# 12. *Magneto-hydrodynamic Oscillations of a Conducting Fluid Sphere in a Uniform Magnetic Field.*

By Tsuneji RIKITAKE,

Earthquake Research Institute.

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**Summary**

In order to examine the couplings between poloidal or toroidal magnetic fields of different type, magneto-hydrodynamic oscillations of a fluid sphere under the influence of a uniform magnetic field are studied. A set of simultaneous differential equations for the oscillation is obtained in a general form which contains harmonic constituents of successive degrees. Although the writer can not solve the equation rigorously, some features of the coupling are qualitatively examined. A simple normal mode of the free oscillation is also studied by taking into account only the $S_1^0$ and $S_3^0$ type magnetic fields.

## 1. Introduction

The writer[1] has studied magneto-hydrodynamic oscillations in the earth's core. It seems likely that the possible magneto-hydrodynamic oscillations play an important role for the explanation of the secular geomagnetic variations. In the previous study, only the $S_1^0$-type magnetic field was examined ignoring the mutual dependence of various harmonic constituents. In order to study the problem in more detail, however, we have to refine the theory by taking into account the mutual dependence because the interaction between certain fluid motions and magnetic fields would cause many types of electric currents which in turn give rise to many other types of magnetic field. Therefore we have to study oscillations of every harmonic constituent which is coupling with each other.

This sort of oscillations has also been a topic in astrophysics. M. Schwarzschild[2] discussed, in an approximate way, the magneto-hydrodynamic oscillation of a highly conducting star in a uniform magnetic field. His study was the first approach to the possible magneto-

1) T. RIKITAKE, *Bull. Earthq. Res., Inst.*, **33** (1955), 1.
2) M. SCHWARZSCHILD, *Ann. d'Astrophys.*, **12** (1949), 148.

hydrodynamic oscillations in a celestial body. Though interesting, he did not study the behaviour of a particular harmonic constituent. Accordingly, his result can not be applied to the examination of the mutual coupling of each harmonic constituent which is the main object of the present paper. V. C. A. Ferraro[3] and his colleagues also studied the oscillation of a magnetic star in which the steady magnetic field is considered as due to a magnetic pole at the centre. In that case, there was no coupling between harmonic constituents. The oscillations of this kind seem to be related with the reversal of magnetic field of magnetic variable stars though, as was pointed out by T. G. Cowling[4], no satisfactory explanation between the observed reversal and oscillations has been made yet.

The writer is going to study magneto-hydrodynamic oscillations of a liquid sphere in a uniform magnetic field as was studied by Schwarzschild. But the boundary conditions are different from those in his study because the writer would like to apply the result to the earth's core. As may be seen in the writer's previous paper[1], we may presume that the steady magnetic field would be of the $S_1^0$, $T_2^0$, $T_2^{2c}$ and $T_2^{2s}$ type provided we adopt the simple dynamo which would maintain the earth's permanent magnetic field. However, it is not easy to take into account all of these magnetic fields because of mathematical complexity. So we may here take into account only the $S_1^0$-type steady field for the purpose of examining the mutual coupling of various harmonics. The steady fluid motions are also assumed to be non-existing in the present model. Thus the problem becomes identical to the magneto-hydrodynamic oscillations of a spherical liquid body under the influence of an external uniform magnetic field. It is the purpose of this paper to study the mutual coupling of each oscillation in terms of poloidal and toroidal fields.

Starting from the fundamental equations for electromagnetism and fluid motion, the writer obtains a differential equation for the pressure, the velocity of the fluid motion being obtained from the solutions of this equation. The magnetic field produced by the interaction between the motion and the permanent magnetic field is also calculated. If we take a typical type of the magnetic field, poloidal or toroidal, we can construct an equation for the radial part of the magnetic field. The

3) V. C. A. FERRARO and D. J. MEMORY, *M.N.R.A.S.*, **112** (1952), 361.
C. PLUMPTON and V. C. A. FERRARO, *M.N.R.A.S.*, **113** (1953), 647.

4) T. G. COWLING, *M.N.R.A.S.*, **112** (1952), 527.

general form of the equation suggests that the coupling occurs between five constituents of successive degrees. It is also shown that we have the coupling between fields of poloidal and toroidal types in general cases.

