BULLETIN OF THE EARTHQUAKE
RESEARCH INSTITUTE
Vol. 37 (1959), pp. 245-264

16. *Forced Oscillations of the Earth's Dynamo.*

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(Read Feb. 24, 1959.—Received Feb. 24, 1959.)

Summary

A theory, based on magneto-hydrodynamics, that the geomagnetic secular variations are caused by forced oscillations of small amplitude superposing on the steady state of the earth's dynamo is tried. With drastic simplifications, the driving force due to buoyancy is obtained for hypothetical variations of a 100-*year* period. The time-dependent thermal field responsible for such a force field is also obtained. It is shown that the amplitude of the field would be $10^{-4}{}^\circ C$ in order to account for the S_1^0 field whose amplitude is assumed as 1000 *gammas* at the pole.

1. Introduction

It has become widely believed that geomagnetic secular variations are caused by fluid motions in the earth's core wherein magnetic fields are maintained by the dynamo action. Although it is not known whether or not the earth's dynamo is steady or non-steady, it is possible to study secular variations as small disturbances given to the dynamo which is regarded as nearly steady during a period longer than those of secular variations.

The first study in this line has been published by the present writer[1] by taking Bullard's dynamo[2] as the steady state. As far as the influence of the Coriolis force due to the earth's rotation is disregarded, the study led to a conclusion that the dynamo, whose toroidal magnetic fields are as large as those estimated by Bullard, would not be stable for small disturbances. It is also suggested that geomagnetic secular variations might be explained by the free magneto-hydrodynamic oscillations of the system.

1) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **33** (1955), 1.

2) E. C. BULLARD and H. GELLMAN, *Phil. Trans. Roy. Soc. London A*, **247** (1954), 213.

The above study would not be adequately applied to the earth's dynamo because the Coriolis force, which usually plays an important role in studies of rotating fluid mass of large dimension, is completely ignored. The present writer[3)] extended his study to the case in which the core is subjected to a uniform rotation. It has then turned out that the influence of the Coriolis force is so strong that the dynamo considered is stable for small disturbances unless the steady magnetic fields are as large as 10^5 *gauss* or more. We therefore see that the previous view that the secular variations might be caused by free magneto-hydrodynamic oscillations in the core is not acceptable when we take into account the effect of the Coriolis force.

In the light of the above, it is naturally required to suppose some time-dependent driving force in order that we can account for the secular variation as a result of magneto-hydrodynamic processes prevailing in the earth's core. The most likely cause of such a force would be the buoyancy force of thermal origin. The purpose of this paper is to study forced oscillations of the dynamo, of which the steady state has been studied by Bullard and others, and also to investigate the force field which gives rise to the dipole part of geomagnetic secular variation. Owing to the mathematical complexity, however, only fluid motion, driving force and thermal field which are related to secular variation of the S_1^0 type will be taken into account.

2. Equations to be solved

Maxwell's equations can be written as

$$\vec{I}=\sigma(\vec{E}+\vec{V}\wedge\vec{H}) \,, \qquad (1)$$

$$\mathrm{curl}\,\vec{E}=-\partial\vec{H}/\partial t \,, \qquad (2)$$

$$\mathrm{curl}\,\vec{H}=4\pi\vec{I} \,, \qquad (3)$$

where $\vec{I}$, $\vec{E}$, $\vec{H}$, σ and $\vec{V}$ denote respectively the electric current density, electric field, magnetic field, electrical conductivity and velocity of fluid motion. The magnetic permeability is assumed as unity in electromagnetic unit.

Meanwhile the equation of motion can be written as

$$\rho\partial\vec{V}/\partial t+\rho(\vec{V}\cdot\mathrm{grad})\vec{V}+2\rho(\vec{\omega}_0\wedge\vec{V})=\vec{I}\wedge\vec{H}-\mathrm{grad}\,\vec{P}+\vec{G} \qquad (4)$$

3) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **34** (1956), 283.

where ρ, $\vec{\omega}_0$, P and $\vec{G}$ denote respectively the density, rotation vector of uniform rotation about $\theta=0$ axis, pressure and non-electromagnetic force.

Let us put

$$\left.\begin{array}{l}\vec{I}=\vec{I}_0+\vec{i},\ \vec{E}=\vec{E}_0+\vec{e},\ \vec{H}=\vec{H}_0+\vec{h}\,, \\ \vec{V}=\vec{V}_0+\vec{v},\ P=P_0+p,\ \vec{G}=\vec{G}_0+\vec{g}\,,\end{array}\right\} \tag{5}$$

in which the quantities with subscript 0 specify the steady state, while those denoted by small letters are the departures from the steady state and are regarded as the first order small quantities. Putting (5) into the equations from (1) to (4), we have a system of linear differential equations such as

$$\vec{i}=\sigma(\vec{e}+\vec{v}\wedge\vec{H}_0+\vec{V}_0\wedge\vec{h})\,, \tag{6}$$

$$\mathrm{curl}\,\vec{e}=-\partial\vec{h}/\partial t\,, \tag{7}$$

$$\mathrm{curl}\,\vec{h}=4\pi\vec{i}\,, \tag{8}$$

$$\rho\partial\vec{v}/\partial t+2\rho(\vec{\omega}_0\wedge\vec{v})=\vec{i}\wedge\vec{H}_0+\vec{I}_0\wedge\vec{h}-\mathrm{grad}\,p+\vec{g}\,, \tag{9}$$

provided the second order small quantities are ignored.

On the assumption that $\vec{H}_0$, $\vec{V}_0$, $\vec{I}_0$ and $\vec{\omega}_0$ are known, we are going to solve these equations under suitable boundary conditions. The only difference of these equations from those treated in the previous paper[3)] is the point that the non-electromagnetic force $\vec{g}$ is not ignored here.

