1. *Notes on the Theory of Vibration Analyser.*

By Hirosi KAWASUMI,

Earthquake Research Institute.

(Read May 24, 1955.—Received Dec. 28, 1955.)

The spectrum of seismic waves has recently become a matter of increasing interest to seismologists and structural engineers. In view of the trouble of numerical harmonic analysis without instrumental means, other simpler means of approach to this problem have been frequently sought, among which must be mentioned the frequency curve of different periods examined by many investigators. Of late years, however, electric analog computors and other instruments for this purpose have been introduced and used conveniently. But so far as the writer is aware, the basic principles of these vibration analysers are not yet solved completely. He therefore presents some notes on this subject.

Fundamental theory

It is well known that a simple linear oscillator which is subjected to an external force serves as an analyser for the applied force-function. This is based on the fact that the solution of the equation of motion of the oscillator

$$\ddot{y}+2\varepsilon\dot{y}+n^2y=+Vx \tag{1}$$

is

$$y=\frac{-V}{2\pi}\int_{-\infty}^{\infty}\frac{X(\omega)e^{i\omega t}}{(\omega-\alpha)(\omega-\beta)}d\omega \tag{2}$$

provided

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-i\omega t}dt=A-iB \tag{3}$$

and

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=i\varepsilon\pm\gamma,\quad \gamma=\sqrt{n^2-\varepsilon^2}$$

where $X(\omega)$ is the so called spectrum function of $x(t)$.

In the case when $X(\omega)$ satisfies the Paley-Wiener criterion (that is $X(\omega)$, considered as a function of a complex variable ω, is analytic in the half plane defined by $\mathrm{Im}(\omega)<0$) as in the physically realisable filters in the information theory, then (2) is integrated into

$$y=\frac{-iV}{2\gamma}\{X(\alpha)e^{i\alpha t}-X(\beta)e^{i\beta t}\}-iV\sum_j Y(\omega_j)\frac{\varphi(\omega_j)}{\Phi'(\omega_j)}e^{i\omega_j t} \tag{4}$$

where

$$Y(\omega)=\frac{1}{(\omega-\alpha)(\omega-\beta)},\quad X(\omega)=\frac{\varphi(\omega)}{\Phi(\omega)} \tag{5}$$

and $\Phi(\omega_j)=0$. From the above assumption, Im $(\omega_j)>0$, and $\lim_{t\to\infty} e^{i\omega_j t}=0$. It follows when $\varepsilon\to 0$ and t is large

$$y=+\frac{V}{n}(A\sin nt-B\cos nt) \tag{6}$$

Thus we perceive that the amplitude of this oscillator in this stage

$$|y|=\frac{V}{n}\sqrt{A^2+B^2}=\frac{V}{n}|X(n)| \tag{7}$$

is the function of circular frequency n of the oscillator, and represents the spectrum of $\int_0^t x(t)dt$.

On the Spectrums Obtained by the SMAC Response Analyser

By means of the above fundamental theory we can analyse the force-function $x(t)$ from the motion of a set of linear oscillators with different periods. On the other hand, instead of using many oscillators, we can also analyse the force-function with one oscillator if we can change the time rate of the variation of the force-function.

Since the spectrum-function of $x(ct)$ is

$$X'(\omega)=\int_{-\infty}^{\infty}x(c\tau)e^{-i\omega\tau}d\tau=\frac{1}{c}\int_{-\infty}^{\infty}x(c\tau)e^{-i(\omega/c)c\tau}d(c\tau)=\frac{1}{c}X\left(\frac{\omega}{c}\right) \tag{8}$$

we have by (7)

$$|y'|_m=\frac{V}{n}|X'(\omega)|=\frac{V}{n^2}\left(\frac{n}{c}\right)\left|X\left(\frac{n}{c}\right)\right| \tag{9}$$

The resulting spectrum is, therefore, that of the first derivative with respect to time of the force-function. This result applies to the SMAC response analyser and the difference of the last case from the preceding one was first noticed by R. Takahasi, the above result also being independently obtained by him.

