19. *Range of Possible Existence of Rayleigh- and Sezawa-Waves in a Stratified Medium.*

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1. Introduction

The problem of the possible existence of surface waves in a stratified medium was investigated by a number of authors and its nature was elucidated pretty well.

Of the various kinds of surface waves which are propagated upon the stratified medium, Love-waves have very simple property and will require no explanation today.

However, the range of existence of Rayleigh- as well as Sezawa-waves (M_2-waves), which consist of P- and SV-waves has not been clarified enough to satisfy us and make it possible for us to see into the details of the subject. The investigation was given this direction by K. Kanai[1] some years ago, and his graphs show the critical boundary of existence and non-existence of surface waves under a given condition of L/H, μ'/μ and $(v/v')^2$.

(Where L; wave length
H; thickness of the layer
μ', μ; rigidity of the layer and the substratum respectively
v', v; velocity of S-waves in the layer and the substratum respectively.)

Now, we will calculate this critical boundary more precisely, giving the parameters with smaller intervals.

2. The fundamental equation

Here, we will follow the investigation performed by K. Sezawa and K. Kanai.

1) K. KANAI, "On the M_2-waves (Sezawa waves)," *Bull. Earthq. Res. Inst.*, **29** (1951), 39.

The equation of dispersion curves of surface waves transmitted through a stratified body is, as presented originally by the above two authors[2])

$$\begin{aligned}&\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cosh s'H\\&+\frac{r'}{f}\varphi\left\{\frac{s}{f}\left(2-\frac{k'^2}{f^2}\right)^2-\frac{4rs'^2}{f^3}\right\}\cosh r'H\sinh s'H\\&+\frac{s'}{f}\varphi\left\{\frac{r}{f}\left(2-\frac{k'^2}{f^2}\right)^2-\frac{4sr'^2}{f^3}\right\}\sinh r'H\cosh s'H\\&+\left\{\left(2-\frac{k'^2}{f^2}\right)^2\vartheta+\frac{4r'^2s'^2}{f^4}\zeta\right\}\sinh r'H\sinh s'H=0\,,\qquad\ldots.(1)\end{aligned}$$

where

$$\varphi=\frac{\mu' k^2k'^2}{\mu f^4}\,,\quad \zeta=\frac{4rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)^2-\alpha^2\,,\quad \eta=\frac{2rs}{f^2}\left(\frac{\mu'}{\mu}-1\right)\beta-\alpha\gamma,$$

$$\vartheta=\frac{rs}{f^2}\beta^2-\gamma^2,\quad \alpha=\frac{2\mu'}{\mu}-\left(2-\frac{k^2}{f^2}\right),\quad \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2,$$

$$\gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-\left(2-\frac{k^2}{f^2}\right),$$

$$r^2=f^2-h^2\,,\quad s^2=f^2-k^2\,,\quad r'^2=f^2-h'^2\,,\quad s'^2=f^2-k'^2,$$

$$h^2=\rho p^2/(\lambda+2\mu)\,,\quad k^2=\rho p^2/\mu\,,\quad h'^2=\rho' p^2/(\lambda'+2\mu')\,,\quad k'^2=\rho' p^2/\mu',$$

ρ and ρ' are the density and $f=2\pi/L$. ....(2)

The critical condition of the existence of surface waves is that one of the functions $\exp(rz)$ and $\exp(sz)$, which give the vertical distribution of amplitude, fails to be a real function that decreases exponentially and becomes a circular function[3]). Since k^2 is larger than h^2, the above condition is given by the relation

$$s=0\,,\qquad \text{or}\qquad f^2=k^2\,.$$

Putting this into the equation (1), we have[4])

2) K. SEZAWA and K. KANAI, "Discontinuity in the Dispersion Curves of Rayleigh Waves," *Bull. Earthq. Res. Inst.*, **13** (1935), 237.

3) In the semi-infinite part the type of waves must be [E]. (Cf. Y. SATÔ, "Definition and Classification of Surface Waves," *Bull. Earthq. Res. Inst.*, **32** (1954), 161.)