A more detailed study is made for magnetic fields of zonal distribution. In that case, there is no coupling between poloidal and toroidal fields. It is also shown that there is no coupling between constituents of odd degree and those of even degree. Although the writer can not solve the coupling equation quite rigorously, a normal mode of the oscillation is approximately obtained by taking into account only the $S_1^0$ and $S_3^0$ type magnetic fields.

## 2. Fundamental equations

As we assume that there are no fluid motion and no electric current at the equilibrium state, the fundamental equations can be written as

$$\vec{i}=\sigma(\vec{e}+\vec{v}\wedge\vec{H}_0)\ , \tag{1}$$

$$\mathrm{curl}\,\vec{e}=-\partial\vec{h}/\partial t\ , \tag{2}$$

$$\mathrm{curl}\,\vec{h}=4\pi\vec{i}\ , \tag{3}$$

$$\rho\partial\vec{v}/\partial t=\vec{i}\wedge\vec{H}_0-\mathrm{grad}\ p\ , \tag{4}$$

where $\vec{i}$, $\vec{e}$, $\vec{h}$, $\vec{H}_0$, $\sigma$, $\rho$, $p$ and $\vec{v}$ denote respectively the electric current density, electric field, magnetic field, steady magnetic field applied from outside, electrical conductivity, density, pressure and velocity. It is also assumed (1) that the magnetic permeability is unity, (2) that there is no non-electromagnetic force that depends on time and (3) that the inertia term of fluid motion can be neglected. The fluid is also assumed to be incompressible, so that we have the following relation;

$$\mathrm{div}\,\vec{v}=0\ . \tag{5}$$

By taking curl of (1) and eliminating $\vec{e}$ with the aid of (2), we obtain

$$\mathrm{curl}\,\vec{i}=\sigma\{-D\vec{h}+\mathrm{curl}\,(\vec{v}\wedge\vec{H}_0)\} \tag{6}$$

where we write $D$ in place of $\partial/\partial t$. On the other hand we have from (3)

$$4\pi\,\mathrm{curl}\,\vec{i}=\mathrm{curl}\,\mathrm{curl}\,\vec{h}=-\nabla^2\vec{h}\ . \tag{7}$$

From (6) and (7), we obtain

$$\{D-(4\pi\sigma)^{-1}\nabla^2\}\vec{h}=\mathrm{curl}\,(\vec{v}\wedge\vec{H}_0)\ . \tag{8}$$

We also have from (4)

$$\rho D\vec{v}=(4\pi)^{-1}(\mathrm{curl}\,\vec{h}\wedge\vec{H}_0)-\mathrm{grad}\,p\ . \tag{9}$$

If we make div of (9), we obtain

$$\nabla^2 p=-(4\pi)^{-1}(\mathrm{curl}\,\vec{h}\ \mathrm{curl}\,\vec{H}_0-\vec{H}_0\,\mathrm{curl}\,\mathrm{curl}\,\vec{h})\ .$$

As $\vec{H}_0$ is applied from outside, we have the relation

$$\mathrm{curl}\,\vec{H}_0=0\ . \tag{10}$$

Therefore the above expression becomes

$$\nabla^2 p=-(4\pi)^{-1}(\vec{H}_0\nabla^2\vec{h})\ . \tag{11}$$

## 3. Poloidal magnetic field

In general, $\vec{h}$ can be expressed as

$$\vec{h}=\sum_{n,m}\vec{h}^m_{s,n}+\sum_{n,m}\vec{h}^m_{t,n} \tag{12}$$

where $\vec{h}^m_{s,n}$ is of the poloidal type, the $r$, $\theta$ and $\phi$ components in the spherical coordinate being written as

$$\vec{h}^m_{s,n}=\left\{\begin{array}{l} -n(n+1)s^m_n(r)r^{n-1}Y^m_n\ , \\ -\left[r\dfrac{ds^m_n}{dr}+(n+1)s^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\partial\theta}\ , \\ -\left[r\dfrac{ds^m_n}{dr}+(n+1)s^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\ , \end{array}\right. \tag{13}$$

and $\vec{h}^m_{t,n}$ is of the toroidal type which is written as

$$\vec{h}^m_{t,n}=\left\{\begin{array}{l} 0\ , \\ -t^m_n(r)r^n\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\ , \\ t^m_n(r)r^n\dfrac{\partial Y^m_n}{\partial\theta}\ , \end{array}\right. \tag{14}$$

where

$$Y^m_n=P^m_n(\cos\theta)\begin{array}{c}\cos\\ \sin\end{array}m\phi\ . \tag{15}$$