We may further assume that the fluid is incompressible, so that

$$\mathrm{div}\,\vec{v}=0\,. \tag{10}$$

By taking curl of (6) and eliminating $\vec{e}$ with the aid of (7), we obtain

$$\mathrm{curl}\,\vec{i}=\sigma\{-D\vec{h}+\mathrm{curl}\,(\vec{v}\wedge\vec{H}_0)+\mathrm{curl}\,(\vec{V}_0\wedge\vec{h})\}\,, \tag{11}$$

where we write D in place of $\partial/\partial t$. On the other hand we have from (8)

$$4\pi\,\mathrm{curl}\,\vec{i}=\mathrm{curl}\,\mathrm{curl}\,\vec{h}=-\nabla^2\vec{h}\,, \tag{12}$$

because $\mathrm{div}\,\vec{h}=0$.

From (11) and (12), we can eliminate $\vec{i}$ getting

$$\{D-(4\pi\sigma)^{-1}\nabla^2\}\vec{h}=\text{curl}\,(\vec{V}_0\wedge\vec{h})+\text{curl}\,(\vec{v}\wedge\vec{H}_0)\ . \tag{13}$$

We also have from (9)

$$\rho D\vec{v}+2\rho(\vec{\omega}_0\wedge\vec{v})=(4\pi)^{-1}\{(\text{curl}\,\vec{h}\wedge\vec{H}_0)+(\text{curl}\,\vec{H}_0\wedge\vec{h})\}-\text{grad}\,p+\vec{g}\ . \tag{14}$$

(13) and (14) are regarded as the simultaneous equations for $\vec{h}$ and $\vec{v}$.

3. Solution for pressure

Let us assume that $\vec{g}$ is the buoyancy force. In that case it is obvious that its θ and ϕ components are to be zero, the radial component of $\vec{g}$ being written as g hereafter.

If we make div of (14) we obtain

$$\nabla^2 p=-(4\pi)^{-1}(\vec{H}_0\cdot\nabla^2\vec{h}+\vec{h}\cdot\nabla^2\vec{H}_0+2\text{curl}\,\vec{h}\cdot\text{curl}\,\vec{H}_0)$$
$$+2\rho\vec{\omega}_0\cdot\text{curl}\,\vec{v}+r^{-2}\frac{d(r^2g)}{dr} \tag{15}$$

because of (10). $\text{curl}\,\vec{\omega}_0=0$ which is derived from the condition of uniform rotation is also taken into account.

In general, $\vec{h}$ is to be expressed with the sum of poloidal and toroidal magnetic fields of various degree. As has been done in the previous papers[1),3)], however, we shall assume $\vec{h}$ is of the S_1^0 type and ignore the fields of other type which will appear in the righthand-side of (13), otherwise the problem becomes far from tractable because many fields of various type come out through the induction processes.

If we take only the S_1^0 field, we may write as

$$\left.\begin{aligned}
\vec{h}&=\begin{cases}-2s(r)P_1\\[4pt] -\left(r\dfrac{ds}{dr}+2s\right)\dfrac{dP_1}{d\theta}\\[4pt] 0\,,\end{cases}\\
\text{curl}\,\vec{h}&=\begin{cases}0\\[4pt] 0\\[4pt] -\left(r\dfrac{d^2s}{dr^2}+4\dfrac{ds}{dr}\right)\dfrac{dP_1}{d\theta}\,,\end{cases}\\
\nabla^2\vec{h}&=\begin{cases}-2\left(\dfrac{d^2s}{dr^2}+\dfrac{4}{r}\dfrac{ds}{dr}\right)P_1\\[4pt] -\left(r\dfrac{d^3s}{dr^3}+6\dfrac{d^2s}{dr^2}+\dfrac{4ds}{rdr}\right)\dfrac{dP_1}{d\theta}\\[4pt] 0\,.\end{cases}
\end{aligned}\right\} \tag{16}$$

As for the steady magnetic fields and fluid motions, we take the simplest combination of Bullard's dynamo[2)], so that the motion consists of the T_1^0 and S_2^{2c} types while the magnetic fields contain the S_1^0, T_2^0, T_2^{2c} and T_2^{2s} types. It then follows that

$$\left.\begin{aligned} \vec{H}_0&=\vec{H}_1+\vec{H}_2+\vec{H}_3+\vec{H}_4\,, \\ \vec{V}_0&=\vec{V}_1+\vec{V}_2\,, \end{aligned}\right\} \tag{17}$$

