22. *Analysis of the Dispersion Curves of Love-Waves.*

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1. Introduction

A characteristic feature of Love-wave is that it shows dispersion owing to the stratified structure or more generally to the heterogeneity of the medium. It is usual to assume two or more homogeneous layers and choose the values of their density and elastic constants so as to explain best the observed dispersion curve.

But it must be remarked that within the errors of observation the result can not be unique—we can suppose different structures which equally fit the observations[1]. In other words, we can conclude only with ambiguity where and how the discontinuities exist in the earth's crust.

It seems, then, rather natural to assume that the density and elastic constants vary continuously in the medium. This might be looked upon as the result of averaging over various surface layers which the Love-wave encounters while travelling long distance from its source to the place where it is observed.

In this paper, we shall first develop the general theory of dispersion under the assumed heterogeneity, next the inverse problem—given the dispersion curve to determine the structure of the medium—will be treated.

2. Dispersion of Love-type waves propagated over heterogeneous medium

We shall treat the problem in a two-dimensional form. The x-axis indicates the surface and z-axis is taken vertically downward. Density ρ and rigidity μ are thought to be the analytic positive functions of z. The displacement v occurs in the vertical direction both to x- and z-axes.

Then the equation of motion is expressed as

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1) Y. SATÔ, *Bull. Earthq. Res. Inst.*, **29** (1951), 519.

$$\rho(z)\frac{\partial^2 v}{\partial t^2}=\mu(z)\frac{\partial^2 v}{\partial x^2}+\frac{\partial}{\partial z}\left(\mu(z)\frac{\partial v}{\partial z}\right). \qquad (1)$$

Assuming now $v=\exp(ipt-ifx)V(z)$,
the equation (1) reduces itself to

$$\frac{d}{dz}\left(\mu\frac{dV}{dz}\right)+\mu s^2 V=0 \qquad (2)$$

with

$$\left.\begin{array}{l} s^2=(\rho/\mu)p^2-f^2=f^2\{(v_Q/v_s(z))^2-1\} , \\ v_Q=p/f , \quad v_s(z)=\sqrt{\mu/\rho} . \end{array}\right\} \qquad (3)$$

Let the variables be changed by the substitution

$$\left.\begin{array}{l} \theta=\int_0^z s dz , \\ Z(\theta)=\sqrt{\mu s}\,V(z) , \end{array}\right\} \qquad (4)$$

then the equation (2) is transformed into

$$d^2Z/d\theta^2+\{1+Q(\theta)\}Z=0 , \qquad (5)$$

with

$$Q(\theta)=-(\mu s)^{-1/2}(d/d\theta)^2(\mu s)^{1/2}=-(\mu s)^{-1/2}(d/sdz)^2(\mu s)^{1/2} . \qquad (6)$$

We now make following assumptions:

(i) $v_s(z)$ is a increasing function of z.

(ii) $v_Q>v_s(0)$. Then s^2 has only one zero-point $z=H$, and $v_s(H)=v_Q$. (7)

(iii) $|Q(\theta)|\ll 1$, if $|z-H|$ is large.

It is easy, then, to observe that the asymptotic solutions of the equation (5) are

(a) sinusoidal, if $z\ll H$ ($v_Q>v_s(z)$ and θ is real),

or

(b) exponential, if $z\gg H$ ($v_Q<v_s(z)$ and θ is imaginary).

We need the solution which converges to zero as $z\to\infty$. To obtain the corresponding sinusoidal function, we must remark that, owing to the many-valued character of s, the so-called Stokes' phenomenon happens. H. Jeffreys was the first who showed how to connect the above two asymptotic solutions each belonging to different regions[2]. According to him, the sinusoidal solution at $z \ll H$, which decreases exponentially for large z, is expressed as

$$Z(\theta) \sim \cos\{\pi/4-(\theta_H-\theta)\}\ , \tag{8}$$

or

$$V(z) \sim (\mu s)^{-1/2} \cos\left(\pi/4-\int_z^H s dz\right)\ . \tag{9}$$

In case the assumption (iii) is fulfilled near the surface, we can apply to the expression (9) the boundary condition

