18. A Short Note on a Graphical Solution of the Spectral Response of the Ground.

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1. When an infinite train of harmonic elastic waves of unit amplitude is propagated from beneath to a stratified ground, the amplitude of oscillation at the surface of the ground is a function of the wave length, and is called the spectral response of the ground.

The spectral response is a very important function because it enables us to calculate, in the Fourier manner, the response of the ground to an arbitrary disturbance. Another important point is that the spectral response explains the dominant periods of the ground which appear on occasions of earthquakes and microtremors.

2. The layers are numbered from top to bottom, so the quantities related to the uppermost layer are denoted by the suffix 1. The semi-infinite bedrock is denoted by the suffix n. The amplitude of the incident wave is u_0, and the amplitude of the ground motion at the surface is $2u$. v, τ and c mean respectively the particle velocity, the shearing stress and the propagation velocity of the elastic waves.

Since the incident wave train is infinite in length, waves propagated respectively upwards and downwards result in a stationary oscillation of the layer, because we assume the dissipation of the wave energy in the layers to be negligibly small.

In the layer i, we have

$$v_i = \frac{v_{i0}}{\cos \theta_i} \cos\left(\frac{\omega x_i}{c_i} + \theta_i\right) , \tag{1}$$

$$\tau_i = \frac{v_{i0} z_i}{\cos \theta_i} \sin\left(\frac{\omega x_i}{c_i} + \theta_i\right) , \tag{2}$$

in which v_{i0} is the value of v_i at the upper boundary of the layer from which x_i is measured downwards. z_i is equal to $jc_i\rho_i$, ρ_i being the density of the soil and $j=\sqrt{-1}$. ω is the circular frequency of the waves. θ_i is the angle which is determined by the relation

$$z_i \tan \theta_i = z_{i-1} \tan\left(\frac{\omega H_{i-1}}{c_{i-1}} + \theta_{i-1}\right) . \tag{3}$$

This relation must hold at the upper boundary of the layer i, because v_i, τ_i must be continuous respectively with v_{i-1}, τ_{i-1} at the boundary. Since τ_1 must vanish at the ground surface $x_1 = 0$, we must have $\theta_1 = 0$.

3. As can be seen in the expressions (1) and (2), τ_i/z_i and v_i are proportional to the sine and cosine of $\left(\dfrac{\omega x_i}{c_i} + \theta_i \right)$, so they can be represented by the ordinate and abscissa of a point which moves on a circle of radius

$$v_{i0}/\cos\theta_i = |v_i + j\tau_i/z_i| . \tag{4}$$

As for the phase angle at the lower boundary of the layer i, we have the relation

$$\frac{\omega H_i}{c_i} + \theta_i = 2\pi \frac{H_1}{L_1} \left(\frac{H_i}{c_i} \Big/ \frac{H_1}{c_1} \right) + \theta_i . \tag{5}$$

We can therefore determine in turn the values $v_{20}/\cos\theta_2, \cdots v_{n0}/\cos\theta_n$,—the maximum amplitudes of the stationary oscillation in each layer—, provided that we know $\rho_1, \rho_2 \cdots \rho_n$; $c_1, c_2, \cdots c_n$ and the thickness of each layer $H_1, H_2 \cdots H_{n-1}$. v_{10} is assumed to be unity.

Since $v_{n0}/\cos\theta_n$ is the maximum velocity amplitude of the stationary oscillation in the bedrock, the velocity amplitude due to the incident wave is a half of this value. We have therefore

$$\frac{2u}{u_0} = \frac{v_{10}}{\frac{1}{2}(v_{n0}/\cos\theta_n)} = \frac{2}{\dfrac{v_{n-1,0}}{\cos\theta_{n-1}} \left| \cos\varphi_{n-1} + j\dfrac{z_{n-1}}{z_n}\sin\varphi_{n-1} \right|} \tag{6}$$

$$= 2 \Big/ \left| \cos\varphi_{n-1} + j\frac{z_{n-1}}{z_n}\sin\varphi_{n-1} \right| \cdots \left| \cos\varphi_1 + j\frac{z_1}{z_2}\sin\varphi_1 \right|, \tag{7}$$

where $\varphi_i = \dfrac{\omega H_i}{c_i} + \theta_i$. The last expression can easily be obtained from (1), (2) and (3).

