38. *Magneto-hydrodynamic Oscillations of Finite Amplitude of a Conducting Fluid Sphere.*

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Summary

Finite magneto-hydrodynamic oscillations of perfectly conducting fluid sphere are studied. No motion and no magnetic field are assumed at the equilibrium state. Ignoring higher spherical harmonic constituents, only the field that gives the dipole-like one outside the sphere is considered. Although it seems difficult to solve the non-linear integro-differential equations which are deduced from the fundamental equations of electromagnetism and fluid motion, a crude examination of the equations suggests that there exists a finite oscillation of a special type. The result also suggests that the earth's dynamo might perform a finite oscillation even though the small oscillations of the dynamo are unstable. In that case, it is of interest to note that one might expect reversals of the earth's magnetic field.

1. Introduction

Magneto-hydrodynamic oscillations of small amplitude have been investigated by M. Schwarzschild[1], V. C. A. Ferraro[2),3] and others with applications to magnetically variable stars, while the present writer[4),5] has been investigating possible oscillations of the same sort in relation to the stability of the earth's dynamo. All of these studies dealt with small oscillations about a steady state. The steady state has been assumed to be a magnetic field uniformly given from the outside to the conducting liquid body in question, or a magnetic field given by a magnetic pole at the centre of the body, or one generated by the self-exciting process with suitable fluid motions. The writer's investigation has suggested that the steady state of the earth's dynamo considered

1) M. SCHWARZSCHILD, *Ann. d'Astrophys.*, **12** (1949), 148.
2) V. C. A. FERRARO and D. J. MEMORY, *M.N.R.A.S.*, **112** (1952), 361.
3) C. PLUMPTON and V. C. A. FERRARO, *M.N.R.A.S.*, **113** (1953), 647.
4) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **33** (1955), 1.
5) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **33** (1955), 175.

by E. C. Bullard and H. Gellman[6] would be unstable for small disturbances. According to Bullard, however, the earth's dynamo might perform a finite oscillation though its steady state is unstable for small disturbances. Bullard's discussion[7] is based on possible analogies between homogeneous and homopolar dynamos, and finite oscillations of the latter are fully studied by Bullard himself. However, in order to see whether or not the supposed earth's dynamo performs oscillations of such sort, it is highly desirable to investigate the possibility of magneto-hydrodynamic oscillations of finite amplitude in the earth's core. This sort of oscillation would be, if it does occur, closely related to the hypothetical reversals of the earth's magnetic field in the past.

It will be of great difficulty to establish, in a complete form, the theory of finite magneto-hydrodynamic oscillations of a conducting body because it has been shown that even small oscillations are quite complicated. In order to render the problem tractable, it is assumed here that the fluid has no motion and no magnetic field at the equilibrium state. It is also assumed that only the S_1^0 type field may be taken into account, the fields of other types being ignored. Non-linear integro-differential equations concerning the radial part of the fluid will be obtained from Maxwell's equations and the equation of fluid motion. The equations thus obtained will be solved with rough approximation.

2. Theory

Since we assume that there are no fluid motion and no magnetic field at the equilibrium state, the fundamental equations can be written as

$$\vec{E}+\vec{V}\wedge\vec{H}=0\,, \tag{1}$$

$$\mathrm{curl}\,\vec{E}=-\partial\vec{H}/\partial t\,, \tag{2}$$

$$\mathrm{curl}\,\vec{H}=4\pi\vec{I}\,, \tag{3}$$

$$\rho\left\{\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\mathrm{grad})\vec{V}\right\}=\vec{I}\wedge\vec{H}-\mathrm{grad}\,p\,, \tag{4}$$

where $\vec{I}$, $\vec{E}$, $\vec{H}$, ρ, p and $\vec{V}$ denote respectively the electric current

6) E. C. BULLARD and H. GELLMAN, *Phil. Trans. Roy. Soc. London A*, **247** (1954), 213.

