44. *Steady State Non-linear Vibration of Two Story Buildings.*

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Introduction

For the past several years, vibration tests have been performed on full size structures of new fire-resisting constructions to investigate their aseismic properties. In each test a small or a gigantic shaking machine was installed on the floor or a building so that it might produce forced vibration through the centrifugal shaking force. When the amplitude of the test structure is small, its dynamical behaviour can be regarded as linear but, if the amplitude becomes so large as to do some damage to the building, the response features are no longer linear but present non-linearity. Of these problems the authors already presented two papers[1)2)] in which they dealt with some theoretical and experimental problems on the masonry and two-story frame structures.

In this paper the analysis is shown for the steady state non-linear vibration of the two-storied frame structure which is subjected to shaking force by a machine and its application was investigated on the test results obtained from a pre-stressed concrete frame structure.

In steady state vibration of a building, the relation between the restoring force and the relative displacement of each story would usually make a figure of loop in their co-ordinate but, in the analysis shown in this paper, the relation is assumed as a curve passing through the origin of the co-ordinate, and the damping proportional to the velocity of relative deformation is considered. The vibrational properties of the test building including dampings in large vibrations are computed from this analysis and compared with the ones in small amplitudes.

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1) T. HISADA and K. NAKAGAWA, "An Analysis of Vibration of Masonry Buildings," *Report of the Building Research Institute*, **12** (1953).

2) T. HISADA, "Non-Linear Vibration of Two Story Buildings" *Trans. Architectural Institute of Japan*, **50** (1955).

1. Forced Vibration by Shaking Machine

Assumptions

In this analysis, the following assumptions are employed:

The building is considered as a system of two degrees of freedom and the restoring force of each story has terms of displacement having odd power (up to 7 in this paper).

The damping force of each story is considered as viscous and proportional to the relative velocity of deformation.

Notations

The following notation is used in this paper:

m_i = mass, $i=1, 2$

$\bar{m}=m_2/m_1$

x_i = instantaneous displacement of mass m_i, $i=1, 2$

c_i = coefficient of the linear term in the algebraic expression for a non-linear restoring force.

$\mu_i^2, \nu_i^4, \xi_i^6$ = ratios of the coefficients of the non-linear terms to that of the linear term in the algebraic expression for a non-linear restoring force, $i=1, 2$

$\kappa_i^2=c_i/m_i$, $i=1, 2$

$\bar{\kappa}^2=\kappa_2^2/\kappa_1^2$

$\bar{\mu}^2=\mu_2^2/\mu_1^2$

$n_i=10\nu_i^4/9\mu_1^4$, $i=1, 2$

$l_i=35\xi_i^6/27\mu_1^6$, $i=1, 2$

b_i = coefficient of damping force, $i=1, 2$

$D_i=b_i/2m_i\kappa_i$ = dimensionless relative damping, $i=1, 2$

Q_i = amplitude of harmonically varying displacement of mass m_i, $i=1, 2$

$\bar{Q}_i=\sqrt{3/4}\mu_1 Q_i$ = dimensionless amplitude of displacement of mass m_i, $i=1, 2$

M = eccentric mass of shaking machine

r = arm of eccentric mass

$s=Mr/m_1$

$\bar{s}=Mr/m_1\cdot\sqrt{3/4}\mu_1$

α_i = difference of phase of the sinusoidal motion of mass m_i $i=1, 2$

Ω = circular frequency of shaking machine

$\eta=\Omega/\kappa_1$

t = time

$\tau=\Omega t$

General Relations

The differential equations of motion of the system shown in Fig. 1 are

$$m_1\ddot{x}_1 + c_1[x_1 - \mu_1^2 x_1^3 + \nu_1^4 x_1^5 - \xi_1^6 x_1^7] - c_2[(x_2-x_1) - \mu_2^2(x_2-x_1)^3 + \nu_2(x_2-x_1)^5 - \xi_2^6(x_2-x_1)^7] + b_1\dot{x}_1 - b_2(\dot{x}_2-\dot{x}_1) = 0 \qquad (1.1)$$

$$m_2\ddot{x}_2 + c_2[(x_2-x_1) - \mu_2^2(x_2-x_1)^3 + \nu_2^4(x_2-x_1)^5 - \xi_2^6(x_2-x_1)^7] + b_2(\dot{x}_2-\dot{x}_1) - Mr\Omega^2\cos\Omega t = 0 \qquad (1.2)$$