4. Instead of seeking $2u/u_0$ directly, we will find out u_0/u, which is by (6) equal to $v_{n0}/\cos\theta_n$. The construction for getting this value is as follows: Referring to Fig. 1,

i) Take OA equal to unity on the x-axis.

ii) Give H_1/L_1 a convenient value. Draw a circular arc AB such that the angle $\angle AOB = 360° \times H_1/L_1$. Drop a perpendicular BC on OA. Then $OC = v_{20}$ and $BC = \tau_{20}/z_1$.

iii) Find a point D on BC such that $CD=\dfrac{c_1\rho_1}{c_2\rho_2}BC$. Then $\angle DOC$ $=\theta_2$, and OD is equal to the maximum amplitude of the stationary waves in the layer 2.

iv) Draw a circular arc DE and let $\angle EOD=\angle AOB\times\dfrac{H_2/c_2}{H_1/c_1}$. Drop a perpendicular EF on OC. The coordinates of the point E are v_{30} and τ_{30}/z_2.

v) Find a point G on EF such that $GF=\dfrac{c_2\rho_2}{c_3\rho_3}EF$. Then
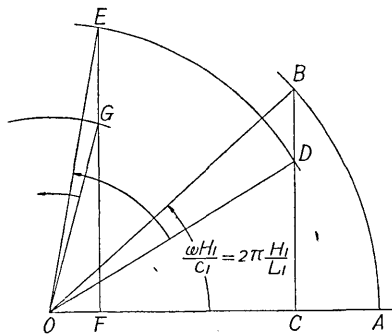

Fig. 1. $OA=1$
$DC: BC=c_1\rho_1 : c_2\rho_2$
$GF : EF=c_2\rho_2 : c_3\rho_3$
$\angle AOB=\varphi_1=2\pi H_1/L_1$
$\angle AOD=\theta_2$
$\angle DOE=2\pi\dfrac{H_1}{L_1}\left(\dfrac{H_2}{c_2}\Big/\dfrac{H_1}{c_1}\right)$
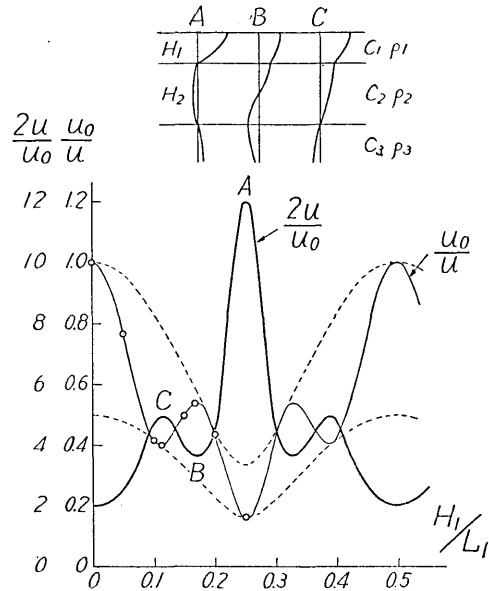$\angle AOE=\varphi_2$
$\angle AOG=\theta_3$

Fig. 2. $\rho_1=\rho_2=\rho_3$
$c_1 : c_2 : c_3=1 : 3 : 6$
$H_1 : H_2 : H_3=1 : 6 : \infty$
u_0: Amplitude of the incident waves
$2u$: Amplitude at the ground surface

$\angle GOF=\theta_3$ and the segment OG is equal to the maximum amplitude of the stationary oscillation in the layer 3.

vi) The procedure will be repeated until we get the amplitude of the stationary waves in the bedrock n. Then the amplitude gives us the value of u_0/u.

vii) The above procedures, i) to vi), are repeated, giving H_1/L_1 different values.