In the first place, we shall take a poloidal field and calculate $p$, $\vec{v}$,

$\vec{v}\wedge\vec{H}_0$ and curl $(\vec{v}\wedge\vec{H}_0)$.

From (13), we obtain

$$\operatorname{curl}\vec{h}^m_{s,n}=\begin{cases} 0\,, \\ \left[r\dfrac{d^2s^m_n}{dr^2}+2(n+1)\dfrac{ds^m_n}{dr}\right]r^{n-1}\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\,, \\ -\left[r\dfrac{d^2s^m_n}{dr^2}+2(n+1)\dfrac{ds^m_n}{dr}\right]r^{n-1}\dfrac{\partial Y^m_n}{\partial\theta}\,, \end{cases} \tag{16}$$

and

$$\nabla^2\vec{h}^m_{s,n}=\begin{cases} -n(n+1)\left[\dfrac{d^2s^m_n}{dr^2}+\dfrac{2(n+1)}{r}\dfrac{ds^m_n}{dr}\right]r^{n-1}Y^m_n\,, \\ -\left[r\dfrac{d^3s^m_n}{dr^3}+3(n+1)\dfrac{d^2s^m_n}{dr^2}+\dfrac{2n(n+1)}{r}\dfrac{ds^m_n}{dr}\right]r^{n-1}\dfrac{\partial Y^m_n}{\partial\theta}\,, \\ -\left[r\dfrac{d^3s^m_n}{dr^3}+3(n+1)\dfrac{d^2s^m_n}{dr^2}+\dfrac{2n(n+1)}{r}\dfrac{ds^m_n}{dr}\right]r^{n-1}\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\,. \end{cases} \tag{17}$$

If we assume that the external field is uniform and parallel to the $\theta=0$ axis, $\vec{H}_0$ is given as

$$\vec{H}_0=\begin{cases} H\cos\theta\,, \\ -H\sin\theta\,, \\ 0\,. \end{cases} \tag{18}$$

From (17) and (18), we obtain

$$\vec{H}_0\cdot\nabla^2\vec{h}^m_{s,n}=H\Bigg[-n(n+1)\left\{\frac{d^2s^m_n}{dr^2}+\frac{2(n+1)}{r}\frac{ds^m_n}{dr}\right\}Y^m_n\cos\theta$$
$$+\left\{r\frac{d^3s^m_n}{dr^3}+3(n+1)\frac{d^2s^m_n}{dr^2}+\frac{2n(n+1)}{r}\frac{ds^m_n}{dr}\right\}\frac{\partial Y^m_n}{\partial\theta}\sin\theta\Bigg]r^{n-1}\,.$$

Taking into account the following recurrence formulae for Neumann's spherical surface harmonics

$$\cos\theta P^m_n=\frac{1}{2n+1}\{(n-m+1)P^m_{n+1}+(n+m)P^m_{n-1}\}\,,$$

$$\sin\theta\frac{dP^m_n}{d\theta}=\frac{1}{2n+1}\{n(n-m+1)P^m_{n+1}-(n+1)(n+m)P^m_{n-1}\}\,,$$

the above expression can be written as follows;

$$\vec{H}_0\cdot\nabla^2\vec{h}^m_{s,n}=\frac{Hr^{n-1}}{2n+1}\Big[n(n-m+1)\Big(r\frac{d^3s^m_n}{dr^3}+2(n+1)\frac{d^2s^m_n}{dr^2}-\frac{2(n+1)}{r}\frac{ds^m_n}{dr}\Big)Y^m_{n+1}$$
$$-(n+1)(n+m)\Big(r\frac{d^3s^m_n}{dr^3}+(4n+3)\frac{d^2s^m_n}{dr^2}+\frac{4n(n+1)}{r}\frac{ds^m_n}{dr}\Big)Y^m_{n-1}\Big]. \tag{19}$$