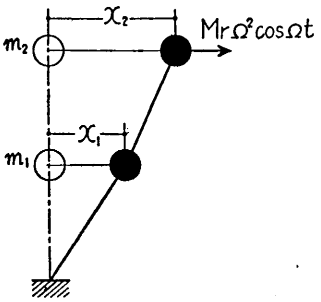

Fig. 1.

or

$$E_1 \equiv \ddot{x} + \kappa_1^2[x_1 - \mu_1^2 x_1^3 + \nu_1^4 x_1^5 - \xi_1^6 x_1^7] - \overline{m}\kappa_2^2[(x_2-x_1) - \mu_2^2(x_2-x_1)^3 + \nu_2^4(x_2-x_1)^5 - \xi_2^6(x_2-x_1)^7] + 2D_1\kappa_1\dot{x}_1 - 2\overline{m}D_2\kappa_2(\dot{x}_2-\dot{x}_1) = 0 \qquad (1.3)$$

$$E_2 \equiv \ddot{x}_2 + \kappa_2^2[(x_2-x_1) - \mu_2^2(x_2-x_1)^3 + \nu_2^4(x_2-x_1)^5 - \xi_2^6(x_2-x_1)^7] + 2D_2\kappa_2(\dot{x}_2-\dot{x}_1) - s\Omega^2/\overline{m}\cdot\cos\Omega t = 0 \qquad (1.4)$$

We assume the following periodic solution:

$$\tilde{x}_1 = Q_1\cos(\tau-\alpha_1)\ , \qquad \tilde{x}_2 = Q_2\cos(\tau-\alpha_2) \qquad (1.5)$$

The Ritz averaging method[3] gives the following four conditions:

$$\left.\begin{aligned} &\int_0^{2\pi} E_1(\tilde{x}_1,\tilde{x}_2)\cos\tau d\tau = 0\ , \qquad \int_0^{2\pi} E_1(\tilde{x}_1,\tilde{x}_2)\sin\tau d\tau = 0 \\ &\int_0^{2\pi} E_2(\tilde{x}_1,\tilde{x}_2)\cos\tau d\tau = 0\ , \qquad \int_0^{2\pi} E_2(\tilde{x}_1,\tilde{x}_2)\sin\tau d\tau = 0 \end{aligned}\right\} \qquad (1.6)$$

Introducing (1.5) into (1.3) and (1.4), and furthermore putting those two equations in (1.6), we obtain the following four equations:

$$\begin{aligned} &[(1+\overline{m}\bar{\kappa}^2-\eta^2)\overline{Q}_1 - (1+\overline{m}\bar{\kappa}^2\bar{\mu}^2)\overline{Q}_1^3 + (n_1+\overline{m}\bar{\kappa}^2 n_2)\overline{Q}_1^5 - (l_1+\overline{m}\bar{\kappa}^2 l_2)\overline{Q}_1^7]{\cos \atop \sin}\alpha_1 \\ &+[-\overline{m}\bar{\kappa}^2\overline{Q}_2 + \overline{m}\bar{\kappa}^2\bar{\mu}^2\overline{Q}_2^2 - \overline{m}\bar{\kappa}^2 n_2\overline{Q}_2^5 + \overline{m}\bar{\kappa}^2 l_2\overline{Q}_2^7]{\cos \atop \sin}\alpha_2 \\ &+\overline{m}\bar{\kappa}^2\bar{\mu}^2\overline{Q}_1^2\overline{Q}_2\left[2{\cos \atop \sin}\alpha_2 + {\cos \atop \sin}(2\alpha_1-\alpha_2)\right] - \overline{m}\bar{\kappa}^2\bar{\mu}^2\overline{Q}_1\overline{Q}_2^2\left[2{\cos\alpha_1 \atop \sin\alpha_1} + {\cos \atop \sin}(2\alpha_2-\alpha_1)\right] \\ &+\overline{m}\bar{\kappa}^2 n_2\overline{Q}_1\overline{Q}_2^4\left[2\ {\cos \atop -\sin}(\alpha_1-2\alpha_2) + 3{\cos \atop \sin}\alpha_1\right] - \overline{m}\bar{\kappa}^2 n_2\overline{Q}_1^4\overline{Q}_2\left[2\ {\cos \atop -\sin}(\alpha_2-2\alpha_1)\right. \\ &\left.+3{\cos \atop \sin}\alpha_2\right] + \overline{m}\bar{\kappa}^2 n_2\overline{Q}_1^3\overline{Q}_2^2\left[6{\cos \atop \sin}\alpha_1 + {\cos \atop -\sin}(2\alpha_2-3\alpha_1) + 3{\cos \atop \sin}(2\alpha_2-\alpha_1)\right] \end{aligned}$$