5. In Figs. 2–4 are shown spectral responses of grounds of various structures. In Fig. 2, the constants of layers are

$$\rho_1 : \rho_2 : \rho_3 = 1 : 1 : 1,$$
$$c_1 : c_2 : c_3 = 1 : 3 : 6,$$
$$H_1 : H_2 : H_3 = 1 : 6 : \infty.$$

In this case the spectral response is symmetric about the value of $H_1/L_1=0.25$ and has the period of $H_1/L_1=0.5$. Small circles in the Figure is the point obtained by the present method. Two dotted curves are envelopes of the u_0/u curve. As can be seen from Fig. 1, the upper envelope represents the radius vector OD of the ellipse which is described by the point D. This envelope can therefore be expressed as $\sqrt{1-k^2 \sin \varphi}$, where $k^2 = 1 - \left(\dfrac{c_1\rho_1}{c_2\rho_2}\right)^2$ and $\varphi = 2\pi \dfrac{H_1}{L_1}$. The lower envelope is merely $c_2\rho_2/c_3\rho_3$ times the upper envelope. These envelopes represent the effect of the first layer.
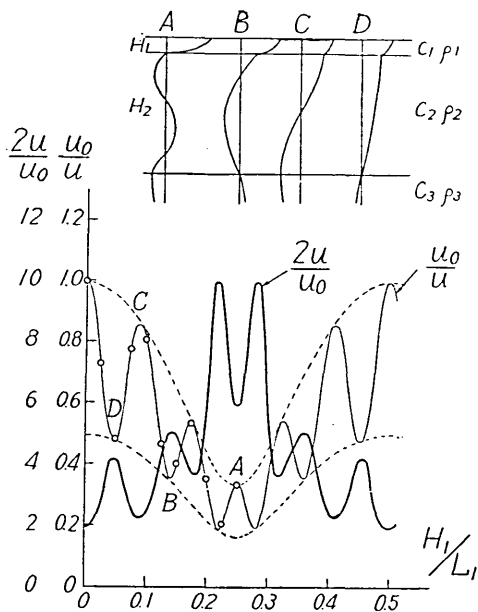

Fig. 3. $\rho_1 = \rho_2 = \rho_3$
$c_1 : c_2 : c_3 = 1 : 3 : 6$
$H_1 : H_2 : H_3 = 1 : 15 : \infty$
u_0: Amplitude of the incident waves
$2u$: Amplitude at the surface.

The u_0/u curve contacts with the upper envelope as at B when OG in Fig. 1 is equal to OE. This occurs only when φ_2 is zero or a multiple of 180°. This is when the stationary oscillation has a loop at the lower boundary of the layer 2 and we have $\tau_{30}=0$ there.

When the u_0/u curve contacts with the lower envelope as at C, we have $OE = \dfrac{z_2}{z_3} \times OG$, so the angle φ_2 must be a multiple of 90°. We have then $v_{30}=0$ which shows that there occurs a node at the inter-surface between the layers 2 and 3.

A in Fig. 2 is the special point where we have a node at each boundary between layers. This case occurs only when φ_1 and φ_2 are mutiples of 90° at the same time.

The points A, B, C do not generally coincide with the extremums of the u_0/u curve but are very near to them.

In the upper part of Fig. 2, modes of oscillation of the ground are shown for cases A, B and C, but they are not to scale.