where

$$\left.\begin{aligned}
\vec{H}_1&=\begin{cases} -2S_1(r)P_1 \\ -\left(r\dfrac{dS_1}{dr}+2S_1\right)\dfrac{dP_1}{d\theta} \\ 0\,, \end{cases} \\
\operatorname{curl}\vec{H}_1&=\begin{cases} 0 \\ 0 \\ -\left(r\dfrac{d^2S_1}{dr^2}+4\dfrac{dS_1}{dr}\right)\dfrac{dP_1}{d\theta} \end{cases} \\
\nabla^2\vec{H}_1&=\begin{cases} -2\left(\dfrac{d^2S_1}{dr^2}+\dfrac{4}{r}\dfrac{dS_1}{dr}\right)P_1 \\ -\left(r\dfrac{d^3S_1}{dr^3}+6\dfrac{d^2S_1}{dr^2}+\dfrac{4}{r}\dfrac{dS_1}{dr}\right)\dfrac{dP_1}{d\theta} \\ 0\,, \end{cases}
\end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned}
\vec{H}_2&=\begin{cases} 0 \\ 0 \\ (r/a)^2\,T_2(r)\dfrac{dP_2}{d\theta}\,, \end{cases} \\
\operatorname{curl}\vec{H}_2&=\begin{cases} -6a^{-2}rT_2P_2 \\ -a^{-2}\left(r\dfrac{dT_2}{dr}+3T_2\right)r\dfrac{dP_2}{d\theta} \\ 0\,, \end{cases} \\
\nabla^2\vec{H}_2&=\begin{cases} 0 \\ 0 \\ a^{-2}\left(\dfrac{d^2T_2}{dr^2}+\dfrac{6}{r}\dfrac{dT_2}{dr}\right)r^2\dfrac{dP_2}{d\theta}\,, \end{cases}
\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}
\vec{H}_3 &= \begin{cases} 0 \\ -(r/a)^2 T_3(r)\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi} \\ (r/a)^2 T_3(r)\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\,, \end{cases} \\
\operatorname{curl}\vec{H}_3 &= \begin{cases} -6a^{-2}rT_3P_2^2\cos 2\phi \\ -a^{-2}\left(r\dfrac{dT_3}{dr}+3T_3\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta} \\ -a^{-2}\left(r\dfrac{dT_3}{dr}+3T_3\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\,, \end{cases} \\
\nabla^2\vec{H}_3 &= \begin{cases} 0 \\ -a^{-2}\left(\dfrac{d^2T_3}{dr^2}+\dfrac{6}{r}\dfrac{dT_3}{dr}\right)r^2\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi} \\ a^{-2}\left(\dfrac{d^2T_3}{dr^2}+\dfrac{6}{r}\dfrac{dT_3}{dr}\right)r^2\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\,, \end{cases}
\end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned}
\vec{H}_4 &= \begin{cases} 0 \\ -(r/a)^2 T_4(r)\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi} \\ (r/a)^2 T_4(r)\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta}\,, \end{cases} \\
\operatorname{curl}\vec{H}_4 &= \begin{cases} -6a^{-2}rT_4P_2^2\sin 2\phi \\ -a^{-2}\left(r\dfrac{dT_4}{dr}+3T_4\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta} \\ -a^{-2}\left(r\dfrac{dT_4}{dr}+3T_4\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi}\,, \end{cases} \\
\nabla^2\vec{H}_4 &= \begin{cases} 0 \\ -a^{-2}\left(\dfrac{d^2T_4}{dr^2}+\dfrac{6}{r}\dfrac{dT_4}{dr}\right)r^2\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi} \\ a^{-2}\left(\dfrac{d^2T_4}{dr^2}+\dfrac{6}{r}\dfrac{dT_4}{dr}\right)r^2\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta}\,, \end{cases}
\end{aligned}\right\} \tag{21}$$

$$\vec{V}_1 = \begin{cases} 0 \\ 0 \\ (r/a)V_1(r)\dfrac{dP_1}{d\theta}\,, \end{cases} \tag{22}$$

$$
\vec{V}_2 = \begin{cases} -6(r/a)V_2 P_2^2 \cos 2\phi \\ -(r/a)\left(r\dfrac{dV_2}{dr} + 3V_2\right)\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta} \\ -(r/a)\left(r\dfrac{dV_2}{dr} + 3V_2\right)\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi} \, . \end{cases} \tag{23}
$$

With the above expressions, the first three terms of the righthand-side of (15) are calculated as follows;

$$
\begin{aligned}
\vec{H}_0\cdot\nabla^2\vec{h} &= \frac{2}{3}\left\{2S_1\left(\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)+\left(r\frac{dS_1}{dr}+2S_1\right)\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)\right\} \\
&\quad+\frac{2}{3}\left\{4S_1\left(\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)-\left(r\frac{dS_1}{dr}+2S_1\right)\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)\right\}P_2 \\
&\quad+2a^{-2}T_3r^2\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)P_2^2\sin 2\phi \\
&\quad-2a^{-2}T_4r^2\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)P_2^2\cos 2\phi \, ,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\vec{h}\cdot\nabla^2\vec{H}_0 &= \frac{2}{3}\left\{2\left(\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)s+\left(r\frac{d^3S_1}{dr^3}+6\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)\left(r\frac{ds}{dr}+2s\right)\right\} \\
&\quad+\frac{2}{3}\left\{4\left(\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)s-\left(r\frac{d^3S_1}{dr^3}+6\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)\left(r\frac{ds}{dr}+2s\right)\right\}P_2 \\
&\quad+2a^{-2}\left(r^2\frac{d^2T_3}{dr^2}+6r\frac{dT_3}{dr}\right)\left(r\frac{ds}{dr}+2s\right)P_2^2\sin 2\phi \\
&\quad-2a^{-2}\left(r^2\frac{d^2T_4}{dr^2}+6r\frac{dT_4}{dr}\right)\left(r\frac{ds}{dr}+2s\right)P_2^2\cos 2\phi \, ,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\operatorname{curl}\vec{h}\cdot\operatorname{curl}\vec{H}_0 &= \frac{2}{3}\left(r\frac{d^2S_1}{dr^2}+4\frac{dS_1}{dr}\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right) \\
&\quad-\frac{2}{3}\left(r\frac{d^2S_1}{dr^2}+4\frac{dS_1}{dr}\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2 \\
&\quad+2a^{-2}\left(r^2\frac{dT_3}{dr}+3rT_3\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2^2\sin 2\phi \\
&\quad-2a^{-2}\left(r^2\frac{dT_4}{dr}+3rT_4\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2^2\cos 2\phi \, .
\end{aligned} \tag{26}
$$

We thus see that the electromagnetic force considered gives rise to pressure of which the distribution is described with spherical surface harmonics P_0, P_2, $P_2^2\cos 2\phi$ and $P_2^2\sin 2\phi$.

In the next place, we have to examine the role of $\vec{\omega}_0\cdot\mathrm{curl}\,\vec{v}$. Since the velocities that give rise to the S_1^0 magnetic field through curl $(\vec{v}\wedge\vec{H}_0)$ in (13) are only the S_2^0, S_2^{2c} and S_2^{2s} types, it is sufficient to consider the following velocities;

$$\vec{v}=\vec{v}_1+\vec{v}_2+\vec{v}_3\ , \tag{27}$$

where

$$\vec{v}_1=\begin{cases} -6\xi_2(r)rP_2 \\ -\left(r\dfrac{d\xi_2}{dr}+3\xi_2\right)r\dfrac{dP_2}{d\theta} \\ 0\ , \end{cases} \tag{28}$$

$$\vec{v}_2=\begin{cases} -6\xi_2^{2c}(r)rP_2^2\cos 2\phi \\ -\left(r\dfrac{d\xi_2^{2c}}{dr}+3\xi_2^{2c}\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta} \\ -\left(r\dfrac{d\xi_2^{2c}}{dr}+3\xi_2^{2c}\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\ , \end{cases} \tag{29}$$

$$\vec{v}_3=\begin{cases} -6\xi_2^{2s}(r)rP_2^2\sin 2\phi \\ -\left(r\dfrac{d\xi_2^{2s}}{dr}+3\xi_2^{2s}\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta} \\ -\left(r\dfrac{d\xi_2^{2s}}{dr}+3\xi_2^{2s}\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi}\ . \end{cases} \tag{30}$$