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3) K. KLOTTER, "Non-linear Vibration Problems Treated by the Averaging Method of W. Ritz," *Proc. First U. S. National Congress of Applied Mechanics*, 1951, ASME.

$$-\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1\overline{Q}_2^6\left[3\genfrac{}{}{0pt}{}{\cos}{-\sin}(\alpha_1-2\alpha_2)+4\genfrac{}{}{0pt}{}{\cos}{\sin}\alpha_1\right]+\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^2\overline{Q}_2^5\left[3\genfrac{}{}{0pt}{}{\cos}{-\sin}(2\alpha_1-3\alpha_2)\right.$$

$$\left.+6\genfrac{}{}{0pt}{}{\cos}{\sin}(2\alpha_1-\alpha_2)+12\genfrac{}{}{0pt}{}{\cos}{\sin}\alpha_2\right]-\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^3\overline{Q}_2^4\left[\genfrac{}{}{0pt}{}{\cos}{-\sin}(3\alpha_1-4\alpha_2)\right.$$

$$\left.+4\genfrac{}{}{0pt}{}{\cos}{\sin}(3\alpha_1-2\alpha_2)+12\genfrac{}{}{0pt}{}{\cos}{-\sin}(\alpha_1-2\alpha_2)+18\genfrac{}{}{0pt}{}{\cos}{\sin}\alpha_1\right]$$

$$+\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^4\overline{Q}_2^3\left[\genfrac{}{}{0pt}{}{\cos}{-\sin}(3\alpha_2-4\alpha_1)+4\genfrac{}{}{0pt}{}{\cos}{\sin}(3\alpha_2-2\alpha_1)+12\genfrac{}{}{0pt}{}{\cos}{-\sin}(\alpha_2-2\alpha_1)\right.$$

$$\left.+18\genfrac{}{}{0pt}{}{\cos}{\sin}\alpha_2\right]-\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^5\overline{Q}_2^2\left[3\genfrac{}{}{0pt}{}{\cos}{-\sin}(2\alpha_2-3\alpha_1)+6\genfrac{}{}{0pt}{}{\cos}{\sin}(2\alpha_2-\alpha_1)+12\genfrac{}{}{0pt}{}{\cos}{\sin}\alpha_1\right]$$

$$+\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^6\overline{Q}_2\left[3\genfrac{}{}{0pt}{}{\cos}{-\sin}(\alpha_2-2\alpha_1)+4\genfrac{}{}{0pt}{}{\cos}{\sin}\alpha_2\right]+2D_1\eta\overline{Q}_1\genfrac{}{}{0pt}{}{\sin}{-\cos}\alpha_1$$

$$+2\overline{m}D_2\bar{\kappa}\eta\overline{Q}_1\genfrac{}{}{0pt}{}{\sin}{-\cos}\alpha_1-2\overline{m}D_2\bar{\kappa}\eta\overline{Q}_2\genfrac{}{}{0pt}{}{\sin}{-\cos}\alpha_2=0 \qquad \cdots\cdots\left\{\begin{matrix}(1.7.1)\\(1.7.2)\end{matrix}\right.$$

$$[-\overline{m}\bar{\kappa}^2\overline{Q}_1+\overline{m}\bar{\kappa}^2\bar{\mu}^2\overline{Q}^3-\overline{m}\bar{\kappa}^2 n_2\overline{Q}_1^5+\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^7]\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_1$$

$$+[\overline{m}(\bar{\kappa}^2-\eta^2)\overline{Q}_2-\overline{m}\bar{\kappa}^2\bar{\mu}^2\overline{Q}_2^3+\overline{m}\bar{\kappa}^2 n_2\overline{Q}_2^5-\overline{m}\bar{\kappa}^2 l_2\overline{Q}_2^7]\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_2$$

$$-\overline{m}\bar{\kappa}^2\bar{\mu}^2\overline{Q}_1^2\overline{Q}_2\left[2\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_2+\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(2\alpha_1-\alpha_2)\right]$$

$$+\overline{m}\bar{\kappa}^2\bar{\mu}^2\overline{Q}\,\overline{Q}_2^2\left[2\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_1+\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(2\alpha_2-\alpha_1)\right]$$