Thus we see that a linear oscillator with no damping serves as an analyser of a continuous spectrum but not a line spectrum. For the analysis of the latter some other device must be used.

Use of an Analyser with Finite Damping

Theoretically speaking, an oscillator with finite damping cannot generally be used as an analyser in the above sense. There is, however, a case when such an oscillator can be used rigorously. It is the case of a line spectrum, to which analysis an oscillator without damping has been proved to be inapplicable owing to its behavior of resonance.

A line spectrum is here defined for a harmonic series

$$x(t)=\sum a_j \sin(p_j t+\delta_j) \tag{10}$$

as the limit when $T\to\infty$ of

$$X(t,\omega)=\frac{2}{T}\int_{t-T/2}^{t+T/2} x(\tau)e^{-i\omega\tau}d\tau$$
$$=\sum\frac{a_j}{i}\left[e^{i\{(p_j-\omega)t+\delta_j\}}Sa\left(\frac{p_j-\omega}{2}T\right)-e^{-i\{(p_j+\omega)t+\delta_j\}}Sa\left(\frac{p_j+\omega}{2}T\right)\right] \tag{11}$$

in which $Sa(x)$ is the so called sampling function $\frac{\sin x}{x}$. The spectrum function denotes the amplitude a_j and the phase δ_j in the above harmonic series.

If we apply a force $x(ct)=\sum a_j \sin(cp_j t+\delta_j)$ to an oscillator, we have from the equation (1), when t is large,

$$y=\frac{\Sigma Va_j \sin(p_j ct+\delta_j+\phi_j)}{\sqrt{\{n^2-p_j^2c^2\}^2+4\varepsilon^2p_j^2c^2}},\quad \tan\phi_j=\frac{2\varepsilon p_j c}{p_j^2c^2-n^2} \tag{12}$$

If we here assume $c=1$ and n is variable, then we observe, in the stationary state, a maximum amplitude

$$|y|_m \doteqdot \frac{Va_j}{p_j^2 2h\sqrt{1-h^2}} \tag{13}$$

when $n=p_j\sqrt{1-2h^2}$ provided $h=\varepsilon/n$. Thus a set of such damped oscillators give the spectrum of $\iint x(t)dtdt$. On the other hand if c is assumed to be variable and n is constant as in the SMAC response analyser, the maximun stationary amplitude is

$$|y'|_m=\frac{Va_j}{n^2 2h\sqrt{1-h^2}} \tag{14}$$

when $c=\sqrt{n^2-2\varepsilon^2}/p_j$. We perceive that the coefficient of a_j in $|y'|_m$ is constant so that the resulting amplitude as a function of c represents the spectrum of the force-function directly.

Temporal Spectrum at t of the Force-function

R. Takahasi inferred from the well-known solution

$$y=\frac{V}{\gamma}\int_0^t x(\tau)e^{-\varepsilon(t-\tau)}\sin\gamma(t-\tau)d\tau=\frac{V}{\gamma}\int_0^t x(t-u)e^{-\varepsilon u}\sin\gamma u du \qquad (15)$$

of (1) that the amplitude $|y(t)|$ may possibly represent the temporal or local spectrum in the neighbourhood of t of the force-function. This is based on the fact that the main contribution of $x(t-u)e^{-\varepsilon u}$ to y is confined to the portion of $x(t-u)$ near $u=0$. To verify this inference, the writer defined the local or temporal spectrum by (11), with the exception, that $T=\nu\frac{2\pi}{\omega}$, ν being finite in this case. Now if we take, for example, (i) the case when $x(t)$ is approximated by $x_0(t)\sin(pt+\delta)$ where $x_0(t)$ is assumed to be constant in the interval $t-\frac{1}{2}T$ and $t+\frac{1}{2}T$, then, by (11)