4) K. KANAI, *loc. cit.*, 1).

$$\frac{4r's'}{f^2}\left(2-\frac{k'^2}{f^2}\right)\eta-\frac{r's'}{f^2}\left\{4\vartheta+\left(2-\frac{k'^2}{f^2}\right)^2\zeta\right\}\cosh r'H\cosh s'H$$

$$-\frac{4rr's'^2}{f^4}\varphi\cosh r'H\sinh s'H+\frac{rs'}{f^2}\varphi\left(2-\frac{k'^2}{f^2}\right)^2\sinh r'H\cosh s'H$$

$$+\left\{\frac{4r'^2s'^2}{f^4}\zeta+\left(2-\frac{k'^2}{f^2}\right)^2\vartheta\right\}\sinh r'H\sinh s'H=0\ , \qquad \ldots.(3)$$

where

$$\varphi=\frac{\mu'k'^2}{\mu f^2}\ , \qquad \zeta=-\alpha^2\ , \qquad \eta=-\alpha\gamma\ , \qquad \vartheta=-\gamma^2\ ,$$

$$\alpha=\frac{2\mu'}{\mu}-1\ , \quad \beta=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-2\ , \quad \gamma=\frac{\mu'}{\mu}\left(2-\frac{k'^2}{f^2}\right)-1\ ,$$

$$\frac{k'^2}{f^2}=\left(\frac{v}{v'}\right)^2, \quad \frac{r^2}{f^2}=1-\frac{1}{\lambda/\mu+2}\ , \quad \frac{r'^2}{f^2}=1-\frac{1}{\lambda'/\mu'+2}\left(\frac{v}{v'}\right)^2,$$

$$\frac{s'^2}{f^2}=1-\left(\frac{v}{v'}\right)^2. \qquad \ldots.(4)$$

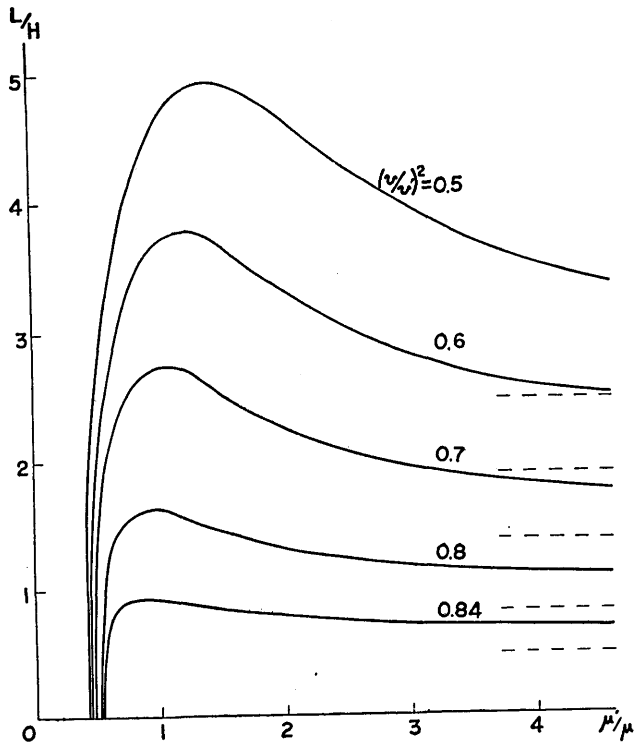

Fig. 1. Critical lines separating the existence and non-existence of Rayleigh waves. Above the curves Rayleigh waves are allowed to exist, while below them they cannot exist. Sezawa waves are not allowed to exist throughout the domain.

Table I. The values of L/H that give the critical boundary conditions μ'/μ and $(v/v')^2$

μ'/μ \ $(v/v')^2$	0.5	0.6	0.7	0.8	0.84	0.8453…	0.85	0.90	0.985	1.0
0						Critical value that separates two conditions. When $(v/v')^2>0.8453\cdots$, Rayleigh waves always exist and Sezawa waves also have the chance of existence, while if $(v/v')^2<0.8453\cdots$, no Sezawa waves are allowed to exist and only Rayleigh waves have the chance of existence.	0.795	1.274	1.606	1.651
0.25							–	–	–	1.648
0.30							0.800	1.264	–	–
0.40							–	–	1.458	1.497
0.419	0						–	–	–	–
0.443	–	0					–	–	–	–
0.472	–	–	0				–	–	–	–
0.5	2.574	2.033	1.417				0.654	0.990	1.246	1.282
0.512	–	–	–	0			–	–	–	–
0.532	–	–	–	–	0		–	–	–	–
0.537	–	–	–	–	–		0	–	–	–
0.555	–	–	–	–	–			0.735	–	–
0.572	–	–	–	–	–			0	–	–
0.60	–	–	–	1.275	0.774				–	–
0.65	–	–	–	–	–				0.677	0.765
0.698	–	–	–	–	–				0	–
0.75	4.012	3.288	2.508	1.562	0.910					0.385
0.9	–	–	–	–	–					0.049
1.0	4.652	3.713	2.739	1.629	0.919					0
1.25	4.909	3.785	2.681	1.543	0.877					0.128
1.5	4.936	3.677	2.533	1.449	0.839					0.295
2.0	4.654	3.321	2.259	1.313	0.781					0.513
2.048	–	–	–	–	–				0	–
2.2	–	–	–	–	–				0.403	–
2.5	4.254	3.021	2.072	1.229	0.745				0.525	–
3.0	3.949	2.811	1.949	1.171	0.712				0.648	0.700
3.5	3.700	2.665	1.863	1.130	0.702				–	–
4.0	3.519	2.558	1.801	1.099	0.690				0.731	0.780
8.47	–	–	–	–	–			0	–	–
114.	–	–	–	–	–		0	–	–	–
∞	2.471	1.895	1.372	0.812	0.469		0.450	0.738	0.946	0.972