$$-\overline{m}\bar{\kappa}^2 n_2\overline{Q}_1\overline{Q}_2^4\left[2\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(\alpha_1-2\alpha_2)+3\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_1\right]$$

$$+\overline{m}\bar{\kappa}^2 n_2\overline{Q}_1^4\overline{Q}_2\left[2\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(\alpha_2-2\alpha_1)+3\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_2\right]$$

$$+\overline{m}\bar{\kappa}^2 n_2\overline{Q}_1^2\overline{Q}_2^3\left[6\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}Q_2+\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(2\alpha_1-3\alpha_2)+3\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(2\alpha_1-\alpha_2)\right]$$

$$-\overline{m}\bar{\kappa}^2 n_2\overline{Q}_1^3\overline{Q}_2^2\left[6\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_1+\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(2\alpha_2-3\alpha_1)+3\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(2\alpha_2-\alpha_1)\right]$$

$$+\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1\overline{Q}_2^6\left[3\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(\alpha_1-2\alpha_2)+4\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_1\right]$$

$$-\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^2\overline{Q}_2^5\left[3\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(2\alpha_1-3\alpha_2)+6\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(2\alpha_1-\alpha_2)+12\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_2\right]$$

$$+\overline{m}\bar{\kappa}^2 l_2\overline{Q}_1^3\overline{Q}_2^4\left[\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(3\alpha_1-4\alpha_2)+4\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(3\alpha_1-2\alpha_2)\right.$$

$$\left.+12\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix}(\alpha_1-2\alpha_2)+18\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\alpha_1\right]$$

$$
\begin{aligned}
&-\overline{m}\overline{\kappa}^2 l_2 \overline{Q}_1^4 \overline{Q}_2^3 \Big[\begin{Bmatrix} \cos \\ -\sin \end{Bmatrix} (3\alpha_2 - 4\alpha_1) + 4 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (3\alpha_2 - 2\alpha_1) \\
&\qquad\qquad + 12 \begin{Bmatrix} \cos \\ -\sin \end{Bmatrix} (\alpha_2 - 2\alpha_1) + 18 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \alpha_2 \Big] \\
&+\overline{m}\overline{\kappa}^2 l_2 \overline{Q}_1^5 \overline{Q}_2^2 \Big[3 \begin{Bmatrix} \cos \\ -\sin \end{Bmatrix} (2\alpha_2 - 3\alpha_1) + 6 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (2\alpha_2 - \alpha_1) + 12 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \alpha_1 \Big] \\
&-\overline{m}\overline{\kappa}^2 l_2 \overline{Q}_1^6 \overline{Q}_2 \Big[3 \begin{Bmatrix} \cos \\ -\sin \end{Bmatrix} (\alpha_2 - 2\alpha_1) + 4 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \alpha_2 \Big] \\
&+2\overline{m} D_2 \overline{\kappa} \eta \overline{Q}_2 \begin{Bmatrix} \sin \\ -\cos \end{Bmatrix} \alpha_2 - 2\overline{m} D_2 \overline{\kappa} \eta \overline{Q}_1 \begin{Bmatrix} \sin \\ -\cos \end{Bmatrix} \alpha_1 - \begin{Bmatrix} \overline{s}\eta^2 \\ 0 \end{Bmatrix} = 0 \qquad \cdots \begin{Bmatrix} (1.7.3) \\ (1.7.4) \end{Bmatrix}
\end{aligned}
$$

For undamped systems, by putting $\alpha_1 = \alpha_2 = 0$ and $D_1 = D_2 = 0$ in (1.7.1) and (1.7.3), we have

$$
\begin{aligned}
&(1-\eta^2)\overline{Q}_1 - \overline{Q}_1^3 + n_1 \overline{Q}_1^5 - l_1 \overline{Q}_1^7 - \overline{m}\overline{\kappa}^2 (\overline{Q}_2 - \overline{Q}_1) + \overline{m}\overline{\kappa}^2 \overline{\mu}^2 (\overline{Q}_2 - \overline{Q}_1)^3 \\
&\quad - n_2 \overline{m}\overline{\kappa}^2 (\overline{Q}_2 - \overline{Q}_1)^5 + l_2 \overline{m}\overline{\kappa}^2 (\overline{Q}_2 - \overline{Q}_1)^7 = 0 \qquad (1.8.1)
\end{aligned}
$$