$$|X(t,\omega)|=\sqrt{|X(t,\omega)\cdot X(t,-\omega)|}$$
$$=x_0(t)\sqrt{\left|Sa^2\left(\frac{p-\omega}{2}T\right)+Sa^2\left(\frac{p+\omega}{2}T\right)-2\cos2(pt+\delta)Sa\left(\frac{p-\omega}{2}T\right)Sa\left(\frac{p+\omega}{2}T\right)\right|}$$
$$\doteqdot x_0(t)Sa\left(\frac{p-\omega}{2}T\right) \qquad (16)$$

when pT is moderately large.

Now the response amplitude of an oscillator for $x(t)$ in the last example is the same as that of the general term in (12), provided T is not too small and the proper motion caused at $t-\frac{1}{2}T$ of the oscillator, which is proportional to $e^{-\frac{1}{2}\varepsilon T}$, may be discarded. The result is therefore the same as that for a line spectrum.

(ii) Next we shall examine a more concrete example

$$x(t)=x_0\sin(at+bt^2) \qquad (17)$$

Then the solution of (1) for this force-function is

$$y=\frac{Vx_0}{4i\gamma}\left[\frac{e^{i\alpha t}}{\alpha-a}\sum_{\kappa=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{i4b}{(\alpha-a)^2}\right\}^{\kappa}-\frac{e^{i\beta t}}{\beta-a}\sum\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{i4b}{(\beta-a)^2}\right\}^{\kappa}\right.$$

$$-\frac{e^{i\alpha t}}{\alpha+a}\sum\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{-i4b}{(\alpha+a)^2}\right\}^\kappa+\frac{e^{i\beta t}}{\beta+a}\sum\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{-i4b}{(\beta+a)^2}\right\}^\kappa$$

$$-\frac{e^{i(at+bt^2)}}{\alpha-a-2bt}\sum\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{i4b}{(\alpha-a-2bt)^2}\right\}^\kappa$$

$$+\frac{e^{i(at+bt^2)}}{\beta-a-2bt}\sum\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{i4b}{(\beta-a-2bt)^2}\right\}^\kappa$$

$$+\frac{e^{-i(at+bt^2)}}{\alpha+a+2bt}\sum\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{-i4b}{(\alpha+a+2bt)^2}\right\}^\kappa$$

$$\left.-\frac{e^{-i(at+bt^2)}}{\beta+a+2bt}\sum\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\kappa\right)}\left\{\frac{-i4b}{(\beta+a+2bt)^2}\right\}^\kappa\right] \tag{18}$$

For a sufficiently large value of t at which $e^{-\varepsilon t}$ is negligible, and when $4b/\varepsilon^2$ is also negligible,

$$y \doteqdot \frac{Vx_0\sin(at+bt^2+\phi)}{\sqrt{\{(a+2bt)^2-n^2\}^2+4\varepsilon^2(a+2bt)^2}},\quad \tan\phi=\frac{2\varepsilon(a+2bt)}{(a+2bt)^2-n^2} \tag{19}$$

Here we take $t=t_0$, and assume that n is variable then the maximum response

$$|y|_m=\frac{Vx_0}{(a+2bt_0)^2 2h\sqrt{1-h^2}}$$

is proportional to the spectrum of $\iint x(t)dtdt$ as in the above example (i). The case for the SMAC response analyser is obtained by introducing ac and bc^2 into (19) for a and b respectively. We see the maximum response amplitude is

$$|y|_m=\frac{Vx_0}{n^2 2h\sqrt{1-h^2}}$$

for the time at which $act+bc^2t^2=at_0+bt_0^2$ or $t=-(a+bt_0)/bc$ when

$c=\dfrac{\sqrt{n^2-2\varepsilon^2}}{a+2bt_0}$. The maximun response amplitude is again directly proportional to the temporal spectrum x_0.