lines of the existence of Rayleigh and Sezawa waves, when are given. $(\lambda=\mu,\ \lambda'=\mu')$.

1.1	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.5	2.6	3.0
1.899	2.098	2.418	2.683	2.913	3.120	3.310	3.479	3.570	3.652	3.957
1.883	2.069	2.368	2.618	2.843	3.055	3.263	3.475	3.581	3.688	4.150
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
1.491	1.666	1.970	2.246	2.513	2.786	3.072	3.384	3.548	3.722	4.479
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	–	–	–	–	–	–
0.945	1.195	1.563	1.878	2.185	2.513	2.868	3.279	3.508	3.752	4.852
–	–	–	–	–	–	–	–	–	–	–
0.811	1.056	1.405	1.701	1.997	2.324	2.704	3.182	3.466	3.779	5.254
–	–	–	–	–	–	–	–	–	–	
0.836	1.046	1.348	1.604	1.859	2.146	2.511	3.036	3.393	3.824	
0.888	1.076	1.353	1.588	1.821	2.081	2.418	2.941	3.336	3.862	
–	–	–	–	–	–	–	–	–	–	
–	–	–	–	–	–	–	–	–	–	
–	–	–	–	–	–	–	–	–	–	
0.960	1.123	1.375	1.591	1.801	2.035	2.338	2.835	3.260	3.919	
–	–	–	–	–	–	–	–	–	–	
1.000	1.152	1.392	1.598	1.799	2.019	2.304	2.787	3.213	3.961	
–	–	–	–	–	–	–	–	–	–	
–	–	–	–	–	–	–	–	–	–	
1.127	1.252	1.460	1.640	1.814	2.000	2.231	2.622	3.037	4.260	

3. Numerical calculation

Assuming various values of $(v/v')^2$ and putting them into the above equation, we have calculated that relation between L/H and μ'/μ which gives the critical condition of the existence of Rayleigh and Sezawa waves. (In this calculation λ and λ' are always assumed to be equal to μ and μ' respectively.)

The result is illustrated in Figs. 1, 2 and 3 and the numerical values are found in Table I. (In Fig. 3 the unit of abscissa is reciprocal to that of Fig. 2.)

4. Discussion of the result

Three of the curves in the Figs. 1 and 2 (curves corresponding to

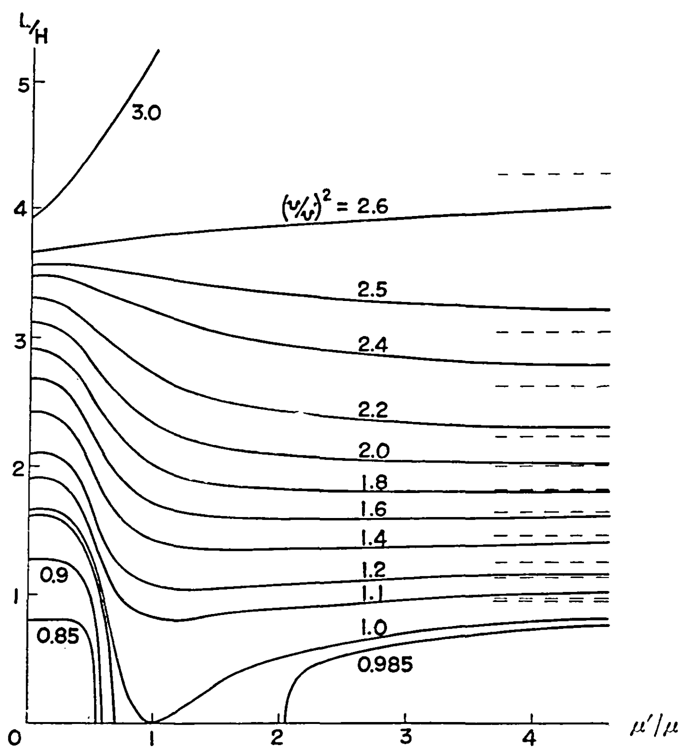

Fig. 2. Above the critical lines only Rayleigh waves can exist, while below the curves both Rayleigh and Sezawa waves are allowed to exist.