$$
\begin{aligned}
&\overline{m}\overline{\kappa}^2 (\overline{Q}_2 - \overline{Q}_1) - \overline{m}\overline{\kappa}^2 \overline{\mu}^2 (\overline{Q}_2 - \overline{Q}_1)^3 + n_2 \overline{m}\overline{\kappa}^2 (\overline{Q}_2 - \overline{Q}_1)^5 \\
&\quad - l_2 \overline{m}\overline{\kappa}^2 (\overline{Q}_2 - \overline{Q}_1)^7 - \overline{m}\eta^2 \overline{Q}_2 - \overline{s}\eta^2 = 0 \qquad (1.8.2)
\end{aligned}
$$

From those two equations, we can compute the response curves which represent the relations between $\overline{Q}_i$ and η^2 by using graphical method.

To see the behavior of free vibration of undamped systems, we put $\overline{s}=0$ in (1.8.1) and (1.8.2), then we have

$$
\overline{Q}_2 = \frac{(1-\eta^2)\overline{Q}_1 - \overline{Q}_1^3 + n_1 \overline{Q}_1^5 - l_1 \overline{Q}_1^7}{\overline{m}\eta^2} \qquad (1.9)
$$

Introducing (1.9) into (1.8.1), we obtain

$$
\begin{aligned}
&\overline{m}^6 \eta^{11} [(1-\eta^2) - \overline{Q}_1^2 + n_1 \overline{Q}_1^4 - l_1 \overline{Q}_1^6] - \overline{m}^6 \overline{\kappa}^2 \eta^{12} [(1-\eta^2 - \overline{m}\eta^2) - \overline{Q}_1^2 + n_1 \overline{Q}_1^4 - l_1 \overline{Q}^6] \\
&\quad + \overline{m}\overline{\kappa}^2 \overline{\mu}^2 \eta^8 [(1-\eta^2-\overline{m}\eta^2) - \overline{Q}_1^2 + n_1 \overline{Q}_1^4 - l_1 \overline{Q}_1^6]^3 \overline{Q}^2 - \overline{m}\overline{\kappa}^2 n_2 \eta^4 [(1-\eta^2-\overline{m}\eta^2) \\
&\quad - \overline{Q}_1^2 + n_1 \overline{Q}_1^4 - l_1 \overline{Q}_1^6]^5 \overline{Q}_1^4 + \overline{\kappa}^2 l_2 [(1-\eta^2-\overline{m}\eta^2) - \overline{Q}_1^2 + n_1 \overline{Q}_1^4 - l_1 \overline{Q}_1^6]^7 \overline{Q}_1^6 = 0 \qquad (1.10)
\end{aligned}
$$

(1.10) is the formula for "backbone" curves of $\overline{Q}_1$ and represents the relation between amplitude Q_1 and frequency η^2 in free vibration, while (1.9) represents the relationship between $\overline{Q}_1$ and $\overline{Q}_2$ on their "backbone" curves.

For damped systems, we can get the response curves by solving (1.7.1), (1.7.2), (1.7.3) and (1.7.4) simultaneously. The computations take very long and are tedious, so, if the dampings of the system are small, we allow ourselves to infer the general tendency of the damped systems from the response curves of the undamped ones. In this case, it is necessary and most important to determine the behavior of the system at resonance for the first mode. For this, we put $\alpha_1=\alpha_2=\pi/2$ in (1.7) and, by adding (1.7.1) and (1.7.2) and also (1.7.3) and (1.7.4), we obtain

$$\begin{aligned}(1-\eta^2)\overline{Q}_1-\overline{Q}_1^3+n_1\overline{Q}_1^5-l_1\overline{Q}_1^7-\overline{m}\overline{\kappa}^2(\overline{Q}_2-\overline{Q}_1)+\overline{m}\overline{\kappa}^2\overline{\mu}^2(\overline{Q}_2-\overline{Q}_1)^3\\ -\overline{m}\overline{\kappa}^2n_2(\overline{Q}_2-\overline{Q}_1)^5+\overline{m}\overline{\kappa}^2l_2(\overline{Q}_2-\overline{Q}_1)^7+2D_1\eta\overline{Q}_1+2\overline{m}D_2\overline{\kappa}\eta\overline{Q}_1-2\overline{m}D_2\overline{\kappa}\eta\overline{Q}_2=0\end{aligned} \tag{1.11}$$