7) E. C. BULLARD, *Proc. Cambr. Phil. Soc.*, **51** (1955), 744.

density, electric field, magnetic field, density, pressure and velocity. It is also assumed that the magnetic permeability is unity in electromagnetic unit, that the non-electromagnetic force balancing the pressure gradient at the equilibrium state is not dependent on the time, and that the fluid is perfectly conducting. The fluid is also assumed to be incompressible, whence we have

$$\operatorname{div} \vec{V}=0 . \tag{5}$$

If we eliminate $\vec{I}$ and $\vec{E}$ from the equations from (1) to (4), we obtain

$$\frac{\partial \vec{H}}{\partial t}=\operatorname{curl}(\vec{V} \wedge \vec{H}), \tag{6}$$

$$4\pi\rho\left\{\frac{\partial \vec{V}}{\partial t}+(\vec{V}\cdot\operatorname{grad})\vec{V}\right\}=\operatorname{curl}\vec{H}\wedge\vec{H}-4\pi\operatorname{grad}p \tag{7}$$

In the cases of small oscillations, as have been investigated in the writer's previous papers, the time-dependent velocity and the magnetic field are considered to be of the first order small quantities and the second order quantities are ignored. If we take the same approximation here, it becomes clear that the equation for the magnetic field is to be separated from the one for the fluid motion, coupling between the magnetic field and fluid motion being by no means possible. Hence the next step of approximation may be to take into account the electromagnetic force in the right-hand side of (7), while the second term of the left-hand side of (7) may be assumed to be smaller than the first one, so that we may take

$$4\pi\rho\frac{\partial \vec{V}}{\partial t}=\operatorname{curl}\vec{H}\wedge\vec{H}-4\pi\operatorname{grad}p \tag{8}$$

instead of (7). If this approximation is taken for granted, we obtain the following relation by making div. of (8) and using (5).

$$\nabla^2 p=-(4\pi)^{-1}\{\vec{H}\nabla^2\vec{H}+(\operatorname{curl}\vec{H})^2\} . \tag{9}$$

We shall now assume that $\vec{H}$ is of the S_1^0 type that gives only the dipole-like field outside the sphere. In that case, the r, θ and ϕ components of $\vec{H}$ are respectively written as

$$\vec{H}=\begin{cases} -2s(r,\,t)P_1\,, \\ -\left(r\dfrac{\partial s}{\partial r}+2s\right)\dfrac{dP_1}{d\theta}\,, \\ 0\,, \end{cases} \tag{10}$$

from which we also obtain

$$\mathrm{curl}\,\vec{H}=\begin{cases} 0\,, \\ 0\,, \\ -\left(r\dfrac{\partial^2 s}{\partial r^2}+4\dfrac{\partial s}{\partial r}\right)\dfrac{dP_1}{d\theta}\,, \end{cases} \tag{11}$$

$$\nabla^2\vec{H}=\begin{cases} -2\left(\dfrac{\partial^2 s}{\partial r^2}+\dfrac{4}{r}\dfrac{\partial s}{\partial r}\right)P_1\,, \\ -\left(r\dfrac{\partial^3 s}{\partial r^3}+6\dfrac{\partial^2 s}{\partial r^2}+\dfrac{4}{r}\dfrac{\partial s}{\partial r}\right)\dfrac{dP_1}{d\theta}\,, \\ 0\,. \end{cases} \tag{12}$$

With the aid of the expressions from (10) to (12), (9) can be written as

$$\nabla^2 p=f_0+f_2P_2 \tag{13}$$

where

$$\left.\begin{aligned} -4\pi f_0&=\frac{2}{3}\Big[2\left(\frac{\partial^2 s}{\partial r^2}+\frac{4}{r}\frac{\partial s}{\partial r}\right)s+\left(r\frac{\partial^3 s}{\partial r^3}+6\frac{\partial^2 s}{\partial r^2}+\frac{4}{r}\frac{\partial s}{\partial r}\right) \\ &\qquad\times\left(r\frac{\partial s}{\partial r}+2s\right)+\left(r\frac{\partial^2 s}{\partial r^2}+4\frac{\partial s}{\partial r}\right)^2\Big]\,, \\ -4\pi f_2&=\frac{2}{3}\Big[4\left(\frac{\partial^2 s}{\partial r^2}+\frac{4}{r}\frac{\partial s}{\partial r}\right)s-\left(r\frac{\partial^3 s}{\partial r^3}+6\frac{\partial^2 s}{\partial r^2}+\frac{4}{r}\frac{\partial s}{\partial r}\right) \\ &\qquad\times\left(r\frac{\partial s}{\partial r}+2s\right)-\left(r\frac{\partial^2 s}{\partial r^2}+4\frac{\partial s}{\partial r}\right)^2\Big]\,. \end{aligned}\right\} \tag{14}$$