Putting (19) into (11), we obtain a differential equation for the pressure which can be written as

$$\nabla^2 p=f_{n+1}Y^m_{n+1}+f_{n-1}Y^m_{n-1}\,, \tag{20}$$

where

$$\left.\begin{aligned}-4\pi f_{n+1}&=H\frac{n(n-m+1)}{2n+1}r^{n-1}\Big(r\frac{d^3s^m_n}{dr^3}+2(n+1)\frac{d^2s^m_n}{dr^2}-\frac{2(n+1)}{r}\frac{ds^m_n}{dr}\Big)\,,\\ 4\pi f_{n-1}&=H\frac{(n+1)(n+m)}{2n+1}r^{n-1}\Big(r\frac{d^3s^m_n}{dr^3}+(4n+3)\frac{d^2s^m_n}{dr^2}+\frac{4n(n+1)}{r}\frac{ds^m_n}{dr}\Big)\,.\end{aligned}\right\} \tag{21}$$

The solution of (20) can be given as

$$p=q^m_{n+1}Y^m_{n+1}+q^m_{n-1}Y^m_{n-1} \tag{22}$$

where $q^m_{n+1}$ and $q^m_{n-1}$ are respectively the solutions of the following differential equations:

$$\left.\begin{aligned}\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dq^m_{n+1}}{dr}\Big)-\frac{(n+1)(n+2)}{r^2}q^m_{n+1}&=f_{n+1}\,,\\ \frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dq^m_{n-1}}{dr}\Big)-\frac{(n-1)n}{r^2}q^m_{n-1}&=f_{n-1}\,.\end{aligned}\right\} \tag{23}$$

It is easily seen that the solutions of (23) which remain finite at $r=0$ are given as

$$\left.\begin{aligned}q^m_{n+1}&=a^{-(n+1)}r^{n+1}k^m_{n+1}+\frac{1}{2n+3}\Big(r^{n+1}\int_0^r r^{-n}f_{n+1}dr-r^{-n-2}\int_0^r r^{n+3}f_{n+1}dr\Big)\,,\\ q^m_{n-1}&=a^{-(n-1)}r^{n-1}l^m_{n-1}+\frac{1}{2n-1}\Big(r^{n-1}\int_0^r r^{-n+2}f_{n-1}dr-r^{-n}\int_0^r r^{n+1}f_{n-1}dr\Big)\,.\end{aligned}\right\} \tag{24}$$

where $k^m_{n+1}$ and $l^m_{n-1}$ are constants which are to be determined later.

Putting (21) into (24), we obtain, after some calculations, the following expressions:

$$\left.\begin{aligned} 4\pi q_{n+1}^m &= a^{-(n+1)} r^{n+1} \alpha_{n+1}^m - H\frac{n(n-m+1)}{2n+1} r^n \frac{ds_n^m}{dr}\,, \\ 4\pi q_{n-1}^m &= a^{-(n-1)} r^{n-1} \beta_{n-1}^m + H\frac{(n+1)(n+m)}{2n+1}\left\{ r^n \frac{ds_n^m}{dr} + (2n+1) r^{n-1} s_n^m \right\}, \end{aligned}\right\} \quad (25)$$

where

$$\alpha_{n+1}^m = 4\pi k_{n+1}^m\,, \qquad \beta_{n-1}^m = 4\pi l_{n-1}^m\,.$$

These constants are to be determined by the boundary conditions.

Thus the solution for the pressure is given by (22) together with (23).

