-x}\ , \\ I_{1/2}(x)&=\left(\frac{2}{\pi x}\right)^{1/2}\sinh x\ , \\ I_{3/2}(x)&=\left(\frac{2}{\pi x}\right)^{1/2}(\cosh x-x^{-1}\sinh x)\ . \end{aligned}\right\} \tag{10}$$

From (4), therefore, we have

$$u_1=\sqrt{\frac{\pi}{2}}A\frac{e^{-k_1r}}{k_1r}\ . \tag{11}$$

Meanwhile, if we take a cylindrical coordinate $(\rho, \phi, z)$ having its origin at the centre of the sphere and taking the $z$-axis upwards, we have a well known relation

$$\frac{e^{-kr}}{r}=\int_0^\infty e^{-z\sqrt{\lambda^2+k^2}}J_0(\lambda\rho)\frac{\lambda d\lambda}{\sqrt{\lambda^2+k^2}} \qquad (z>0), \qquad (12)$$

which can be found in a number of text-books of mathematical physics, so that (11) is transformed to

$$u_1=\sqrt{\frac{\pi}{2}}\,Ak_1^{-1}\int_0^\infty e^{-z\sqrt{\lambda^2+k_1^2}}J_0(\lambda\rho)\frac{\lambda d\lambda}{\sqrt{\lambda^2+k_1^2}}\,. \qquad (13)$$

We then take an additional solution of the following form

$$u'=\int_0^\infty C(\lambda)e^{z\sqrt{\lambda^2+k_1^2}}J_0(\lambda\rho)d\lambda\,. \qquad (14)$$

The general form of boundary condition at the earth's surface is written as

$$\left(K_1\frac{\partial}{\partial z}+h\right)(u_1+u'-u_a)=0 \qquad (15)$$

where $u_a$ is the air temperature and $h$ is a constant. For the earth's surface, where the air is always moving about, $h$ is considered to be fairly large, so that, if we simply assume $u_a=0$, the relation

$$u_1+u'=0 \qquad (16)$$

is a good approximation of (15).

Let us designate the depth of the centre of the sphere by $\zeta$. Putting both (13) and (14) into (16) which is to be satisfied at $z=\zeta$, we obtain

$$C(\lambda)=-\sqrt{\frac{\pi}{2}}\,Ak_1^{-1}e^{-2\zeta\sqrt{\lambda^2+k_1^2}}\frac{\lambda}{\sqrt{\lambda^2+k_1^2}}\,. \qquad (17)$$

Substituting (17) into (14), it follows that

$$u'=-\sqrt{\frac{\pi}{2}}\,Ak_1^{-1}\int_0^\infty e^{-(2\zeta-z)\sqrt{\lambda^2+k_1^2}}J_0(\lambda\rho)\frac{\lambda d\lambda}{\sqrt{\lambda^2+k_1^2}}\,, \qquad (18)$$

which, if we denote the distance from the point $z=2\zeta$ on the $z$-axis by $r'$, can be written as

$$u' = -\sqrt{\frac{\pi}{2}}\, A \frac{e^{-k_1 r'}}{k_1 r'}\,, \tag{19}$$

so that we see that $u'$ is derived from the image source which is placed at the symmetrical position in regard to the earth's surface.

Now we are in a position to interpret the operational equations (11), (19) and (5) in order to obtain expressions which are explicit with respect to $t$. Since the general forms of $A$ and $B$ are so complicated, as given in (7) and (8), that no exact solutions can be deduced, however, some simplifications of the boundary condition and parameters are to be considered here. First of all, we regard $K_1/g$ and $K_2/g$ as infinitesimally small, the interface drop of temperature being ignored in that case. We further assume that the thermal properties inside and outside the sphere are the same. Putting then $K_1 = K_2 = K$, $\kappa_1 = \kappa_2 = \kappa$ and consequently $k_1 = k_2 = k$, it is found, with the aid of the formulas of modified Bessel functions, that (7) and (8) can be written as