$$(dV/dz)_{z=0}=0\ , \tag{10}$$

and obtain

$$\tan\left(\int_0^H s dz-\pi/4\right)=\left\{-\frac{\mu}{2}\frac{d}{dz}\left(\frac{1}{\mu s}\right)\right\}_{z=0}\ , \tag{11}$$

or

$$\int_0^H s dz = n\pi+\frac{\pi}{4}-\mathrm{Tan}^{-1}\left\{\frac{\mu}{2}\frac{d}{dz}\left(\frac{1}{\mu s}\right)\right\}_{z=0}\ , \quad (n=0, 1, 2, \cdots)\ . \tag{12}$$

This equation gives the relation between the velocity v_Q and the wavelength $L(=2\pi/f)$ of the n-th Love-wave, if the functions $\rho(z)$ and $\mu(z)$ are known satisfying the conditions (i)–(iii).

Substituting in (12) the expression (3), we have

$$\int_0^H \{(v_Q/v_s(z))^2-1\}^{1/2} dz = \frac{L}{8}\left\{4n+1-\frac{4}{\pi}\,\mathrm{Tan}^{-1}\left(\frac{\mu}{2}\frac{d}{dz}\left(\frac{1}{\mu s}\right)\right)_0\right\}\ , \tag{12'}$$

from which we can observe that

$$\begin{cases} \text{if } L\to 0, \text{ then } H\to 0 \text{ and } v_Q\to v_s(0). \\ \text{if } L\to\infty, \text{ then } H\to\infty \text{ and } v_Q\to v_s(\infty). \end{cases}$$

To derive the formula (12), it was necessary to assume that $Q(\theta)$ becomes negligibly small near the surface; this condition may be fulfilled when H is of several wave-lengths, that is, when n differs from 0. But the only case $n=0$ being practically observed, we must next obtain the corresponding formula to (12) which is valid when L is not small compared with H.

2) Now called as W.K.B. method:
E. KEMBLE, *Fundamental Principles of Quantum Mechanics* (1937).
M. MOOSE and H. FESHBACH, *Methods of Theoretical Physics* (1953), chap. 9.

2.1. Near the zero point $z=H$, taking the first term of the expansion, s^2 is proportional to $(H-Z)$, then

$$\varphi \equiv \theta_{II} - \theta = \int_z^H s dz \propto \int_0^s s^2 ds = s^3/3 . \tag{13}$$

Putting this relation into (6), we get as the first approximation

$$Q(\theta) = -\varphi^{-1/6}(d^2/d\varphi^2)\varphi^{1/6} = (5/36)\varphi^{-2} . \tag{14}$$

Then the solutions of (5) are expressed as Bessel functions of order $\pm 1/3$ multiplied with $\varphi^{1/2}$. We choose the linear combination of them so as to converge to zero when $z \to \infty$, and obtain

$$V = (\varphi/\mu s)^{1/2}\{J_{1/3}(\varphi) + J_{-1/3}(\varphi)\} . \tag{15}$$

This procedure is due to R. E. Langer[3].

We can rewrite the expression (15) in the following form

$$\begin{aligned} V &= (\varphi/\mu s)^{1/2} \cos(\pi/6)\{\exp(i\pi/6) \cdot H_{1/3}^{(1)}(\varphi) + \exp(-i\pi/6) \cdot H_{1/3}^{(2)}(\varphi)\} \\ &= 2\cos(\pi/6)(\varphi/\mu s)^{1/2} |H_{1/3}^{(1)}| \cos(\pi/6 + \arg H_{1/3}^{(1)}) . \end{aligned} \tag{16}$$

If we apply the boundary condition (10) to (16), we have

$$\begin{aligned} \tan(\pi/6 + \arg H_{1/3}^{(1)}(\varphi_0)) &\{(d/dz) \arg H_{1/3}^{(1)}(\varphi)\}_{z=0} \\ &= \{(d/dz) \log ((\varphi/\mu s)^{1/2} |H_{1/3}^{(1)}(\varphi)|)\}_{z=0} . \end{aligned} \tag{17}$$