Now we are in a position to calculate the velocity according to (9). From (16) and (18), we have

$$\operatorname{curl} \vec{h}_{s,n}^m \wedge \vec{H}_0 = \begin{cases} -H\left[ r\dfrac{d^2 s_n^m}{dr^2} + 2(n+1)\dfrac{ds_n^m}{dr}\right] r^{n-1} \sin\theta \dfrac{\partial Y_n^m}{\partial\theta}\,, \\ -H\left[ r\dfrac{d^2 s_n^m}{dr^2} + 2(n+1)\dfrac{ds_n^m}{dr}\right] r^{n-1} \cos\theta \dfrac{\partial Y_n^m}{\partial\theta}\,, \\ -H\left[ r\dfrac{d^2 s_n^m}{dr^2} + 2(n+1)\dfrac{ds_n^m}{dr}\right] r^{n-1} \cos\theta \dfrac{\partial Y_n^m}{\sin\theta\,\partial\phi}\,. \end{cases} \quad (26)$$

We also calculate grad $p$ from (22) and (23). Putting (26) and grad $p$ thus obtained into (9), the velocity can be calculated. Taking into consideration (5), we see that the velocity can be expressed as a sum of poloidal and toroidal velocity fields, this expression being thought to be convenient. After some lengthy calculations using the recurrence formulae of spherical surface harmonics, we arrive at

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \quad (27)$$

in which

$$4\pi\rho D\vec{v}_1 = \begin{cases} -(n+1)(n+2) A_{n+1}^m r^n Y_{n+1}^m \\ -\left[ r\dfrac{dA_{n+1}^m}{dr} + (n+2) A_{n+1}^m \right] r^n \dfrac{\partial Y_{n+1}^m}{\partial\theta}\,, \\ -\left[ r\dfrac{dA_{n+1}^m}{dr} + (n+2) A_{n+1}^m \right] r^n \dfrac{\partial Y_{n+1}^m}{\sin\theta\,\partial\phi}\,, \end{cases}$$

$$A_{n+1}^m = \frac{a^{-(n+1)}\alpha_{n+1}^m}{n+2} + H\frac{n(n-m+1)}{(2n+1)(n+1)} r^{-1} \frac{ds_n^m}{dr}\,, \quad (28)$$

$$4\pi\rho D\vec{v}_2=\begin{cases} -(n-1)nB_{n-1}^m r^{n-2}Y_{n-1}^m , \\ -\left[r\dfrac{dB_{n-1}^m}{dr}+nB_{n-1}^m\right]r^{n-2}\dfrac{\partial Y_{n-1}^m}{\partial\theta} , \\ -\left[r\dfrac{dB_{n-1}^m}{dr}+nB_{n-1}^m\right]r^{n-2}\dfrac{\partial Y_{n-1}^m}{\sin\theta\partial\phi} , \end{cases}$$

$$B_{n-1}^m=\frac{a^{-(n-1)}\beta_{n-1}^m}{n}+H\frac{(n+1)(n+m)}{(2n+1)n}\left\{r\frac{ds_n^m}{dr}+(2n+1)s_n^m\right\} , \tag{29}$$

$$4\pi\rho D\vec{v}_3=\begin{cases} 0 , \\ -C_n^m r^n\dfrac{\partial\tilde{Y}_n^m}{\sin\theta\partial\phi} , \\ C_n^m r^n\dfrac{\partial\tilde{Y}_n^m}{\partial\theta} , \end{cases}$$

$$C_n^m=-\frac{H}{n(n+1)}\left\{\frac{d^2s_n^m}{dr^2}+2(n+1)r^{-1}\frac{ds_n^m}{dr}\right\} , \tag{30}$$

where

$$\tilde{Y}_n^m=\partial Y_n^m/\partial\phi . \tag{31}$$

So it becomes clear that we have two types of poloidal velocity ($S_{n+1}^{m,c}$ and $S_{n-1}^{m,c}$) and one toroidal velocity ($T_n^{m,s}$ type) through the interaction between $\vec{H}_0$ and $\vec{h}_{s,n}^{m,c}$. $c$ and $s$, which are attached to the shoulders of the symbols here, denote respectively that those velocities or magnetic fiields contain $\cos m\phi$ or $\sin m\phi$. When we take $\vec{h}_{s,n}^{m,s}$ instead of $\vec{h}_{s,n}^{m,c}$, we get velocities of the $S_{n+1}^{m,s}$, $S_{n-1}^{m,s}$ and $T_n^{m,c}$ types.