$$A = \Theta (ka)^{3/2} I_{3/2}(ka)\,, \tag{20}$$

$$B = -\Theta (ka)^{3/2} K_{3/2}(ka)\,. \tag{21}$$

With these expressions, (11), (19) and (5) are reduced to the forms;

$$u_1 = \frac{\Theta}{2r}\Big[a\big(e^{-\kappa^{-1/2}(r-a)p^{1/2}} + e^{-\kappa^{-1/2}(r+a)p^{1/2}}\big) - \kappa^{1/2}p^{-1/2}\big(e^{-\kappa^{-1/2}(r-a)p^{1/2}} - e^{-\kappa^{-1/2}(r+a)p^{1/2}}\big)\Big]\,, \tag{22}$$

$$u' = -\frac{\Theta}{2r'}\Big[a\big(e^{-\kappa^{-1/2}(r'-a)p^{1/2}} + e^{-\kappa^{-1/2}(r'+a)p^{1/2}}\big) - \kappa^{1/2}p^{-1/2}\big(e^{-\kappa^{-1/2}(r'-a)p^{1/2}} - e^{-\kappa^{-1/2}(r'+a)p^{1/2}}\big)\Big]\,, \tag{23}$$

$$u_2 = \Theta - \frac{\Theta}{2r}\Big[a\big(e^{-\kappa^{-1/2}(a-r)p^{1/2}} - e^{-\kappa^{-1/2}(a+r)p^{1/2}}\big) + \kappa^{1/2}p^{-1/2}\big(e^{-\kappa^{-1/2}(a-r)p^{1/2}} - e^{-\kappa^{-1/2}(a+r)p^{1/2}}\big)\Big]\,. \tag{24}$$

On interpreting the above operational equations term by term, we arrive at the solutions which can be written as

$$u_1 = \Theta\left[\frac{1}{2}\left(\mathrm{erf.}\,\frac{r+a}{2\kappa^{1/2}t^{1/2}} - \mathrm{erf.}\,\frac{r-a}{2\kappa^{1/2}t^{1/2}}\right) + \frac{1}{r}\left(\frac{\kappa t}{\pi}\right)^{1/2}\left(e^{-(r+a)^2/4\kappa t} - e^{-(r-a)^2/4\kappa t}\right)\right]\,, \tag{25}$$

$$u' = -\Theta\left[\frac{1}{2}\left(\text{erf.}\,\frac{r'+a}{2\kappa^{1/2}t^{1/2}} - \text{erf.}\,\frac{r'-a}{2\kappa^{1/2}t^{1/2}}\right) + \frac{1}{r'}\left(\frac{\kappa t}{\pi}\right)^{1/2}\left(e^{-(r'+a)^2/4\kappa t} - e^{-(r'-a)^2/4\kappa t}\right)\right], \quad (26)$$

$$u_2 = \Theta\left[\frac{1}{2}\left(\text{erf.}\,\frac{a+r}{2\kappa^{1/2}t^{1/2}} + \text{erf.}\,\frac{a-r}{2\kappa^{1/2}t^{1/2}}\right) + \frac{1}{r}\left(\frac{\kappa t}{\pi}\right)^{1/2}\left(e^{-(a+r)^2/4\kappa t} - e^{-(a-r)^2/4\kappa t}\right)\right]. \quad (27)$$

## 3. Temperature distribution

By making $u_1 + u'$ $(r > a)$ and $u_2 + u'$ $(r < a)$, the temperature at any place and any time can be calculated with the aid of (25), (26) and (27). For example, let us take $\kappa = 0.01$ *c.g.s.*, which may be regarded as usual for rocks, $a = 2s\,km$ ($s$ may take any positive value), while $\zeta$, depth of the centre of the sphere, takes three values, $5s$, $4s$ and $3s\,km$. Corresponding to $\zeta = 5s\,km$, the distributions of the temperature on the vertical line passing through the centre can be calculated for various epochs as are shown in Fig. 2. Since lengths always appear in (25), (26) and (27) as ratio to square root of time, it is noticed that, if we multiply the depth as well as the radius of the sphere by $s$, the corresponding curves of temperature distribution may be regarded as those for the epochs at