In that case, we obtain

$$\left.\begin{aligned} &\vec{\omega}_0\cdot\mathrm{curl}\,\vec{v}_1=0\ , \\ &\vec{\omega}_0\cdot\mathrm{curl}\,\vec{v}_2=2\omega r^2\left(\frac{d^2\xi_2^{2c}}{dr^2}+\frac{6}{r}\,\frac{d\xi_2^{2c}}{dr}\right)P_2^2\sin 2\phi\ , \\ &\vec{\omega}_0\cdot\mathrm{curl}\,\vec{v}_3=-2\omega r^2\left(\frac{d^2\xi_2^{2s}}{dr^2}+\frac{6}{r}\,\frac{d\xi_2^{2s}}{dr}\right)P_2^2\cos 2\phi\ . \end{aligned}\right\} \tag{31}$$

Finally, $\vec{g}$ may be assumed as

$$\vec{g}=\vec{g}_1+\vec{g}_2+\vec{g}_3\ , \tag{32}$$

where

$$\left.\begin{aligned}
\vec{g}_1 &= \begin{cases} F_2(r)P_2 \\ 0 \\ 0, \end{cases} \\
\vec{g}_2 &= \begin{cases} F_2^{2c}(r)P_2^2 \cos 2\phi \\ 0 \\ 0, \end{cases} \\
\vec{g}_3 &= \begin{cases} F_2^{2s}(r)P_2^2 \sin 2\phi \\ 0 \\ 0, \end{cases}
\end{aligned}\right\} \tag{33}$$

while forces of other types have nothing to do with the present problem.

Introducing (24), (25), (26), (31) and (33) to the righthand-side of (15), we obtain the differential equation for the pressure as follows;

$$\nabla^2 p = f_0 + f_2 P_2 + f_2^{2c} P_2^2 \cos 2\phi + f_2^{2s} P_2^2 \sin 2\phi \tag{34}$$

where

$$\left.\begin{aligned}
f_0 &= -(4\pi)^{-1}\frac{4}{3}S_1\left(r\frac{d^3s}{dr^3}+7\frac{d^2s}{dr^2}+\frac{8}{r}\frac{ds}{dr}\right), \\
f_2 &= -(4\pi)^{-1}\frac{4}{3}S_1\left(r\frac{d^3s}{dr^3}+4\frac{d^2s}{dr^2}-\frac{4}{r}\frac{ds}{dr}\right)+\frac{d(r^2F_2)}{r^2dr}, \\
\begin{pmatrix} f_2^{2c} \\ f_2^{2s} \end{pmatrix} &= \pm(4\pi)^{-1}2(r/a)^2\begin{pmatrix} T_4 \\ T_3 \end{pmatrix}\left(r\frac{d^3s}{dr^3}+12\frac{d^2s}{dr^2}+\frac{28}{r}\frac{ds}{dr}\right) \\
&\quad \mp 4\rho\omega\begin{pmatrix} \dfrac{d^2\xi_2^{2s}}{dr^2}+\dfrac{6}{r}\dfrac{d\xi_2^{2s}}{dr} \\ \dfrac{d^2\xi_2^{2c}}{dr^2}+\dfrac{6}{r}\dfrac{d\xi_2^{2c}}{dr} \end{pmatrix}+\begin{pmatrix} \dfrac{d(r^2F_2^{2c})}{r^2dr} \\ \dfrac{d(r^2F_2^{2s})}{r^2dr} \end{pmatrix},
\end{aligned}\right\} \tag{35}$$

in which it is assumed, for the sake of simplicity, that S_1, T_3, and T_4 are independent of r.

The solution of (34) can be expressed as

$$p = \sum_{n,m} q_n^m(r) Y_n^m , \tag{36}$$

where

$$Y_n^m = P_n^m(\cos\theta) \begin{matrix}\cos \\ \sin\end{matrix}\, m\phi .$$

Since $\nabla^2 p$ can be written as

$$\nabla^2 p=\sum_{n,m}\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{n(n+1)}{r^2}\right\}q_n^m(r)Y_n^m\ ,$$

we have a set of ordinary differential equations such as

$$\left.\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)q_0^0=f_0\ ,\\&\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{6}{r^2}\right\}\begin{pmatrix}q_2^0\\q_2^{2c}\\q_2^{2c}\end{pmatrix}=\begin{pmatrix}f_2\\f_2^{2c}\\f_2^{2s}\end{pmatrix}.\end{aligned}\right\}\qquad(37)$$

The solutions of (37) which remain finite at $r=0$ are given as

$$\left.\begin{aligned}q_0^0=&(4\pi)^{-1}\left\{L_0^0-\frac{4}{3}S_1\left(r\frac{ds}{dr}+3s\right)\right\}\ ,\\q_2^0=&(4\pi)^{-1}\left\{(r/a)^2L_2^0+\frac{4}{3}S_1r\frac{ds}{dr}\right\}\\&+\frac{1}{5}\left\{3r^2\int_0^r r^{-2}F_2dr+2r^{-3}\int_0^r r^3F_2dr\right\}\ ,\\\begin{pmatrix}q_2^{2c}\\q_2^{2s}\end{pmatrix}=&(4\pi)^{-1}\left\{(r/a)^2\begin{pmatrix}L_2^{2c}\\L_2^{2s}\end{pmatrix}\pm\begin{pmatrix}T_4\\T_3\end{pmatrix}\left(r^3\frac{ds}{dr}+4r^2s-2r^{-3}\int_0^r r^4s\,dr\right)\right\}\\&\mp 4\rho\omega r^2\begin{pmatrix}\xi_2^{2s}\\\xi_2^{2c}\end{pmatrix}+\frac{1}{5}\left\{3r^2\int_0^r r^{-2}\begin{pmatrix}F_2^{2c}\\F_2^{2s}\end{pmatrix}dr+2r^{-3}\int_0^r r^3\begin{pmatrix}F_2^{2c}\\F_2^{2s}\end{pmatrix}dr\right\}\ ,\end{aligned}\right\}\qquad(38)$$

where L_0^0, L_2^0, L_2^{2c} and L_2^{2s} are constants which shall be determined by the boundary conditions.