In Fig. 3, the structure of the ground is as follows:

$$\rho_1 : \rho_2 : \rho_3 = 1 : 1 : 1$$
$$c_1 : c_2 : c_3 = 1 : 3 : 6$$
$$H_1 : H_2 : H_3 = 1 : 15 : \infty$$

Since H_2/c_2 is large in this case, the u_0/u curve oscillates more frequently between the envelopes, which are entirely the same curves with the envelopes in Fig. 2. Results shown in Figs. 2 and 3 agree with K. Kanai's.[1]

Fig. 4 deals with the case in which

$$\rho_1 : \rho_2 : \rho_3 : \rho_4 : \rho_5 = 1 : 1 : 1 : 1 : 1$$
$$c_1 : c_2 : c_3 : c_4 : c_5 = 1 : 2 : 3 : 4 : 8$$
$$H_1 : H_2 : H_3 : H_4 : H_5 = 1 : 3 : 6 : 6 : \infty$$

In this case curves are symmetric about the ordinate $H_1/L_1 = 0.5$ and have the period of $H_1/L_1 = 1.0$. A in the Fig. 4 indicates a special case in which there is a node at each boundary between layers. Even in such case, the value of $2u/u_0$ will never exceed $2c_n\rho_n/c_1\rho_1$.

6. The propagation of elastic waves in the ground normally to layers is similar to the propagation of electric waves in a transmission line with one end open. The soil particle velocity and stress due to the elastic wave corres-
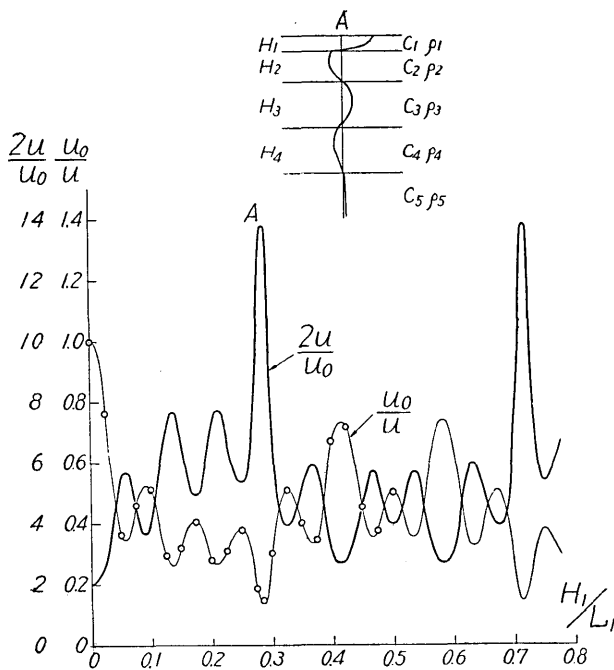

Fig. 4. $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5$
$c_1 : c_2 : c_3 : c_4 : c_5 = 1 : 2 : 3 : 4 : 8$
$H_1 : H_2 : H_3 : H_4 : H_5 = 1 : 3 : 6 : 6 : \infty$

1) K. KANAI, "Relation between the Nature of Surface Layers and the Amplitude of Earthquake Motions. Part II.", *Bull. Earthq. Res. Inst.*, **31** (1953), 219.

pond to the current and voltage in the line. The present method is, in effect, a modification of the well-known method of the position angle but it has its own merit in that it furnishes a quick and intuitive solution of the problem.

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18.　地盤のスペクトル応答を求める図式解法

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　多数層から成る地盤に下方から垂直に無限長の単色弾性波が伝つて来る場合は分布常数線路に交流電圧が伝わる場合と同等であつて位置角の方法によつて計算することができる．

　特に地盤内の粘性による減衰が省略できる場合には地盤内には定常波ができ，且つ同一層内の速度振巾と応力振巾/特性インピーダンスとは余弦と正弦の関係になるから，コンパスと分度器による簡単な作図によつて地表の振幅を単位とした地層内の振巾を次々に求めることができる．その方法は第1図によつて見うる．第2〜4図には二三の例題を示した．

　減衰を考えないから，短周期の振動や高次の振動に対しては次第に実際と喰違つてくるが，地盤の大体の振動特性を簡単に早く推察しうる所に利点がある．

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