$$\begin{aligned}\overline{m}\overline{\kappa}^2(\overline{Q}_2-\overline{Q}_1)-\overline{m}\overline{\kappa}^2\overline{\mu}^2(\overline{Q}_2-\overline{Q}_1)^3+\overline{m}\overline{\kappa}^2n_2(\overline{Q}_2-\overline{Q}_1)^5-\overline{m}\overline{\kappa}^2l_2(\overline{Q}_2-\overline{Q}_1)^7\\ -\overline{m}\eta^2\overline{Q}_2+2\overline{m}D_2\overline{\kappa}\eta(\overline{Q}_2-\overline{Q}_1)-\overline{s}\eta^2=0\end{aligned} \tag{1.12}$$

Adding (1.11) and (1.12) and using the relation in (1.9), we obtain

$$2D_1\overline{Q}_1-\overline{s}\eta=0 \tag{1.13}$$

The crossing point of (1.13) and the backbone curve of $\overline{Q}_1$ represents the amplitude of Q_1 at resonance and we can find the corresponding point of Q_2 on its backbone curve.

As to the dampings, we obtain from (1.11), (1.12) and (1.9) the the following formulas:

$$\begin{aligned}D_1=\{&-\overline{m}\eta^2Q_2+\overline{m}\overline{\kappa}^2(\overline{Q}_2-\overline{Q}_1)-\overline{m}\overline{\kappa}^2\overline{\mu}^2(\overline{Q}_2-\overline{Q}_1)^3+\overline{m}\overline{\kappa}^2n_2(\overline{Q}_2-\overline{Q}_1)^5-\overline{m}\overline{\kappa}^2l_2(\overline{Q}_2-\overline{Q}_1)^7\\ &+2\overline{m}\overline{\kappa}\eta(\overline{Q}_2-\overline{Q}_1)D_2\}/2\eta\overline{Q}_1\end{aligned} \tag{1.14}$$

$$\begin{aligned}D_2=\{&-\overline{m}\overline{\kappa}^2(\overline{Q}_2-\overline{Q}_1)+\overline{m}\overline{\kappa}^2\overline{\mu}^2(\overline{Q}_2-\overline{Q}_1)^3-\overline{m}\overline{\kappa}^2n_2(\overline{Q}_2-\overline{Q}_1)^5+\overline{m}\overline{\kappa}^2l_2(\overline{Q}_2-\overline{Q}_1)^7+\overline{m}\eta^2\overline{Q}_2\\ &+\overline{s}\eta^2\}/2\overline{m}\overline{\kappa}\eta(\overline{Q_2-Q_1})\end{aligned} \tag{1.15}$$

2. Analysis of Large Vibration Tests of A Pre-Stressed Concrete Structure

Vibration Tests[4] were performed on a two-storied frame structure of prestressed concrete construction by installing a small or a large shaking machine on the roof floor. In those tests eccentric moments *Mr* were 137 kg.cm and 3960 kg.cm respectively. In the case of small vibration no damage was done to the structure but, in the first large vibration test, many cracks appeared in the cement mortar filled in the connections of the frame. Keeping the eccentric moment of machine at a constant value, the test was continued and, in the third test, the steady state response features shown in Fig. 2 were obtained.

4) K. NAKAGAWA and others, "Vibration of A Prestressd Concrete Structure," *Report, Architectural Institute of Japan*, **31** (1955).

For this analysis we computed the relation between the restoring force and displacement of each story by using the period and amplitude at resonance, considering the structure as a system of two degrees of freedom, and obtained Fig. 3 where the results calculated frow small and large vibration tests are plotted.

In this figure we assume curves R_1 and R_2 for the first and second stories respectively. The numerical values used here are as follows:

$$m_1 = 10300\,\text{kg}/980\,\text{cm.sec}^{-2} = 10.5\,\text{kg.sec}^2.\text{cm}^{-1}$$

$$m_2 = 15500\,\text{kg}/980\,\text{cm.sec}^{-2} = 15.85\,\text{kg.sec}^2.\text{cm}^{-1}$$