4. Velocity and magnetic field

Since we can obtain the solution for pressure, we go back to (14) in which grad p is calculated from (36) together with (38).

If we ignore the T_1^0, T_3^0, T_3^{2c} and T_3^{2s} velocities which are derived from curl $\vec{H}_0\wedge\vec{h}$, (14) leads to the equations for the T_2^0, T_2^{2c} and T_2^{2s} velocities which are readily obtained as follows;

$$D\rho\xi_2=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^0-\frac{2}{3}a^2S_1r^{-1}\frac{ds}{dr}\right)+\frac{1}{5}\left(\int_0^r r^{-2}F_2dr-r^{-5}\int_0^r r^3F_2\,dr\right)\ ,\qquad(39)$$

$$D\rho\xi_2^{2c}=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2c}+2T_4r^{-5}\int_0^r r^4 s dr\right)-\frac{2}{3}\rho\omega\left(r\frac{d\xi_2^{2s}}{dr}+3\xi_2^{2s}\right)$$

$$+\frac{1}{5}\left(\int_0^r r^{-2}F_2^{2c}dr-r^{-5}\int_0^r r^3F_2^{2c}dr\right), \quad (40)$$

$$D\rho\xi_2^{2s}=(4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2s}-2T_3r^{-5}\int_0^r r^4 s dr\right)+\frac{2}{3}\rho\omega\left(r\frac{d\xi_2^{2c}}{dr}+3\xi_2^{2c}\right)$$

$$+\frac{1}{5}\left(\int_0^r r^{-2}F_2^{2s}dr-r^{-5}\int_0^r r^3F_2^{2s}dr\right). \quad (41)$$

It is also easily seen that the pressure gradient which does not depend on θ and ϕ is cancelled by the electromagnetic force derived from curl $\vec{h}\wedge\vec{H}_1$. L_2^0, L_2^{2c} and L_2^{2s} are to be determined by the conditions that the normal component of the velocities should vanish at $r=0$.

If we calculate curl $(\vec{V}_0\wedge\vec{h})$ and curl $(\vec{v}\wedge\vec{H}_0)$ and pick up only the S_1^0 magnetic field, (13) gives

$$Ds-(4\pi\sigma)^{-1}\left(\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)=-\frac{6}{5}S_1\left(r\frac{d\xi_2}{dr}+5\xi_2\right)-\frac{216}{5}a^{-2}r^2(T_4\xi_2^{2c}-T_3\xi_2^{2s}). \quad (42)$$

The equations (39), (40), (41) and (42) can be regarded as the simultaneous integro-differential equations for ξ_2, ξ_2^{2c}, ξ_2^{2s} and s.

5. Approximate solution of the integro-differential equations

Since the integro-differential equations are complicated, it seems unlikely that exact solutions can be obtained. As has been done in the previous papers[1),3)], let us assume

$$s=\sum_n a_n(r/a)^n,\ \xi_2^{2c}=\sum_n A_n(r/a)^n,\ \xi_2^{2s}=\sum_n B_n(r/a)^n. \quad (43)$$

We shall further assume $F_2=F_2^{2s}=0$. Since the steady state includes the S_2^{2c} motion, the assumption may be justified if we emphasize small fluctuations of the force which drives only the S_2^{2c} motion. F_2^{2c} is also assumed to be expressed as

$$F_2^{2c}=\sum_n \zeta_n(r/a)^{n+1}. \quad (44)$$

When we introduce (43) into (42) from which ξ_2 has been eliminated by use of (39), we obtain

$$4\pi\rho D^2 a^2 \sum_n a_n\left(\frac{r}{a}\right)^n - \rho D\sigma^{-1}\sum_n n(n+3)a_n\left(\frac{r}{a}\right)^{n-2}$$
$$= -\frac{6}{5}S_1\left(\frac{5}{3}L_2^0 - \frac{2}{3}S_1\sum_n n(n+3)\left(\frac{r}{a}\right)^{n-2}\right)$$
$$-\frac{216}{5}(4\pi\rho D a^2)\left(T_4\sum_n A_n\left(\frac{r}{a}\right)^{n+2} - T_3\sum_n B_n\left(\frac{r}{a}\right)^{n+2}\right). \tag{45}$$

From (40) and (41), we also have

$$D\rho\sum_n A_n\left(\frac{r}{a}\right)^n = (4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2c} + 2T_4\sum_n \frac{a_n}{n+5}\left(\frac{r}{a}\right)^n\right)$$
$$-\frac{2}{3}\rho\omega\sum_n (n+3)B_n\left(\frac{r}{a}\right)^n + a^{-1}\sum_n \frac{\zeta_n}{n(n+5)}\left(\frac{r}{a}\right)^n, \tag{46}$$

$$D\rho\sum_n B_n\left(\frac{r}{a}\right)^n = (4\pi a^2)^{-1}\left(\frac{1}{3}L_2^{2s} - 2T_3\sum_n \frac{a_n}{n+5}\left(\frac{r}{a}\right)^n\right)$$
$$+\frac{2}{3}\rho\omega\sum_n (n+3)A_n\left(\frac{r}{a}\right)^n, \tag{47}$$

in which we can write L_2^0, L_2^{2c} and L_2^{2s} in a form of series with respect to a_n by applying the boundary conditions. Eliminating L_2^0, L_2^{2c} and L_2^{2s}, the above three equations become