The solution of (13) becomes

$$p=q_0+q_2P_2\,, \tag{15}$$

in which q_0 and q_2 should satisfy the following differential equations.

$$\left.\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)q_0=f_0\,, \\ &\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{6}{r^2}\right\}q_2=f_2\,. \end{aligned}\right\} \tag{16}$$

The solutions of (16) which do not become infinite at $r=0$ are given as

$$\left.\begin{aligned} q_0 &= K_0 + \int_0^r r f_0\, dr - r^{-1}\int_0^r r^2 f_0\, dr\,, \\ q_2 &= (r/a)^2 K_2 + \frac{1}{5}\left(r^2\int_0^r r^{-1} f_2\, dr - r^{-3}\int_0^r r^4 f_2\, dr\right), \end{aligned}\right\} \tag{17}$$

where K_0 and K_2 are constants which will be determined later. a denotes the radius of the sphere.

After some calculations, q_0 and q_2 are obtained as

$$\left.\begin{aligned} 4\pi q_0 &= 4\pi K_0 - \frac{2}{3}\left[2r\frac{\partial s}{\partial r}s + \frac{1}{2}r^2\left(\frac{\partial s}{\partial r}\right)^2 + 3s^2 + \int_0^r r\left(\frac{\partial s}{\partial r}\right)^2 dr\right], \\ 4\pi q_2 &= 4\pi K_2(r/a)^2 + \frac{2}{3}\left[2r\frac{\partial s}{\partial r}s + \frac{1}{2}r^2\left(\frac{\partial s}{\partial r}\right)^2 - \frac{7}{5}r^{-3}\int_0^r r^4\left(\frac{\partial s}{\partial r}\right)^2 dr \right. \\ &\quad \left. + \frac{12}{5}r^2\int_0^r r^{-1}\left(\frac{\partial s}{\partial r}\right)^2 dr\right]. \end{aligned}\right\} \tag{18}$$

Since we have calculated the pressure, the right-hand side of (8) can be obtained from (10), (11), (15) and (18). The result is as follows:

$$\vec{V} = \left\{\begin{aligned} &-6S(r)rP_2\,, \\ &-\left(r\frac{\partial S}{\partial r} + 3S\right)r\frac{dP_2}{d\theta}\,, \\ &\ 0\,, \end{aligned}\right. \tag{19}$$

where

$$4\pi\rho\frac{\partial S}{\partial t} = \frac{4}{3}\pi K_2 a^{-2} + \frac{2}{3}\left[-r^{-1}\frac{\partial s}{\partial r}s + \frac{7}{10}r^{-5}\int_0^r r^4\left(\frac{\partial s}{\partial r}\right)^2 dr + \frac{4}{5}\int_0^r r^{-1}\left(\frac{\partial s}{\partial r}\right)^2 dr\right]. \tag{20}$$

In obtaining (19), it is easily seen that the term that does not depend on θ vanishes.

On the other hand, by putting (19) into (6), the right-hand side of (6) is obtained as

$$\operatorname{curl}(\vec{V}\wedge\vec{H}) = \left\{\begin{aligned} &-2t_1P_1\,, \\ &-\left(r\frac{\partial t_1}{\partial r} + 2t_1\right)\frac{dP_1}{d\theta} \\ &\ 0\,, \end{aligned}\right. + \left\{\begin{aligned} &-12t_3r^2P_3\,, \\ &-\left(r\frac{\partial t_3}{\partial r} + 4t_3\right)r^2\frac{dP_3}{d\theta}\,, \\ &\ 0\,, \end{aligned}\right. \tag{21}$$

where

$$t_1=-\frac{6}{5}\left\{r\frac{\partial}{\partial r}(sS)+5sS\right\}, \tag{22}$$

$$t_3=\frac{1}{5}r^{-1}\left(6S\frac{\partial s}{\partial r}-4\frac{\partial S}{\partial r}s\right). \tag{23}$$

If it is assumed that we can ignore the magnetic field of the S_3^0 type thus brought forth, (6), (21) and (22) give

$$\frac{\partial s}{\partial t}=-\frac{6}{5}\left\{r\frac{\partial}{\partial r}(sS)+5sS\right\}, \tag{24}$$

whence we see that the radial parts of the magnetic field and velocity are to be determined by solving (20) together with (24) under suitable boundary conditions. This means solving a set of simultaneous non-linear integro-differential equations. It is not known whether or not we can solve them with rigorous accuracy. But the writer would here like to show a simple solution by making use of a way of rough approximation.