On utilizing the relations

$$\arg H_{1/3}^{(1)} = \tan^{-1}(Y_{1/3}/J_{1/3}) ,$$
$$|H_{1/3}^{(1)}| = (J_{1/3})^2 + (Y_{1/3})^2 ,$$
$$J_{1/3} \cdot \frac{d}{d\varphi} Y_{1/3} - Y_{1/3} \cdot \frac{d}{d\varphi} J_{1/3} = \frac{2}{\pi\varphi} ,$$

we get

$$y \equiv (d/dz) \arg H_{1/3}^{(1)} = -2s(\pi\varphi)^{-1}|H_{1/3}^{(1)}|^{-2} . \tag{18}$$

Then the equation (17) is written as

$$\arg H_{1/3}^{(1)}(\varphi_0) + \pi/6 = n\pi + \mathrm{Tan}^{-1}\{(\mu/2)(d/dz)(\mu y)^{-1}\}_{z=0} . \tag{19}$$

When $\varphi \to \infty$, we have $\arg H_{1/3}^{(1)} \to \varphi - (5/12)\pi$, $\sqrt{\varphi}\,|H_{1/3}^{(1)}| \to \sqrt{2/\pi}$ and $y \to -s$. This shows that (19) coincides with (12) when $z \ll H$.

3) R. E. LANGER, *Phys. Rev.*, **51** (1937), 669.
MOOSE and FESHBACH, *loc. cit.*
I. Imai extended the result of Langer:
I. IMAI, *Phys. Rev.* **74** (1948), 113; **80** (1950), 1112.
His formulas are available when a more precise knowledge about the behaviour of the solution is needed.

We next try to bring the expression (19) to a more convenient form, under the condition which reads

$$1 \gg |\{(\mu/2)(d/dz)(\mu y)^{-1}\}_{z=0}| = |-(\mu/2)\{(d/dz)(\mu s)^{-1}\}(\pi\varphi/2)|H_{1/3}^{(1)}|^2 + (\pi/4)(d/d\varphi)(\varphi|H_{1/3}^{(1)}|^2)|_{z=0} . \quad (20)$$

For a small quantity ε compared with φ, we have

$$\arg H_{1/3}^{(1)}(\varphi+\varepsilon) \doteqdot \arg H_{1/3}^{(1)}(\varphi) + \varepsilon(d/d\varphi)\arg H_{1/3}^{(1)}(\varphi) = \arg H_{1/3}^{(1)}(\varphi) + 2\varepsilon(\pi\varphi)^{-1}|H_{1/3}^{(1)}(\varphi)|^{-2} .$$

Then the equation (19) is reduced into

$$n\pi - \pi/6 = \arg H_{1/3}^{(1)}[\varphi_0 - (\pi\varphi_0/2)^2|H_{1/3}^{(1)}(\varphi_0)|^4\{(\mu_0/2)(d/dz)(\mu s)^{-1}\}_0 + (\pi^2/8)\varphi_0|H_{1/3}^{(1)}(\varphi_0)|^2(d/d\varphi)\{\varphi_0|H_{1/3}^{(1)}(\varphi_0)|^2\}] . \quad (21)$$

To solve this equation we can utilize the numerical tables[4] of $\arg H_{1/3}^{(1)}$. In the case $n=0$, the relation

$$\arg H_{1/3}^{(1)}(0.835) = -\pi/6$$

shows that we can take 0.835 for the first approximation value of φ_0.

Again from the tables[4] of $|H_{1/3}^{(1)}|$

$$|H_{1/3}^{(1)}(0.835)| \doteqdot 0.85 ,$$

$$(d/d\varphi)|H_{1/3}^{(1)}(0.835)| \doteqdot -0.475 .$$

With these values the equation (21) becomes

$$\varphi_0 - 0.035 + 0.9 \times \{(\mu/2)(d/dz)(\mu s)^{-1}\}_{z=0} = 0.835$$

or

$$\varphi_0 \equiv \int_0^H s dz \doteqdot 0.87 - 0.9\{(\mu/2)(d/dz)(\mu s)^{-1}\}_{z=0} . \quad (22)$$

Comparing this formula with (12), we see that the principal term $(\pi/4)$ is augmented about 11 per cent.