$$4\pi\rho D^2 a^2 \sum_n a_n\left(\frac{r}{a}\right)^n - \rho D\sigma^{-1}\sum_n n(n+3)a_n\left(\frac{r}{a}\right)^{n-2}$$
$$= -\frac{4}{5}S_1^2\sum_n a_n\left\{5-(n+3)\left(\frac{r}{a}\right)^{n-2}\right\}n$$
$$-\frac{216}{5}(4\pi\rho D a^2)\sum_n (T_4 A_n - T_3 B_n)\left(\frac{r}{a}\right)^{n+2}, \tag{48}$$

$$4\pi\rho D a^2\sum_n A_n\left(\frac{r}{a}\right)^n = 2T_4\sum_n \frac{a_n}{n+5}\left\{\left(\frac{r}{a}\right)^n - 1\right\}$$
$$-\frac{2}{3}(4\pi\rho\omega a^2)\sum_n B_n\left\{(n+3)\left(\frac{r}{a}\right)^n - n\right\}$$
$$+4\pi a\sum_n \frac{\zeta_n}{n(n+5)}\left\{\left(\frac{r}{a}\right)^n - 1\right\}, \tag{49}$$

$$4\pi\rho Da^2 \sum_n B_n\left(\frac{r}{a}\right)^n = -2T_3 \sum_n \frac{a_n}{n+5}\left\{\left(\frac{r}{a}\right)^n - 1\right\}$$

$$+\frac{2}{3}(4\pi\rho\omega a^2)\sum_n A_n\left\{(n+3)\left(\frac{r}{a}\right)^n - n\right\}. \tag{50}$$

By equating the coefficients of the corresponding terms of both the sides of these equations, we obtain

$$\left.\begin{aligned}
&4\pi\rho D^2a^2a_0 - 10\rho D\sigma^{-1}a_2 = -16S_1^2a_4\,,\\
&4\pi\rho D^2a^2a_2 - 28\rho D\sigma^{-1}a_4 = \frac{112}{5}S_1^2a_4 - \frac{216}{5}(4\pi\rho a^2D)(T_4A_0 - T_3B_0)\,,\\
&4\pi\rho Da^2A_0 = -2T_4\left(\frac{a_2}{7}+\frac{a_4}{9}\right) - \frac{2}{3}(4\pi\rho\omega a^2)(3B_0 - 2B_2 - 4B_4)\\
&\qquad\qquad - 4\pi a\left(\frac{\zeta_2}{14}+\frac{\zeta_4}{36}\right),\\
&4\pi\rho Da^2A_2 = 2T_4\frac{a_2}{7} - \frac{10}{3}(4\pi\rho\omega a^2)B_2 + 4\pi a\frac{\zeta_2}{14}\,,\\
&4\pi\rho Da^2A_4 = 2T_4\frac{a_4}{9} - \frac{14}{3}(4\pi\rho\omega a^2)B_4 + 4\pi a\frac{\zeta_4}{36}\,,\\
&4\pi\rho Da^2B_0 = 2T_3\left(\frac{a_2}{7}+\frac{a_4}{9}\right) + \frac{2}{3}(4\pi\rho\omega a^2)(3A_0 - 2A_2 - 4A_4)\,,\\
&4\pi\rho Da^2B_2 = -2T_3\frac{a_2}{7} + \frac{10}{3}(4\pi\rho\omega a^2)A_2\,,\\
&4\pi\rho Da^2B_4 = -2T_3\frac{a_4}{9} + \frac{14}{3}(4\pi\rho\omega a^2)A_4\,,
\end{aligned}\right\} \tag{51}$$

where a_n's, A_n's, B_n's and ζ_n's for $n>4$ are ignored. It is easily seen that a_n's, A_n's, B_n's and ζ_n's for odd n vanish. It should be also noted that ζ_0 is taken to be zero in order to avoid the term for $n=0$ which otherwise becomes indeterminate.

We have one more equation

$$3a_0 + 5a_2 + 7a_4 = 0\,, \tag{52}$$

which is deduced from the condition that the magnetic field is continuous at $r=a$.

Since the force is regarded as that caused by buoyancy, it must be zero at the boundary of the core. This implies $\zeta_2+\zeta_4=0$ for the present approximation, so that

$$F_2^{2c}=\zeta_2\left(\frac{r}{a}\right)^3\left\{1-\left(\frac{r}{a}\right)^2\right\} . \tag{53}$$

Now we are in a position to solve (51) together with (52) in regard to a_0, a_2, a_4, A_0, A_2, A_4, B_0, B_2 and B_4. After some calculations we obtain

$$\left.\begin{aligned}
a\rho\Phi(D)a_0 &= \kappa\Big[\frac{54}{7}\Big(DT_4-\frac{10}{3}\omega T_3\Big)\Big(D^2+\Big(\frac{14}{3}\omega\Big)^2\Big) \\
&\qquad -3\Big(DT_4-\frac{14}{3}\omega T_3\Big)\Big(D^2+\Big(\frac{10}{3}\omega\Big)^2\Big)\Big]\zeta_2 , \\
a_2 &= \frac{2}{5}\kappa^{-1}Da_0 , \\
a_4 &= -\Big(\frac{3}{7}+\frac{2}{7}\kappa^{-1}D\Big)a_0 ,
\end{aligned}\right\} \tag{54}$$

where

$$\begin{aligned}
\Phi(D) &= D^6+5\kappa D^5+\Big(\frac{296}{9}\omega^2-\frac{48}{35}\frac{T_3^2+T_4^2}{\pi\rho a^2}+\frac{15}{2}\kappa^2\Big)D^4 \\
&\quad +\kappa\Big(\frac{1480}{9}\omega^2+\frac{18}{7}\frac{T_3^2+T_4^2}{\pi\rho a^2}\Big)D^3 \\
&\quad +\Big(\frac{19600}{81}\omega^2-\frac{15168}{315}\frac{T_3^2+T_4^2}{\pi\rho a^2}+\frac{740}{3}\kappa^2\Big)\omega^2D^2 \\
&\quad +\kappa\Big(\frac{98000}{81}\omega^2+\frac{200}{7}\frac{T_3^2+T_4^2}{\pi\rho a^2}\Big)\omega^2D+\frac{16000}{9}\kappa^2\omega^4
\end{aligned} \tag{55}$$

and

$$\kappa=(\pi\sigma a^2)^{-1} . \tag{56}$$

In the calculation, S_1 is assumed as zero, because, in the earth's dynamo, the toroidal fields are believed to be larger than the poloidal one.