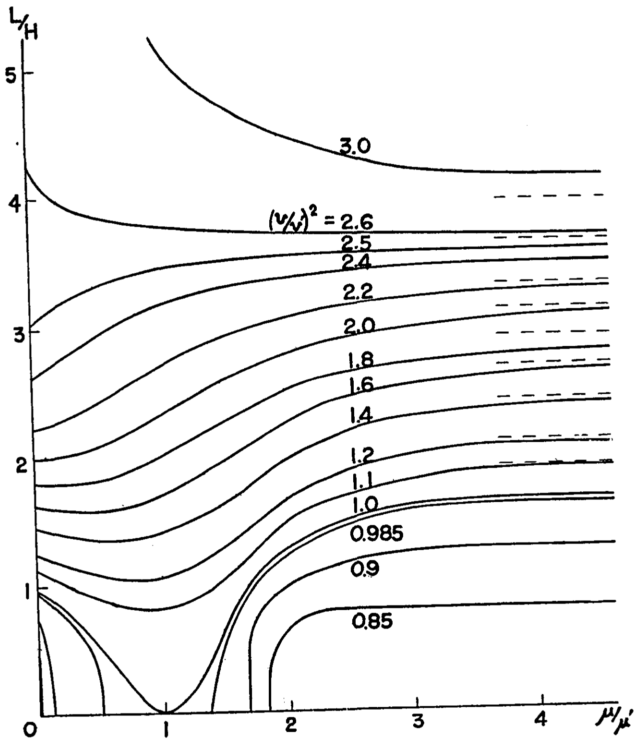

Fig. 3. Above the critical lines only Rayleigh waves can exist, while below the curves both Rayleigh and Sezawa waves are allowed to exist. This figure has the same content with the former one, but the abscissa is reversed.

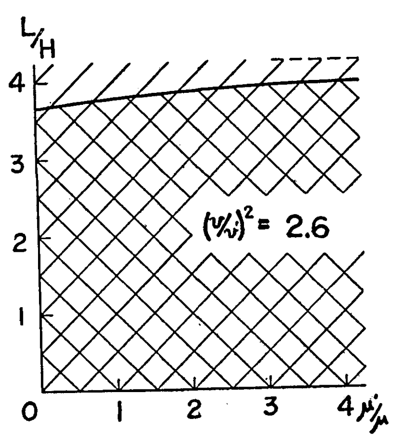

Fig. 4a.

$(v/v')^2=0.5$, 1, 2) were already given by K. Kanai[5], but in that case three curves were isolated and the connection between them was somewhat difficult to grasp. Now, however, the tendency of the shift of curves has been made fairly clear.

At first, when the parameter $(v/v')^2$ is large, namely the velocity of S-waves is much higher in the substratum than in the upper medium, the Rayleigh waves is

5) *ibid.*

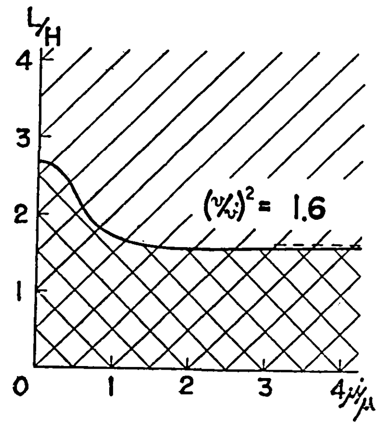

Fig. 4b.

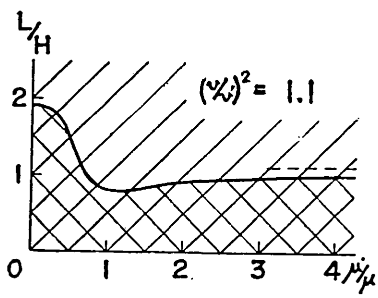

Fig. 4c.

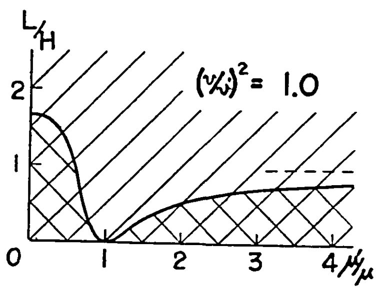

Fig. 4d.