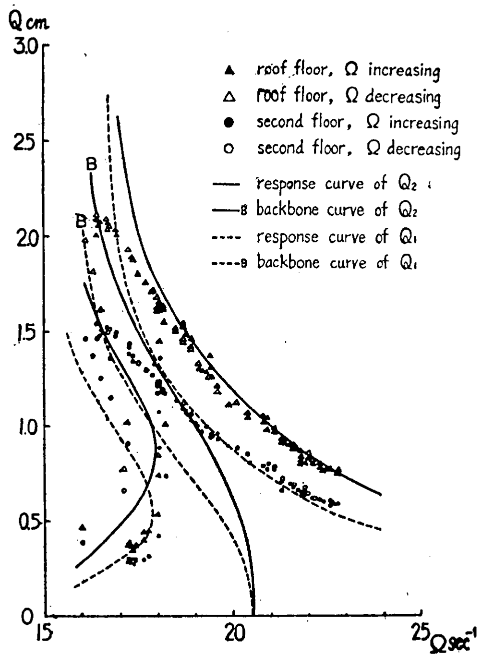

Fig. 2.

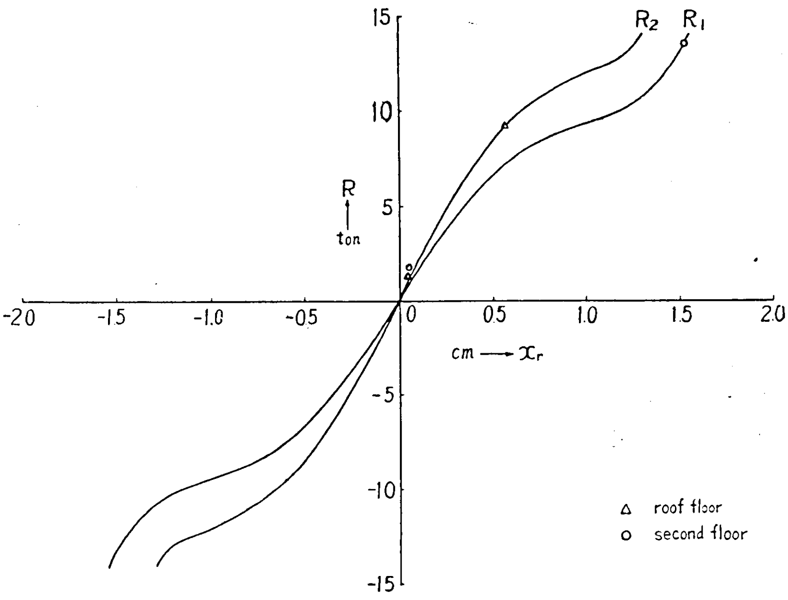

Fig. 3.

$\overline{m}=m_2/m_1=1.51$

$R_1=14.7\ (x_r-0.506x_r^3+0.147x_r^5-0.000267x_r^7)$

$R_2=18.8\ (x_r-0.506x_r^3+0.147x_r^5-0.000267x_r^7)$

R_1, R_2 in ton, x_r in cm.

$\overline{\kappa}^2=\kappa_2^2/\kappa_1^2=c_2/m_2\cdot m_1/c_1=0.846$

$\overline{\mu}^2=\mu_2^2/\mu_1^2=1$

$n_1=10\nu_1^4/9\mu_1^4=0.638$

$n_2=0.638$

$l_1=35\xi_1^6/27\mu_1^6=0.00266$

$l_2=0.00266$

Introducing the above values into (18.1), (18.2) (1.9) and (1.10), we obtained response and backbone curves for the undamped systems as shown in Fig. 2. As may be seen in this figure, the analytical and experimental results show considerable good agreement.

As for the dampings, by putting the resonant circular frequency and the maximum amplitudes into (1.14) and (1.15), we obtained the following values:

$$D_1=0.057 \qquad (1.14')$$

$$D_2=0.094 \qquad (1.15')$$

3. Analysis of Small Vibration Tests of the Same Structure

In the case of forced vibration of small amplitudes, a shaking machine was installed on the second floor, and the behavior of the structure could be regarded as linear oscillations with viscous dampings. Then, we have the following differential equations of motion:

$$\ddot{x}_1+\kappa_1^2x_1-\overline{m}\kappa_2^2(x_2-x_1)+2D_1\kappa_1\dot{x}_1-2\overline{m}D_2\kappa_2(\dot{x}_2-\dot{x}_1)-s\Omega^2\cos\Omega t=0 \qquad (3.1)$$

$$\ddot{x}_2+\kappa_2^2(x_2-x_1)+2D_2\kappa_2(\dot{x}_2-\dot{x}_1)=0 \qquad (3.2)$$