Since we have obtained the velocities, the next step is to calculate $\mathrm{curl}(\vec{v}\wedge\vec{H}_0)$. Applying the recurrence formulae as before, we obtain

$$4\pi\rho D\,\mathrm{curl}(\vec{v}_1\wedge\vec{H}_0)=\begin{cases} -(n+2)(n+3)E_{n+2}^m r^{n+1}Y_{n+2}^m , \\ -\left[r\dfrac{dE_{n+2}^m}{dr}+(n+3)E_{n+2}^m\right]r^{n+1}\dfrac{\partial Y_{n+2}^m}{\partial\theta} , \\ -\left[r\dfrac{dE_{n+2}^m}{dr}+(n+3)E_{n+2}^m\right]r^{n+1}\dfrac{\partial Y_{n+2}^m}{\sin\theta\partial\phi} \end{cases}$$

$$E_{n+2}^m=H\frac{(n+1)(n-m+2)}{(2n+3)(n+2)}r^{-1}\frac{dA_{n+1}^m}{dr}$$

$$+\left\{\begin{array}{l} -n(n+1)F_n^m r^{n-1}Y_n^m \\ -\left[r\dfrac{dF_n^m}{dr}+(n+1)F_n^m\right]r^{n-1}\dfrac{\partial Y_n^m}{\partial\theta} \\ -\left[r\dfrac{dF_n^m}{dr}+(n+1)F_n^m\right]r^{n-1}\dfrac{\partial Y_n^m}{\sin\theta\partial\phi} \end{array}\right.$$

$$F_n^m=H\frac{(n+2)(n+m+1)}{(2n+3)(n+1)}\left\{r\frac{dA_{n+1}^m}{dr}+(2n+3)A_{n+1}^m\right\}$$

$$+\left\{\begin{array}{l} 0 \\ -G_{n+1}^m r^{n+1}\dfrac{\partial\tilde{Y}_{n+1}^m}{\sin\theta\partial\phi} \\ G_{n+1}^m r^{n+1}\dfrac{\partial\tilde{Y}_{n+1}^m}{\partial\theta} \end{array}\right.$$

$$G_{n+1}^m=-\frac{H}{(n+1)(n+2)}\left\{\frac{d^2A_{n+1}^m}{dr^2}+2(n+2)r^{-1}\frac{dA_{n+1}^m}{dr}\right\}, \tag{32}$$

$$4\pi\rho D\,\mathrm{curl}(\vec{v}_2\wedge\vec{H}_0)=\left\{\begin{array}{l} -n(n+1)H_n^m r^{n-1}Y_n^m \\ -\left[r\dfrac{dH_n^m}{dr}+(n+1)H_n^m\right]r^{n-1}\dfrac{\partial Y_n^m}{\partial\theta} \\ -\left[r\dfrac{dH_n^m}{dr}+(n+1)H_n^m\right]r^{n-1}\dfrac{\partial Y_n^m}{\sin\theta\partial\phi} \end{array}\right.$$

$$H_n^m=H\frac{(n-1)(n-m)}{(2n-1)n}r^{-1}\frac{dB_{n-1}^m}{dr}$$

$$+\left\{\begin{array}{l} -(n-2)(n-1)I_{n-2}^m r^{n-3}Y_{n-2}^m \\ -\left[r\dfrac{dI_{n-2}^m}{dr}+(n-1)I_{n-2}^m\right]r^{n-3}\dfrac{\partial Y_{n-2}^m}{\partial\theta} \\ -\left[r\dfrac{dI_{n-2}^m}{dr}+(n-1)I_{n-2}^m\right]r^{n-3}\dfrac{\partial Y_{n-2}^m}{\sin\theta\partial\phi} \end{array}\right.$$

$$I_{n-2}^m=H\frac{n(n+m-1)}{(2n-1)(n-1)}\left\{r\frac{dB_{n-1}^m}{dr}+(2n-1)B_{n-1}^m\right\}$$

$$+\left\{\begin{array}{l} 0 \\ -J_{n-1}^m r^{n-1}\dfrac{\partial\tilde{Y}_{n-1}^m}{\sin\theta\partial\phi} \\ J_{n-1}^m r^{n-1}\dfrac{\partial\tilde{Y}_{n-1}^m}{\partial\theta} \end{array}\right.$$