(iii) As a third example let us examine the case when $x(t)=(A+Bt+Ct^2)\sin(pt+\delta)$. The formula (15) gives the solution of (1) for this case as follows:

$$
\begin{aligned}
y=\frac{V}{4\gamma}\Big[&e^{i\alpha t}\Big\{iA\Big(\frac{1}{p-\alpha}-\frac{1}{p+\alpha}\Big)-B\Big(\frac{1}{(p-\alpha)^2}-\frac{1}{(p+\alpha)^2}\Big) \\
&\qquad -i2C\Big(\frac{1}{(p-\alpha)^3}+\frac{1}{(p+\alpha)^3}\Big)\Big\} \\
&-e^{i\beta t}\Big\{iA\Big(\frac{1}{p-\beta}-\frac{1}{p+\beta}\Big)-B\Big(\frac{1}{(p-\beta)^2}-\frac{1}{(p+\beta)^2}\Big) \\
&\qquad -i2C\Big(\frac{1}{(p-\beta)^3}+\frac{1}{(p+\beta)^3}\Big)\Big\} \\
&+e^{i(pt+\delta)}\Big\{-i(A+Bt+Ct^2)\Big(\frac{1}{p-\alpha}-\frac{1}{p-\beta}\Big) \\
&+(B+2Ct)\Big(\frac{1}{(p-\alpha)^2}-\frac{1}{(p-\beta)^2}\Big)+i2C\Big(\frac{1}{(p-\alpha)^3}-\frac{1}{(p-\beta)^3}\Big)\Big\} \\
&-e^{-i(pt+\delta)}\Big\{i(A+Bt+Ct^2)\Big(\frac{1}{p+\alpha}-\frac{1}{p+\beta}\Big) \\
&+(B+2Ct)\Big(\frac{1}{(p+\alpha)^2}-\frac{1}{(p+\beta)^2}\Big)-i2C\Big(\frac{1}{(p+\alpha)^3}-\frac{1}{(p+\beta)^3}\Big)\Big\}\Big]
\end{aligned}
$$

This equation again reduces to

$$
y\doteqdot\frac{V(A+Bt+Ct^2)}{\sqrt{(p^2-n^2)^2+4\varepsilon^2p^2}}\sin(pt+\delta+\phi)\,,\quad \tan\phi=\frac{2\varepsilon p}{p^2-n^2}
$$

when t is modarately large and B/ε and C/ε^2 are negligible. This is essentially the same as (12) or (19). and a similar conclusion follows as in example (ii).

(iv) Lastly we shall examine the case when the force is a unit impulse given at $t=t_0$. The temporal spectrum in the above definition is

$$
X(t_0,\omega)=\frac{2}{T}e^{-i\omega t_0}=\frac{\omega}{\nu\pi}e^{-i\omega t_0}
$$

while the usual spectrum function is

$$
X(\omega)=e^{-i\omega t_0}
$$

Then the motion of the oscillator is given by (4)

$$y=\frac{-iV}{2\gamma}\{e^{i\alpha(t-t_0)}-e^{i\beta(t-t_0)}\}=+\frac{V}{\gamma}e^{-\varepsilon(t-t_0)}\sin\gamma(t-t_0)$$

and the maximum response, when ε is small, by

$$|y|_m \doteqdot \frac{V}{n}e^{-\frac{\pi}{2}\frac{h}{\sqrt{1-h^2}}}\infty\frac{1}{n^2}X(t_0, n)$$

This also represents the spectrum of $\iint x(t)dtdt$.

As for the SMAC type response analyser, the spectrum function of $x(ct)$ is by (8)

$$X'(\omega)=\frac{1}{c}e^{-i\omega t_0}$$

and

$$|y|_m \doteqdot \frac{V}{n^2}\frac{n}{c}e^{-\frac{\pi}{2}\frac{h}{\sqrt{1-h^2}}}$$

which shows that

$$|y|_m \infty X(t_0, n).$$

From what we have seen we can conclude that the damped SMAC response analyser gives the temporal spectrum of the force-function $x(t)$ itself directly, while the responses of a set of damped oscillators give the temporal spectrum of $\iint x(t)dtdt$.