When the condition (20), or

$$[(\mu/2)(d/dz)(\mu s)^{-1}]_{z=0} \ll 1$$

does not hold, we must use the equation (19). In this case the equation (12) of the former section may be used as the first approximation formula.

2.2. In case the equation (2) has an explicit solution satisfying the

4) G. N. WATSON, *Theory of Bessel Functions* (1922), p. 714.

boundary condition, W.K.B. solution (9) or (15) proves to be an asymptotic function of it.

Sezawa[5] and later Satô[5] treated the case when ρ is constant and μ increases linearly. In this case the solution of (2) which tends to zero for large z is given using Whittaker's function

$$V=\zeta^{-1/2}W_{\kappa,0}(\zeta)\,, \tag{23}$$

where

$$\zeta=2f\mu/\beta\,, \qquad \kappa=\rho p^2/2f\beta\,, \tag{24}$$

and

$$\mu=\mu_0+\beta z\,.$$

When $\kappa\gg 1$ and $\kappa\gg\zeta$, the expression (23) becomes, disregarding the constant factor, asymptotically[6]

$$V\sim\zeta^{-1/4}\cos(\kappa\pi-2\sqrt{\kappa\zeta}-\pi/4)\,. \tag{25}$$

Meanwhile our calculation goes on as follows:

$$s=f\{\rho p^2/\mu f^2-1\}^{1/2}=f(4\kappa/\zeta-1)^{1/2}\,,$$

$$\mu s=\beta\zeta(4\kappa/\zeta-1)^{1/2}\,,$$

$$\int_z^H s dz=(1/2)\int_\zeta^{4\kappa}(4\kappa/\zeta-1)^{1/2}d\zeta$$

$$=-(1/2)(4\kappa\zeta-\zeta^2)^{1/2}+\kappa\,\mathrm{Cos}^{-1}(\zeta/2\kappa-1)\,.$$

Putting these results into (9), we have

$$V=\beta^{-1/2}(4\kappa\zeta-\zeta^2)^{-1/4}\cos\{\pi/4+(1/2)(4\kappa\zeta-\zeta^2)^{1/2}-\kappa\,\mathrm{Cos}^{-1}(\zeta/2\kappa-1)\}\,.$$

We can easily verify that this expression approaches to (23) when $\kappa\gg\zeta$.

We can also observe that with these values equation (15) expresses the asymptotic form of Whittaker's function when κ and ζ are large and $\zeta/\kappa\approx 4$[6].

Y. Satô newly calculated the greatest roots of $dV/d\zeta=0$, according to his formulation of reference 5). His result is shown in Table I and Fig. 1 (dispersion curve).

In our method, we first reduce the expression of φ_0 to the following form:

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5) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **9** (1931), 310.
Y. SATÔ, *Bull. Earthq. Res. Inst.*, **33** (1952), 1.

6) A. ERDÉLYI, *Higher Transcendental Functions*, I. (1953), pp. 279 and 281.

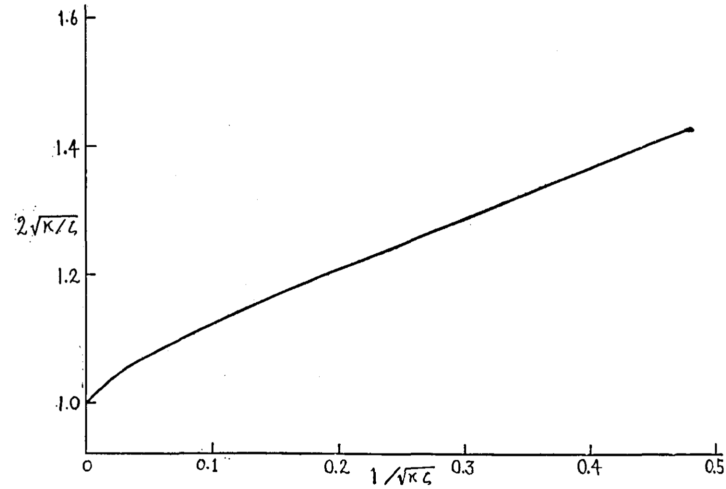

Fig. 1. Ordinate; Phase velocity of surface waves.
Abscissa; Period of surface waves.