Representing the solution in the following forms

$$\left.\begin{aligned} x_1&=Q_1\cos(\Omega t-\gamma_1)\\ x_2&=Q_2\cos(\Omega t-\gamma_2)\end{aligned}\right\} \qquad (3.3)$$

and giving the shaking force and displacements as the form of complex number, we can obtain the solution of (3.1) and (3.2) as follows:

$$Q_1=s\eta^2\sqrt{(C^2+D^2)/(A^2+B^2)} \qquad (3.4.1)$$

$$Q_2=s\eta^2\sqrt{(E^2+F^2)/(A^2+B^2)} \qquad (3.4.2)$$

$$\gamma_1=\tan^{-1}\{(BC-AD)/(AC+BD)\} \qquad (3.4.3)$$

$$\gamma_2=\tan^{-1}\{(BF-AF)/(AE+BF)\} \qquad (3.4.4)$$

where

$$\left.\begin{aligned}A&=\eta^4-(1+\bar{\kappa}^2+\bar{m}\bar{\kappa}^2+4D_1D_2\bar{\kappa})\eta^2+\bar{\kappa}^2\\B&=2\bar{\kappa}(\bar{\kappa}D_1+D_2)\eta-2(D_1+\bar{\kappa}D_2+2\bar{m}\bar{\kappa}D_2)\eta^3\\C&=\bar{\kappa}^2-\eta^2\\D&=2\bar{\kappa}D_2\eta\\E&=\bar{\kappa}^2\\F&=2\bar{\kappa}D_2\eta\end{aligned}\right\}\qquad(3.5)$$

Numerical values for the small vibration test are as follows;

$Q_2=0.1\,\text{cm},\qquad Q_2=0.057\,\text{cm}$

$Mr/m_1=0.0133\,\text{cm}$

$\bar{m}=1.51$

$\bar{\kappa}^2=c_2/(c_1\bar{m})=(29500\,\text{kg/cm})/(30600\,\text{kg/cm})\times\bar{m}=0.638$

$\eta^2=\Omega_r^2m_1/c_1=28.3\times10.5/30600=0.275$

where Ω_r=resonant circular frequency

Introducing the above values into (3.4.1) and (3.4.2), and calculating the values of D_1 and D_2 which satisfy both equations, we have

$$\left.\begin{aligned}D_1&=0.041\\D_2&=0.029\end{aligned}\right\}\qquad(3.6)$$

Comparing these values with those in large vibration, we find that the former is smaller than the latter.

4. General Results

In the forced vibration test of a two-storied pre-stressed concrete structure which was subjected to a large shaking force by a mechanical oscillator, the response behavior of the structure represented non-linear features as shown in Fig. 2 where Q and Ω denote the amplitude of each floor and the circular frequency of shaking machine respectively. The maximum accelerations obtained at roof and second floors at resonance were 580 and 420 gals, and some structural damage was done to the connections and columns of the skeleton.

In analysis of the test results, considerable good agreement between experimental and theoretical values are obtained, if we consider the relation curves between the restoring forces and relative displacements as shown in Fig. 3 and the viscous dampings to be proportional to the relative deformations for the first and second stories of the structure regarded as a system of two degrees of freedom. The fractions of critical damping for the stories computed from the test results are 0.057 and 0.094, and those in the vibration of small amplitudes 0.041 and 0.029 respectively. It is noteworthy that the dampings assumed as viscous in large vibration are larger than those in small amplitudes.

The method presented in this paper may be considered as one of the analytical treatments for the steady state non-linear vibration of two-storied buildings.

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44. 二層建物の非線形定常振動

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プレストレスト・コンクリート造 2 階建建物の屋上に大型起振機を据えて之に大振動（屋上最大振幅約 2 cm, 最大加速度 580 ガル）を与えるとその共振曲線は Fig. 2 に示したように非線形的性状を示し，構造体に相当の損傷を生じた．

この場合各階の復元力と相対変形の関係を Fig. 3 に示すような 7 次の曲線で表わし，且相対変形速度に比例する粘性減衰を各階がもつものとし，Ritz の方法で解を求めると実験結果をかなりよく説明出来るようである．この場合の減衰常数を実験値から求めると 1 階 0.057, 2 階 0.094 となるが同じ建物の大振動前の微振動時（屋上最大振幅 0,1 cm）に於ける値は夫々 0.041, 0.029 であつて大振動時の場合より小さい．

本報文に示した方法及び数値は二層建物の塑性定常振動の解析に関する一資料となるであろう．