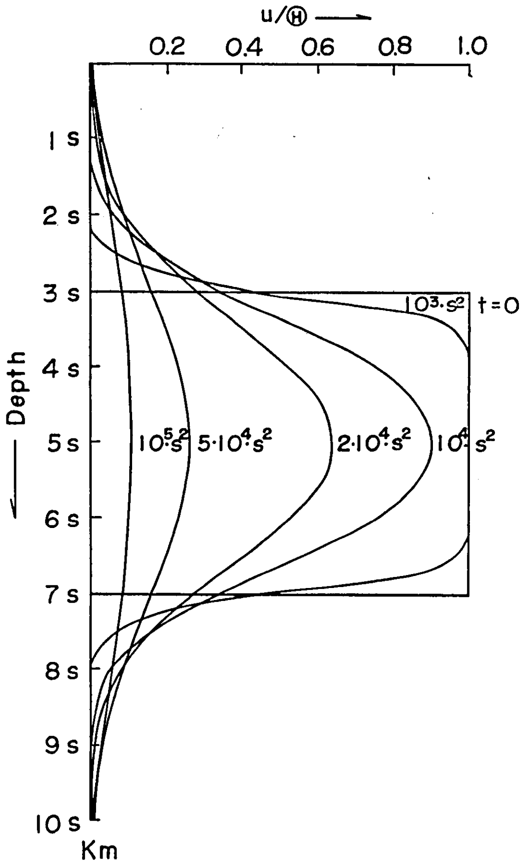


Fig. 2. The distribution of the temperature on the vertical line passing the centre of the sphere for $a = 2s\,km$ and $\zeta = 5s\,km$.

which the times are multiplied by $s^2$.

As can be seen in Fig. 2, the conduction of heat outside the sphere is extremely slow. The influence of the earth's surface is not remarkable for this case, the distribution of temperature is practically symmetric about the centre of the sphere.

In Figs. 3 and 4, however, the depths of the centre are taken respectively as $4s$ and $3s\,km$. In these cases, the influence of the earth's surface becomes considerable, the asymmetry of temperature distribution being enlarged as the sphere approaches the surface.

At the time of the 1950 eruption of Volcano Mihara, the writer has presumed, by analysing the results of repeated geomagnetic surveys, that a roughly spherical hot region of 2 *km* in radius appeared around its centre 5 *km* deep within a period of two *months*. If we suppose such a hot mass in the earth and assume that it is subjected to cooling only by conduction, we see, from what is shown in Fig. 2, that it will take as much as $10^5$ *years* to cool that mass down to a temperature less than 10 *percent* of the initial one. It is also suggested that the temperature gradient near the earth's surface would not be so large in spite of the high temperature at the level of 3 *km* deep.

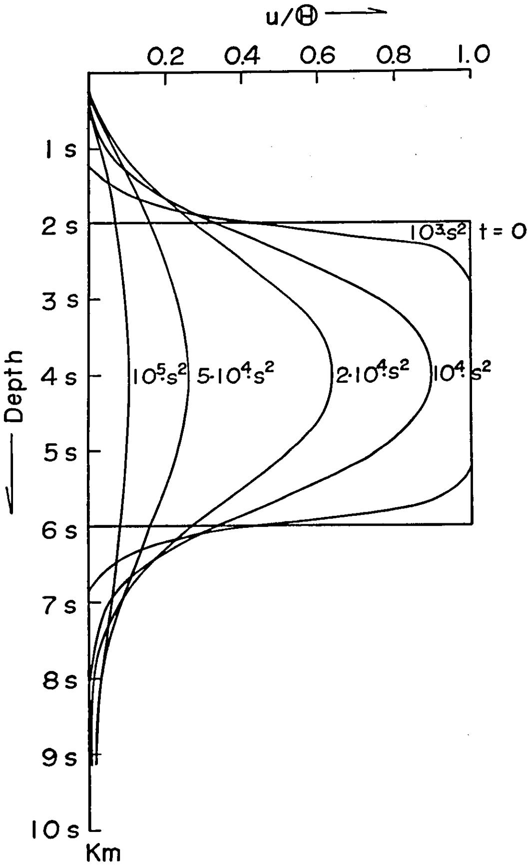


Fig. 3. The distribution of the temperature on the vertical line passing the centre of the sphere for $a=2s\,km$ and $\zeta=4s\,km$.