Unit of velocity is that of S-waves at the free surface.

Unit of length is the distance from the free surface to the depth where the rigidity is twice that of the surface. Unit of time is determined by the ratio of above two quantities. If we assume, say, the velocity of S-waves at the surface to be 4.0 km/sec and the rate of change of rigidity adequately, the unit of time becomes nearly 1 minute.

$$\varphi_0=\kappa(\omega-\sin\omega)\ ,$$

$$\zeta=2\kappa(1+\cos\omega)\ , \qquad (\pi\geqq\omega\geqq0)$$

Then the equation (22) is expressed as

$$\omega-\sin\omega=0.87/\kappa-\varepsilon\ ,$$

with

$$\varepsilon=0.9/\kappa\{\mu/2\cdot d/dz(\mu s)^{-1}\}_0$$

$$=0.9(4\kappa^2)^{-1}\cot\omega(1-\cos\omega)^{-1}\ .$$

Neglecting the small quantity ε, we can solve this equation by help of the numerical table of trigonometric functions. The second approximation is obtained by iteration, on making use of the relation

$$\delta\zeta=-2\kappa\sin\omega\cdot\delta\omega=2\kappa\sin\omega(1-\cos\omega)^{-1}\cdot\varepsilon$$

$$=0.9(2\kappa)^{-1}\cos\omega(1-\cos\omega)^{-2}\ .$$

Table I.

[S] Satô's result
[I] First approximation
[II] Second approximation

κ	ζ[S]	ζ[I]	ζ[II]
1.5	3.00	2.97	2.97
2.5	6.45	6.27	6.35
3.5	10.0	9.73	9.9
4.5	13.7	13.3	13.5
5.5	17.4	16.9	17.2
6.5	21.1	20.6	20.9
7.5	24.9	24.4	24.7
8.5	28.8	28.1	28.5
9.5	32.5	31.8	32.2
11.5	40.2	39.4	39.9
13.5	47.8	47.1	47.6
16.5	59.3	58.5	59.1
19.5	71.1	70.1	70.8
∞	4κ	4κ	4κ

The last two columns of Table I contain the values of ζ calculated in this way.

Comparison with Satô's accurate calculation shows that our method gives satisfactory result and that even the first approximation is capable of use.

3. Analysis of the dispersion curve

Now we proceed to solve the inverse problem—to analyse the dispersion curve obtained as the result of observation.

If the surface values of ρ, μ and their derivatives are known, and the wave-length L is given as the function of v_Q from the dispersion curve, equations (12) and (22) are expressed in the form of the integral equation

$$\varphi_0/f \equiv \int_0^H \{(v_Q/v_s(z))^2-1\}^{1/2}dz = I(v_Q)\ . \tag{26}$$

For instance, if we can neglect the last term of the right side of (22), we have for $n=0$

$$I(v_Q)=(1.11/8)L(v_Q)\ . \tag{27}$$

We now set

$$\left.\begin{aligned}&(v_s(0)/v_Q)^2=t\ ,\\ &(v_s(0)/v_s(z))^2=\tau\ ,\quad (0\leqq t,\ \tau\leqq 1)\end{aligned}\right\} \tag{28}$$

and write $I(v_Q)$ merely as $I(\tau)$, then the equation (26) is transformed into the following integral equation of Abel's type

$$-\int_\tau^1 \{(t-\tau)/\tau\}^{1/2}(dz/dt)dt = I(\tau)\ , \tag{29}$$

whose solution is obtained in the usual manner as

$$z(t) = -(2/\pi)(d/dz)\int_t^1 \{t/(t-\tau)\}^{1/2} I(\tau)d\tau\ . \tag{30}$$

Integrating the right side of (30) by parts, we get

$$z(t)=(2/\pi)\int_{\tau=1}^{\tau=t}(\tau-t)^{-1/2}d(\tau^{1/2}I(\tau))\ . \tag{31}$$