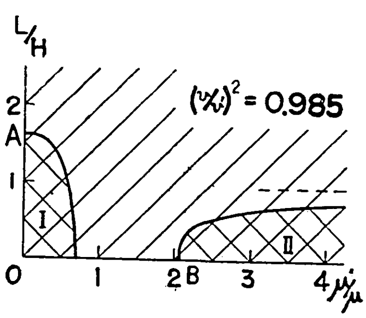

Fig. 4e.

allowed to exist throughout the domain of $\frac{\mu'}{\mu}$, $\frac{L}{H}$-plane while the Sezawa waves can exist only below the critical boundary line. (See Figs. 4a and 4b. In these figures, the range of possible existence of Rayleigh waves is indicated by the hatched area, while in the double hatched area Rayleigh and Sezawa waves can both exist.) As the value decreases, the boundary line shifts downward and its middle part droops (see Fig. 4c), and finally when $(v/v')^2$ becomes unity it touches the axis of μ'/μ. (Cf. Fig. 4d and K. Kanai's paper Fig. 9.). If the value $(v/v')^2$ still decreases, the domain of the existence of Sezawa waves diminishes further and is separated into two areas (see Fig. 4e) until it vanishes when $(v/v')^2$ takes the value $0.8453\cdots$[6]. And in this case the tendency of the shift of curves can be seen better in Fig. 3 than in Fig. 2.

If $(v/v')^2$ is smaller than $0.8453\cdots$, no Sezawa waves are allowed to exist, and even the existence of Rayleigh-waves is somewhat

6) This condition implies that the velocity of Rayleigh waves in the layer is equal to that of S-waves in the substratum.

As the value $(v/v')^2$ diminishes, the point A in Fig. 4e approaches to 0 and the area I vanishes, while the point B tends to $+\infty$ and thus the area II becomes null. This property may be proved from the expression (3) by a little calculation.

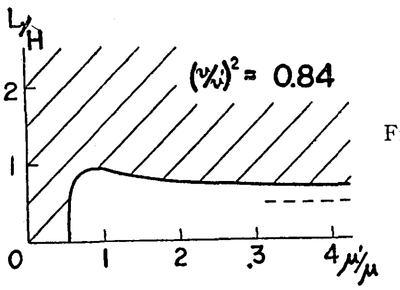

Fig. 4f.

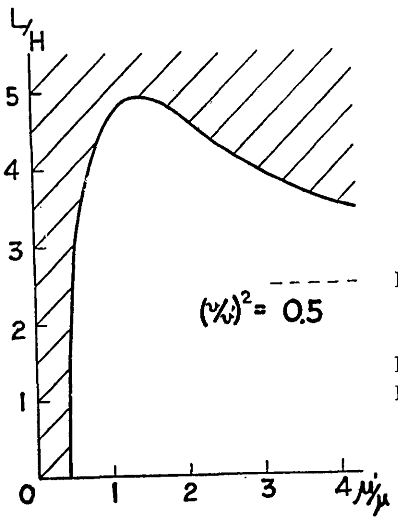

Fig. 4g.

Hatched area; Rayleigh waves only can exist.
Double hatched area; Rayleigh and Sezawa waves can both exist.

limited. If $(v/v')^2$ decreases further, the domain of the existence of Rayleigh waves above the critical boundary line diminishes, namely the blank area (domain without surface waves) increases. (Cf. Figs. 4f and 4g.) The details of the nature of this matter will be found in Fig. 1.

This investigation was performed under the kind guidance of Dr. K. Kanai, to whom the authors' heartiest thanks are due.

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19. 表面層をもつ媒質内の Rayleigh 波と妹沢波の存在限界

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表面層をもつ媒質内を伝わる分散性 Rayleigh 波と妹沢波 (M_2 波) の存在限界をしらべた.

v, v' をそれぞれ下層, 表面層の S 波の速度とすると, $(v/v')^2=0.5$, 1.0, 2.0, の代表的な場合については, $\lambda/\mu=\lambda'/\mu'=1$ としたときの計算が既に金井によつて報告されている[1)].

ここではその方法に準じて, より精しく数値計算を行い図で見られるような結果を得た.

これによつて存在限界の変動を明確に把えるとともに, 上の代表的な場合をも充分に結びつけて理解することが出来た.

特に注意すべきは下層の S 波の速度が上層の Rayleigh 波の速度に等しくなる (すなわち $(v/v')^2=0.8453\cdots$) ときである. Rayleigh 波について言えば, これを境にして $(v/v')^2\geqq 0,8453\cdots$ では常に存在するが, $(v/v')^2<0.8453\cdots$ のときは比較的短波長の範囲に存在しない部分が出来る. 妹沢波については $(v/v')^2>0.8453\cdots$ の場合は比較的短波長の範囲に存在するようになるが, $(v/v')^2\leqq 0.8453\cdots$ のときは常に存在しない.

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