Let us assume that

$$\left.\begin{aligned} s&=\sum_n a_n(r/a)^n,\\ S&=\sum_n b_n(r/a)^n, \end{aligned}\right\} \tag{25}$$

in which the terms for $n>4$ are ignored. It is also assumed that $a_1=0$, otherwise one of the integrals in (20) is divergent. After introducing (25) into (20) and (24), the coefficients of the corresponding terms are equated, so that we obtain

$$\begin{aligned} &a_3=b_1=b_3=0,\\ &\frac{da_0}{dt}=-6a_0b_0,\\ &\frac{da_2}{dt}=-\frac{42}{5}(a_0b_2+a_2b_0),\\ &\frac{da_4}{dt}=-\frac{54}{5}(a_0b_4+a_2b_2+a_4b_0),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &4\pi\rho\frac{db_0}{dt}=\frac{2}{3}a^{-2}(2\pi K_2-2a_0a_2),\\ &4\pi\rho\frac{db_2}{dt}=-\frac{2}{3}a^{-2}\cdot 4a_0a_4,\\ &4\pi\rho\frac{db_4}{dt}=-\frac{2}{3}a^{-2}\cdot\frac{14}{9}a_2a_4,\\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{26}$$

while K_2 is to be determined from the condition that the normal component of the velocity vanishes or $b_0+b_2+b_4=0$ at the boundary, that is, the right-hand side of (20) becomes zero at $r=a$. Taking into account this condition, K_2 is given as

$$2\pi K_2=2a_0a_2+4a_0a_4+\frac{14}{9}a_2a_4\,. \tag{27}$$

Further, the solutions should satisfy the boundary condition that the magnetic field is continuous at $r=a$, so that the following relation must be satisfied as has been shown in the previous papers:

$$3a_0+5a_2+7a_4=0\,. \tag{28}$$

With the aid of (27) and (28), we obtain simultaneous differential equations with respect to a_0, a_2, b_0 and b_2 as follows:

$$\left.\begin{aligned}\frac{da_0}{dt}&=-6a_0b_0\,,\\ \frac{da_2}{dt}&=-\frac{42}{5}(a_0b_2+a_2b_0)\,,\\ 4\pi\rho a^2\frac{db_0}{dt}&=-\frac{4}{21}\left(2a_0+\frac{7}{9}a_2\right)(3a_0+5a_2)\,,\\ 4\pi\rho a^2\frac{db_2}{dt}&=\frac{8}{21}a_0(3a_0+5a_2)\,.\end{aligned}\right\} \tag{29}$$

The writer is going to show a solution of (29) for the special case of $a_2=b_2=0$. In that case, b_0 may be easily eliminated by obtaining a non-linear differential equation for a_0 as

$$a_0\frac{d^2a_0}{dt^2}-\left(\frac{da_0}{dt}\right)^2-f^2a_0^4=0\,, \tag{30}$$

where

$$f^2=\frac{12}{7\pi\rho a^2}\,. \tag{31}$$

If we put $da_0/dt=\xi$, (30) can be written as

$$a_0\xi\frac{d\xi}{da_0}=\xi^2+f^2a_0^4\,, \tag{32}$$

which can be integrated as

$$\xi^2=Aa_0^2+f^2a_0^4\,, \tag{33}$$

whence we have

$$\frac{da_0}{dt} = \pm a_0 \sqrt{A + f^2 a_0^2} \,. \tag{34}$$

Taking the plus sign of the right-hand side of (34) and putting

$$k^2 = -A f^{-2}, \tag{35}$$

(34) can be integrated as

$$ft = \frac{1}{k} \tan^{-1} \sqrt{\frac{a_0^2 - k^2}{k^2}} + c$$

that gives

$$a_0 = k \sec k(ft - c) \,. \tag{36}$$

We see therefore that the magnetic field performs a periodic oscillation given by (36). At $t = c/f$, we have

$$(a_0)_{t=f/c} = k \,, \quad \left(\frac{da_0}{dt}\right)_{t=f/c} = 0 \,.$$

The period is given by

$$T = \frac{2\pi}{kf} = \sqrt{\frac{7}{3}} \, \frac{1}{k} \pi^{\frac{3}{2}} \rho^{\frac{1}{2}} a \,. \tag{38}$$