We then reduce the variables to v_Q and $v_s(z)$ by (28) and have finally

$$z=(2/\pi)\int_{v_Q=v_s(0)}^{v_Q=v_s(z)} v_s(z)\{(v_s(z)/v_Q)^2-1\}^{-1/2}d(I(v_Q)/v_Q)\ . \tag{32}$$

When the expression (27) is used, this formula is written as

$$z(v_s)=(1.11/4\pi)v_s\int_0^{T_s}\{(v_s/v_Q(T))^2-1\}^{-1/2}dT\ , \tag{33}$$

with

$$\left.\begin{aligned} T&=L/v_Q\,, \\ v_Q(T_s)&=v_s\,. \end{aligned}\right\} \qquad (34)$$

3.1. It is difficult to determine the surface values of the derivatives of ρ and μ. They are zero when the earth's crust has actually homogeneous surface layer. Moreover, Love-waves of short wave-length being hardly observed, we assume for the present that derivatives are all zero at the surface. Then we may be allowed to use the equation (33).

As an example of numerical calculation, we took up one of the dispersion curves of Love-waves from the Assam earthquake of 1950 gained by Akima (Fig. 2)[7].

Using the values read from his curve we performed the numerical integration of the right side of the equation (33). (Fig. 3 and Table II).

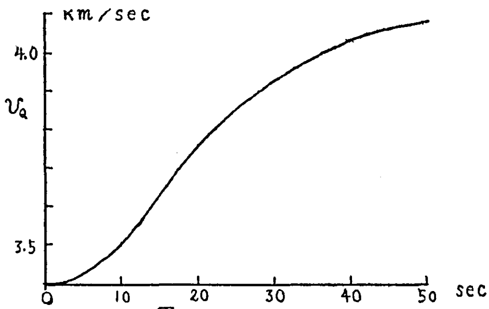

Fig. 2. Dispersion curve gained by Akima.

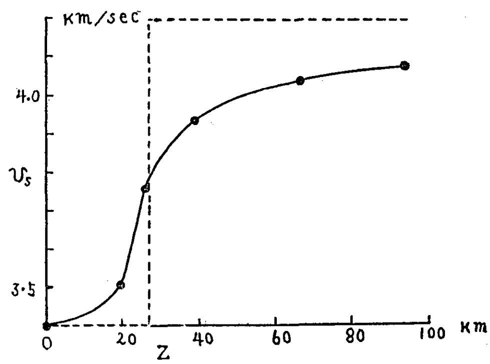

Fig. 3. Full line indicates our result. Dotted line shows the structure employed by Akima.

Table II.

T_s sec	0	10	20	30	40	50
v_s km/sec	3.40	3.51	3.76	3.94	4.04	4.08
z km	0	19.9	26.0	39.0	66.0	94.0

The result seems to give us information about the structure of the earth's crust, at least in the sense inferred in the introduction. In the next stage we should like to treat Rayleigh-waves.

7) T. AKIMA, *Bull. Earthq. Res. Inst.*, **30** (1952), 237.

Finally the author wishes to express his thanks to Mr. Y. Satô who kindly took the labour of troublesome numerical calculation to be compared with our result, and, still more, whose knowledge on surface waves has been of great aid to the author.

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22. Love 波の分散曲線の解析

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Love 波の分散曲線から地殼の構造を定めるとき観測誤差内で一意的に定められないことが知られている．そこで平均的な意味で ρ 及び μ が連続的に変化するものとして所謂 W.K.B. 法の考えによつて近似的な分散公式を求めた．

ρ が一定で μ が深さの一次函数である場合の厳密解を使つて佐藤泰夫が数値計算を行つた結果に対して我々の方法が良好な近似値を与えることを確めた (第I表).

次に分散公式を積分方程式と考えてそれを解くことによつて分散曲線から地殼の構造を求める公式を得た．計算例として秋間の分散曲線の解析を行つた (第3図).