$$J_{n-1}^m = -\frac{H}{(n-1)n}\left\{\frac{d^2B_{n-1}^m}{dr^2} + 2nr^{-1}\frac{dB_{n-1}^m}{dr}\right\}, \tag{33}$$

$$4\pi\rho D\,\mathrm{curl}(\vec{v}_3\wedge\vec{H}_0) = \begin{cases} -n(n+1)K_n^m r^{n-1}Y_n^m \\ -\left[r\frac{dK_n^m}{dr} + (n+1)K_n^m\right]r^{n-1}\frac{\partial Y_n^m}{\partial\theta} \\ -\left[r\frac{dK_n^m}{dr} + (n+1)K_n^m\right]r^{n-1}\frac{\partial Y_n^m}{\sin\theta\partial\phi} \end{cases}$$

$$K_n^m = -\frac{Hm^2}{n(n+1)}C_n^m$$

$$+\begin{cases} 0 \\ -L_{n+1}^m r^{n+1}\frac{\partial\tilde{Y}_{n+1}^m}{\sin\theta\partial\phi} \\ L_{n+1}^m r^{n+1}\frac{\partial\tilde{Y}_{n+1}^m}{\partial\theta} \end{cases}$$

$$L_{n+1}^m = -H\frac{(n-1)(n-m+1)}{2n+1}\left\{r^{-1}\frac{dC_n^m}{dr} + (n+2)r^{-2}C_n^m\right\}$$

$$+\begin{cases} 0 \\ -M_{n-1}^m r^{n-1}\frac{\partial\tilde{Y}_{n-1}^m}{\sin\theta\partial\phi} \\ M_{n-1}^m r^{n-1}\frac{\partial\tilde{Y}_{n-1}^m}{\partial\theta} \end{cases}$$

$$M_{n-1}^m = H\frac{(n+2)(n+m)}{2n+1}\left\{r\frac{dC_n^m}{dr} + (n+2)C_n^m\right\}. \tag{34}$$

Therefore the interaction between the fluid motions and $\vec{H}_0$ induces the electric currents which give rise to the magnetic fields of the $S_{n+2}^{m,c}$, $S_n^{m,c}$, $S_{n-2}^{m,c}$, $T_{n+1}^{m,s}$ and $T_{n-1}^{m,s}$ types for the first case which corresponds to $\vec{h}_{s,n}^{m,c}$. If we take $\vec{h}_{s,n}^{m,s}$, the final magnetic fields become of the $S_{n+2}^{m,s}$, $S_n^{m,s}$, $S_{n-2}^{m,s}$, $T_{n+1}^{m,c}$ and $T_{n-1}^{m,c}$ types.

On the other hand, putting $\vec{h}_{s,n}^m$ in place of $\vec{h}$, the left-hand side of (8) becomes

$$\{D-(4\pi\sigma)^{-1}\nabla^2\}\vec{h}^m_{s,n}=\begin{cases} -n(n+1)\gamma^m_n r^{n-1}Y^m_n\,, \\ -\left[r\dfrac{d\gamma^m_n}{dr}+(n+1)\gamma^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\partial\theta}\,, \\ -\left[r\dfrac{d\gamma^m_n}{dr}+(n+1)\gamma^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\,, \end{cases}$$

$$\gamma^m_n=Ds^m_n-(4\pi\sigma)^{-1}\left\{\frac{d^2s^m_n}{dr^2}+2(n+1)r^{-1}\frac{ds^m_n}{dr}\right\}\,. \tag{35}$$

## 4. Toroidal magnetic field

A similar calculation will be made for the toroidal magnetic field $\vec{h}^m_{t,n}$ which is given in (14). From (14), we have

$$\operatorname{curl}\vec{h}^m_{t,n}=\begin{cases} -n(n+1)t^m_n r^{n-1}Y^m_n\,, \\ -\left[r\dfrac{dt^m_n}{dr}+(n+1)t^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\partial\theta}\,, \\ -\left[r\dfrac{dt^m_n}{dr}+(n+1)t^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\,, \end{cases} \tag{36}$$

and

$$\nabla^2\vec{h}^m_{t,n}=\begin{cases} 0\,, \\ -\left[\dfrac{d^2t^m_n}{dr^2}+2(n+1)r^{-1}\dfrac{dt^m_n}{dr}\right]r^n\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\,, \\ \left[\dfrac{d^2t^m_n}{dr^2}+2(n+1)r^{-1}\dfrac{dt^m_n}{dr}\right]r^n\dfrac{\partial Y^m_n}{\partial\theta}\,. \end{cases} \tag{37}$$