## 4. Thermal gradient and heat flow at the earth's surface

The surface geothermal gradient right up the hot mass is calculated by differentiating $u_1+u'$ with respect to $z$. We obtain

$$-\left[\frac{\partial(u_1+u')}{\partial z}\right]_{\rho=0,z=\zeta}=\Theta\frac{a}{\zeta}\left(\frac{1}{\pi\kappa t}\right)^{1/2}\left[\left(1+\frac{2\kappa t}{a\zeta}\right)e^{-\frac{(\zeta+a)^2}{4\kappa t}}+\left(1-\frac{2\kappa t}{a\zeta}\right)e^{-\frac{(\zeta-a)^2}{4\kappa t}}\right]. \quad (28)$$

Corresponding to the three cases calculated in the last section, the changes in the surface temperature gradient with time is calculated and shown in Fig. 5 for $\zeta=5s$, $4s$ and $3s\,km$, while $a$ is assumed to be always equal to $2s\,km$. If we assume that $\Theta$ amounts to $1300°C$, the peaks of the gradient become roughly 40, 100 and $350°C/km$ for $\zeta=5$, 4 and $3\,km$. If the thermal conductivity is assumed to take a usual value for rocks, $5\times10^{-3}\,cal/cm\,sec\,°C$ say, these gradients give heat flows amounting to 2, 5 and $17.5\times10^{-6}\,cal/cm^2\,sec$ respectively.

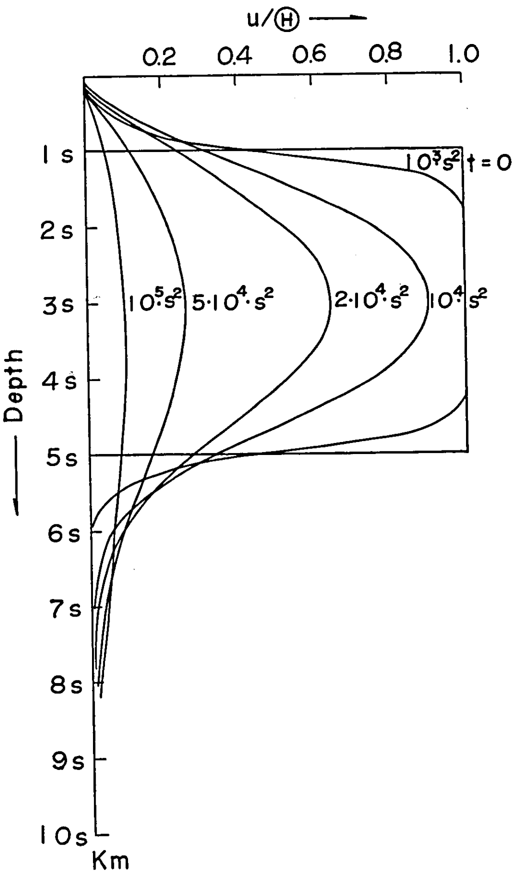


Fig. 4. The distribution of the temperature on the vertical line passing the centre of the sphere for $a=2s\,km$ and $\zeta=3s\,km$.

We see from the above results that the heat flow due to an underground hot magma mass is not so large except for shallow one. For instance, the $1300°C$ magma of $2\,km$ in radius, the depth of its top being $3\,km$, gives rise to a heat flow as small as $2\times10^{-6}\,cal/cm^2\,sec$, that is, only double the mean heat flow of crustal origin. If a magma of the same mass intrudes in a farther depth, no marked anomaly in heat flow can be observed.