The equations (54) seem still complicated. We shall make further simplifications on the basis of physical consideration. We may first put

$$T_3=T_4=T \tag{57}$$

for the order-of-magnitude estimate. Taking $\omega=7.5\times10^{-5}\,sec^{-1}$, $a=3.5$

$\times 10^8\,cm$ and $\rho=10\,g/cm^3$, we see that the relation

$$\omega^2 \gg \frac{T^2}{\pi\rho a^2} \tag{58}$$

holds for $T<10^5\,gauss$. Since T is thought to amount to the order of $10\,gauss$ in the core, (58) is safely assumed.

If $\sigma=10^{-6}\,e.m.u.$, which is usually believed for the core, we have

$$\kappa=2.6\times 10^{-12}\,sec^{-1}\,,$$

hence

$$\omega \gg \kappa\,. \tag{59}$$

Let us consider a variation of a 100-*year* period. In that case, we may put

$$D=i\alpha \quad \alpha=2.0\times 10^{-9}\,sec^{-1}\,, \tag{60}$$

whence we see that

$$\alpha \gg \kappa\,. \tag{61}$$

Taking into account (57), (58), (59) and (61), the solutions can be simplified as

$$\left.\begin{aligned} a_0&=0\,,\\ a_2&=\frac{6T}{5a\rho}\,\frac{\frac{11}{7}D^3-\frac{82}{21}\omega D^2+\frac{404}{9}\omega^2 D-\frac{3640}{27}\omega^3}{D\left(D^4+\frac{296}{9}\omega^2D^2+\frac{19600}{81}\omega^4\right)}\,\zeta_2\,,\\ a_4&=-\frac{5}{7}a_2\,. \end{aligned}\right\} \tag{62}$$

If ζ_2 is given as a function of time, the second equation of (62) is the differential equation of forced oscillation for a_2. As is well known in the case of forced oscillation of a simple pendulum, a_2 is obtained by putting $D=i\alpha$ when the free oscillations are damped out. Assuming a 100-*year* period, we obtain

$$|a_2|=1.3\times 10^3\,T\zeta_2 \tag{63}$$

where $|a_2|$ denotes the modulus of a_2, while a and ρ are taken as before.

6. Driving force and thermal field

Now it is possible from (63) to obtain the coefficients of the driving force which causes the 100-*year* period secular variation of the S_1^0 magnetic field. (63) leads to

$$\zeta_2 = 7.9 \times 10^{-4} \frac{|a_2|}{T} . \tag{64}$$

In the following, let us estimate the order of ζ_2. First of all, $|a_2|$ should be estimated. According to the repeated analyses of the earth's magnetic field, it is well known that the strength of the earth's dipole underwent a decrease amounting to 0.5×10^{25} *e.m.u.* during the past 100 *years*. Although it is not known whether or not the change is periodic, we may, for a rough estimate, suppose that there is a 100-*year* period secular variation of which the amplitude is 0.01 *gauss* at the pole. In order to have this order of magnetic field at the earth's surface, an amplitude of 0.07 *gauss* should take place at the core-mantle boundary. This figure must become slightly larger if we take the shielding effect of the mantle into account. Accordingly, s in (16) takes a value around 0.09 *gauss*. From the third relation of (62), a_2 is then estimated at 0.3 *gauss*.

As for the steady magnetic fields of the earth's dynamo, Bullard[2] estimated that the T_2^0 field amounts to 480 *gauss*. On the other hand the S_1^0 field would be only a few *gauss*. The T_2^{2c} and T_2^{2s} fields would take intermediate intensities. From these considerations, a probable estimate of $|a_2|/T$ would be

$$|a_2|/T \sim 10^{-2} , \tag{65}$$

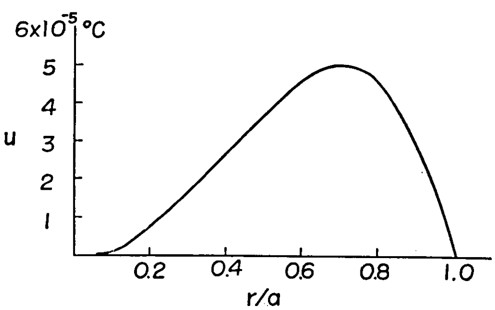

Fig. 1. The distribution of u.

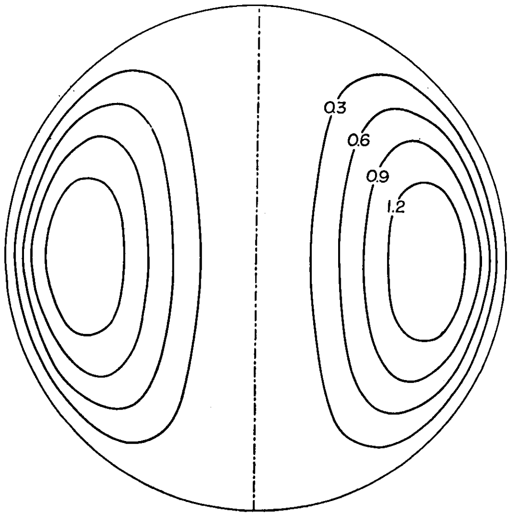

Fig. 2. The distribution pattern of temperature departure in the $\phi=0$ meridian plane in units of 10^{-4}°C.