On G. W. Housner's Definition of the Spectrum of an Earthquake

From the above fundamental theory we may conclude, in a case when a seismometer or a one mass structure with no damping is subjected to an earthquake acceleration $x(t)$, the final amplitude y, after the earthquake is over, represents the spectrum function of the earthquake velocity. Cautious readers will have already become aware of the seeming contradiction of the above inference with G. W. Housner's denominations of the displacement, velocity and acceleration spectrum of an earthquake for y, ny, and n^2y respectively, where y is the response amplitude of the vibration analyser. His omission of the words "motion of a structure due to" before "an earthquake" is presumed to be for the sake of avoiding clumsiness or a matter of the nuance of the word "an earthquake".

Housner's idea of response spectrum is correct and very useful in earthquake engineering, and his use of the maximum amplitude for the spectrum is also correct in itself, although the amplitude of the remanent or final motion of the oscillator after the earthquake must be read in order to obtain the spectrum of the force-function from the mathematical or physical point of view.

---

1. 振動解析器の理論について

地震研究所 河角 廣

最近振動の解析が理工学の各分野に於て盛んに行われるようになり，その為に簡便な共振系が利用されているが，その理論に就いては必ずしも明かにされて居ない点がある．次にその 2〜3 に就いて述べる．

振動系に解析すべき函数 x を力として与え

$$\ddot{y}+2\varepsilon\dot{y}+n^2y=Vx\ ,$$

そのレスポンスによつて x のスペクトル

$$X(\omega)=\int_{-\infty}^{\infty}x(\tau)e^{-i\omega\tau}d\tau$$

を求めるのであるが，その時 $\varepsilon\to0$ とし振動数 n の異る振動系（第 1 型解析器）の振幅 $|y|$ を n の函数と見れば之は $\int_0^t x(t)dt$ のスペクトルを与えるが，若し SMAC レスポンス解析器の如く一つの振動系に $x(t)$ の時間のスケールをかえて作用させ，そのレスポンス振幅 $|y'|$ を求めると，それは $\dfrac{dx}{dt}$ のスペクトルを与える．

上の理論では damping のない振動系が必要であるが，実際若し $x(t)$ が純周期函数を含む場合には damping がなくては解析が出来ない．Damping がある場合には第 1 型解析器では $|y|_m$ は $\iint xdtdt$ のスペクトルを与え，第 2 の SMAC 型のものの $|y'|_m$ は $x(t)$ のスペクトルそのものを与える．

又 $x(t)$ が純周期函数を含まない場合にも damping のある解析器の振幅 $|y(t)|$ は近似的に $x(t)$ の時刻 t の辺のスペクトルを与えると考えられる．その様なスペクトルを Temporal or local spectrum と名づけ，幾つかの例についてその有用性を示した．その結果によると，第 1 型の解析器では上の純周期函数の場合と同様に $|y(t)|_m$ は $\iint xdtdt$ のスペクトルを与え第 2 型のでは $|y'(t)|_m$ は 直接 x のスペクトルを与える事が証明出来た．

尚いう迄もない事であるが，Housner その他米国の地震工学者は地震動の加速度 x を第 1 種の解析器 ($\varepsilon=0$) にかけそのレスポンス $|y|$, $n|y|$, 及び $n^2|y|$ を夫々地震の変位，速度，及び加速度スペクトルと呼んでいるが，上の結果から見ればそれ等は夫々地震動の速度，加速度及び加速度の時間に関する一次の微係数のスペクトルである筈である．Damping のある解析器では米国の学者のいう所は正しい．

勿論 damping のない場合にも米国の学者は地震による建物の振動の変位，速度及び加速度を考えたのであるから，その意味では決して間違いではないが，我々の方で誤解のない様注意しなければならない．