3. Discussions and concluding remarks

Under the assumption that the finite oscillations of the S_1^0 field can occur independently of those of the other harmonics, a simple mode of magneto-hydrodynamic oscillation is obtained in the last section. The oscillation, which is given by (36), is shown in Fig. 1. Since many approximations are made use of in order to get at (36), it is not known to what extent (36) can be applied for the real earth. Especially, the infinite growth of the field causes doubt. In that case, the couplings between a_0 and b_0 and the ignored quantities such as a_2, b_2, a_4, b_4, ... would become appreciable, so that the differential equation (30) would be seriously affected. For all the crude treatments, however, it is of interest to note that the study suggests possible reversals of the S_1^0 field.

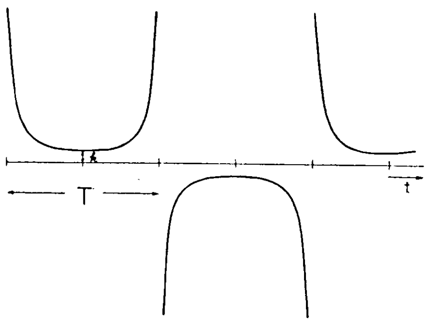

Fig. 1. The oscillation of the S_1^0 field.

The finite oscillation of a disc dynamo studied by Bullard does not give reversals after a field has once been established in a given direction. The finite oscillations of a conducting fluid sphere, however, might give rise to reversals, judging from the study, though incomplete, in this paper.

We see that the period of oscillation is proportional to the square root of the density, the radius of the sphere and the inverse of the initial field as shown in (38). This is thought to be of the right form as made clear in the studies of small oscillations of a conducting fluid sphere and finite one of a disc dynamo. Since the period depends on the initial value of the field, the period may be given any value by assuming suitable values for the initial field. In order to give a period of 10^5 *years*, k is taken to be 5×10^{-3} *gauss*, that means the initial magnetic field at the pole should be 10^{-2} *gauss*.

Since the electrical conductivity has been assumed to be infinity throughout the mathematical treatments in this paper, the effect of finite conductivity on the magneto-hydrodynamic oscillations concerned is not known. But it is clear that the loss of energy as Joule's heat will after all damp down the oscillation. If we suppose that quite a small field is given at the initial state, it is hardly likely that the field will grow without limit following the course as shown in Fig. 1, because of the ohmic dissipation. The situation would be, however, quite different if there are steady fluid motions of certain distribution at the equilibrium state. Actually, the writer[8)] has shown that a small field given to a conducting fluid sphere, in which fluid motions supposed by Bullard and Gellman are existing, can grow. Once the field grows up to an appreciable magnitude, the non-linear terms would become to play an important role. In that case, we might expect finite oscillations of the system. Even reversals of the S_1^0 field might occur at this stage.

The above discussions, of course, are incomplete because the writer can not solve the problem with steady motion at the equilibrium state. It is also not known to what extent the results will be affected by the neglect of higher harmonic constituents. The investigation of the mutual couplings of various harmonic constituents are difficult even in the case of small oscillations. In spite of these defects, the fact that we might expect reversals of the earth's magnetic field when we take into account magneto-hydrodynamic oscillations of finite amplitude still seems to be of importance and interest.

8) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **33** (1955), 571.

38. 地球核内の有限振幅電磁流体振動

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従来地球核内の微小振幅振動を取扱つて地磁気ダイナモの安定性を論じて来たが，この議論を有限振幅の場合に拡張した．数学的困難をさけるために，平衡状態に於ては磁場および流体運動がともに零であるとし，核外に双極子磁場としてあらわれるものにのみ着目し，それ以外の磁場は一切無視する．かくして流体運動の運動方程式および Maxwell の基礎方程式より，磁場ならびに速度に関する非線型連立積分微分方程式を求めた．この方程式を適当な省略をほどこして解く時は，非常に簡単な場合として振動性の解が求められる．この解は磁場が正負の両符号をとり得ることを示し，振動周期は密度の自乗根と核半径に比例し初期磁場の振幅に逆比例する．もしこのような振動が実在する時には，地球磁場の逆転も起り得ることになる．実際問題としては磁場が小さい間は非電磁的流体運動による増幅作用が行われ，磁場がかなり大きくなると本論文にのべたような電磁力の作用がいちじるしくなるものと思われる．