We should pay attention to the fact, however, that the changes in the gradient are extremely slow. Since the heat flow, which is not greatly different from its maximum, continues over a long period, $10^5$ *years* say, we may well expect repeated intrusions of a similar sort. In that case,

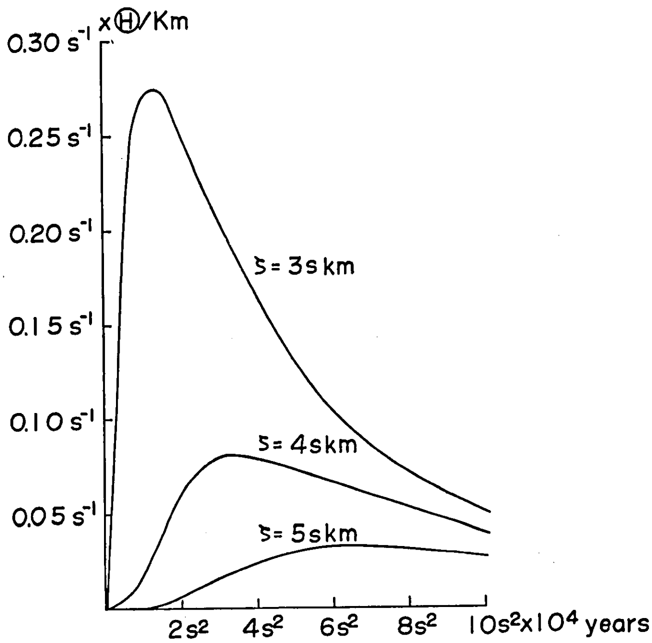


Fig. 5. The changes in the geothermal gradient at the earth's surface right up the magma.

it happens that we observe the sum of heat flows that occurred from time to time during a long period, so that there might be a possibility that the present heat flow becomes an appreciable amount.

In the case of a shallow intrusion, the top of the hot region is 1 *km* deep say, a marked anomaly of heat flow can be observed. The large heat flows observed in some geothermal areas would be of this origin though the high temperature may be likely to be caused by hot gases coming from below.

## 5. Concluding remarks

On the basis of a theory of heat conduction concerning the cooling of a spherical region embedded in a semi-infinite solid, it turns out that a magma intrusion does not produce a large anomaly of heat flow unless the intrusion occurs in a shallow depth, 1 *km* or so. If we assume that the radius of the sphere is 2 *km*, the depth of its centre 5 *km*, we can expect a heat flow right up the sphere amounting to $2\times 10^{-6}$ *cal/cm sec* which is only double the general heat flow from within the earth. It would be of difficulty to detect a high temperature mass of that size by measuring heat flow at the earth's surface provided the depth exceeds 5 *km*.

There is no doubt, however, that, if the top of hot region reaches somehow a depth of 1 *km* or thereabout, we can observe a large anomaly of heat flow, 10 times or more of the general one.

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## 15. 地球熱学 (第 2 報) 岩漿貫入と地表の熱の流れ

地震研究所　力　武　常　次

地下の球状部分が突然高温になつて，その後熱伝導によりどのように冷却するかを理論的に調べた．地表温度が常に零に保たれるとした場合，地表の地温勾配はだんだんと増加して極大に達し，以後徐々に減少する．高温部分の半径を 2 *km*, 中心の深さを 5 *km*, 初期温度を 1300°*C* とすれば，地温勾配の極大は $6\times10^4$ 年後に起り，40°*C*/*km* の値をとる．

この計算は通常の岩石に対する熱拡散率を仮定した結果で，地表の熱流量に換算すれば $2\times10^{-6}$ $cal/cm^2\,sec$ ということになる．この値は平均の熱流量の倍にすぎないので，熱流量の測定からこの程度の大きさおよび深さの岩漿溜を検出することは困難であると予想される．

上記の結果より，通常の火山地域では，熱流量が必ずしも異常に大きいとは考えられないことになる．また第 1 報において報告した草津白根山の測定例は，熱源がずつと地表に近い場合であることが推定される．

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