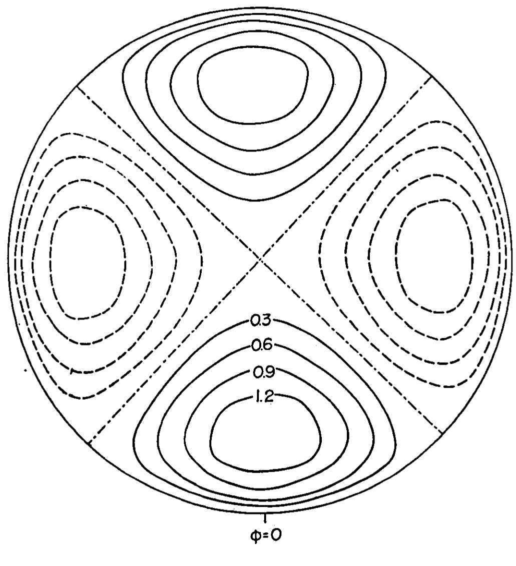

Fig. 3. The distribution pattern of temperature departure in the equatorial plane in units of 10^{-4}°C.

so that (64) gives

$$\zeta_2 \sim 8\times 10^{-6}\, dyn/cm^3 \qquad (66)$$

and, from (53), we obtain

$$F_2^{2c} \sim 8\times 10^{-6}\left(\frac{r}{a}\right)^3\left\{1-\left(\frac{r}{a}\right)^2\right\} dyn/cm^3\,. \qquad (67)$$

If we identify the driving force as the buoyancy one due to the temperature inequality in the core, F_2^{2c} is given by

$$F_2^{2c} = \rho\gamma u r \qquad (68)$$

as discussed by T. Namikawa[4], where u is the temperature departure from the steady state and

$$\gamma = \frac{4}{3}\pi\rho k^2\alpha'\,. \qquad (69)$$

k^2 and α' denote respectively the universal constant of gravitation and coefficient of cubical expansion, the latter being estimated by Bullard[2] at $4.5\times 10^{-6\circ}/C$. From (67) and (68) together with (69), u responsible for the secular variation considered is obtained as

$$u \sim 2\times 10^{-4}\left(\frac{r}{a}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\right\}{}^\circ C\,, \qquad (70)$$

so that the periodic thermal field is given by

$$uP_2^2(\cos\theta)\cos 2\phi\,.$$

The distribution of u against r/a is shown in Fig. 1. Meanwhile the distributions of temperature departure in the meridian plane $\phi=0$ and equatorial plane are respectively shown in Figs. 2 and 3.

7. Discussion

Bullard[2] has estimated for his steady dynamo that the temperature difference between the top and bottom of the core would be $450^\circ C$. However, the difference in temperature between rising and falling convective currents has been also estimated by him at $3\times 10^{-4\circ}C$ from the consideration that the buoyancy forces must be comparable with the

4) T. NAMIKAWA, *Journ. Geomagn. Geoelectr.*, **9** (1957), 182.

electromagnetic forces. The figure was also approved by examining thermodynamical efficiency of the dynamo. Even if we assume that most of the heat generated in the core is carried by convection, the temperature difference amounts only to $6 \times 10^{-3} {}^\circ C$[5].

It is noticeable that the magnitude of the time-dependent thermal field studied above is of the same order as the temperature difference of the steady dynamo. Hence, it might be possible to expect such a fluctuation in the thermal field. In that case, there arises a time-dependent force due to buoyancy, which drives the fluid motion causing the geomagnetic secular variation as we have been discussing in the foregoing sections. If we consider a secular variation of longer period, 1000 *years* say, the amplitude of the thermal field would become smaller.

The magnetic lines of force of the secular variation are easily obtained with the coefficients given in (62). In Fig. 4 are shown the magnetic lines of force of the S_1^0 field ignoring the shielding effect in the mantle. The pattern should grow and diminish with a 100-*year* period. It is also possible to obtain the fluid motions though no attempts of that sort have been made.

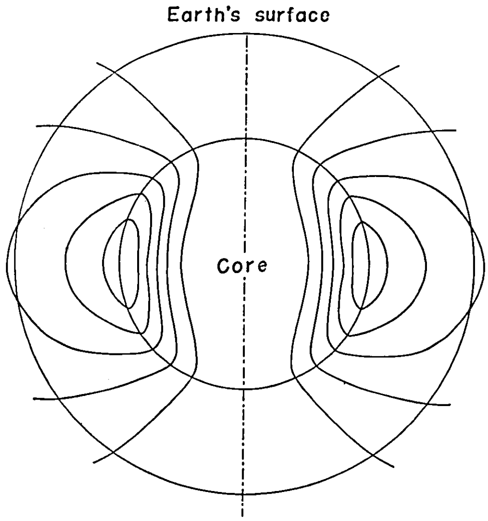

Fig. 4. The magnetic lines of force of the S_1^0 field.

No account is taken of the cause of the time-dependent thermal field supposed here. It might be due to oscillations caused by couplings

5) E. C. BULLARD, *Proc. Roy. Soc. London A*, **197** (1949), 433.

between magnetic fields, fluid motions and heat conduction. This point should be studied later.

8. Concluding remarks

A theory that the geomagnetic secular variations might be caused by forced oscillations of magneto-hydrodynamical nature is tried in this paper. The oscillations are treated as small perturbations superposing on the steady state of the earth's dynamo which is at work in the core. From an extremely simplified treatment, the buoyancy force which gives rise to the hypothetical 100-*year* period secular variation of the S_1^0 field is obtained. If the amplitude of the S_1^0 field is assumed as 1000 *gammas* at the pole, the corresponding temperature departure from the steady state must amount to the order of 10^{-4}°C at its maximum. This figure may be acceptable if we take into account the thermal state of the steady dynamo as has been discussed by Bullard.

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16. 地磁気ダイナモの強制振動

地震研究所　力 武 常 次

地球回転のコリオリ力を考えると，地球核内において，電磁流体力学的自由振動は起りにくいことが筆者によつて指摘されている．したがつて地磁気永年変化に対応する核内の電磁流体力学的過程を考えるためには，時間的に変化する駆動力を考慮する必要がある．本報においては，その駆動力は温度差に起因する浮力によるという場合を考え，極において振幅 1000 ガンマ，周期 100 年の双極子磁場の永年変化を想定すると，核内には定常状態に重畳して 10^{-4}°C 程度の振幅をもつ温度差の分布があればよいという結論が得られた．