Corresponding to (19), it becomes

$$\vec{H}_0\cdot\nabla^2\vec{h}^m_{t,n}=H\left[\frac{d^2t^m_n}{dr^2}+2(n+1)r^{-1}\frac{dt^m_n}{dr}\right]r^n\tilde{Y}^m_n\,. \tag{38}$$

The differential equation for the pressure is given as

$$\nabla^2 p=g_n\tilde{Y}^m_n \tag{39}$$

where

$$-4\pi g_n=H\left(\frac{d^2t^m_n}{dr^2}+2(n+1)r^{-1}\frac{dt^m_n}{dr}\right)r^n\,. \tag{40}$$

The solution of (39) can be given as

$$p=\bar{q}_n^m \tilde{Y}_n^m \tag{41}$$

where $\bar{q}_n^m$ satisfies the following differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\bar{q}_n^m}{dr}\right)-\frac{n(n+1)}{r^2}\bar{q}_n^m=g_n \,. \tag{42}$$

(42) can be solved as

$$\bar{q}_n^m=a^{-n}r^n\bar{k}_n^m+\frac{1}{2n+1}\left(r^n\int_0^r r^{-(n-1)}g_n dr-r^{-n-1}\int_0^r r^{n+2}g_n dr\right) \tag{43}$$

which is written as

$$4\pi\bar{q}_n^m=a^{-n}r^n\bar{\alpha}_n^m-Hr^n t_n^m \tag{44}$$

where

$$\bar{\alpha}_n^m=4\pi\bar{k}_n^m \,.$$

After calculating grad $p$ and curl $\vec{h}_{t,n}^m\wedge\vec{H}_0$, we obtain the velocities as before. They are given as

$$\vec{v}=\vec{v}_1+\vec{v}_2+\vec{v}_3 \tag{45}$$

in which

$$4\pi\rho D\vec{\bar{v}}_1=\left\{\begin{array}{l} -n(n+1)\bar{A}_n^m r^{n-1}\tilde{Y}_n^m \,, \\ -\left[r\dfrac{d\bar{A}_n^m}{dr}+(n+1)\bar{A}_n^m\right]r^{n-1}\dfrac{\partial\tilde{Y}_n^m}{\partial\theta} \,, \\ -\left[r\dfrac{d\bar{A}_n^m}{dr}+(n+1)\bar{A}_n^m\right]r^{n-1}\dfrac{\partial\tilde{Y}_n^m}{\sin\theta\partial\phi} \,, \end{array}\right.$$

$$\bar{A}_n^m=\frac{a^{-n}\bar{\alpha}_n^m}{n+1}+\frac{H}{n(n+1)}t_n^m \,, \tag{46}$$

$$4\pi\rho D\vec{\bar{v}}_2=\left\{\begin{array}{l} 0 \,, \\ -\bar{B}_{n+1}^m r^{n+1}\dfrac{\partial Y_{n+1}^m}{\sin\theta\partial\phi} \,, \\ \bar{B}_{n+1}^m r^{n+1}\dfrac{\partial Y_{n+1}^m}{\partial\theta} \,, \end{array}\right.$$

$$\bar{B}_{n+1}^m=H\frac{n(n-m+1)}{(n+1)(2n+1)}r^{-1}\frac{dt_n^m}{dr} \,, \tag{47}$$

$$4\pi\rho D\vec{v}_3=\begin{cases} 0\,, \\ -\bar{C}^m_{n-1}r^{n-1}\dfrac{\partial Y^m_{n-1}}{\sin\theta\partial\phi}\,, \\ \bar{C}^m_{n-1}r^{n-1}\dfrac{\partial Y^m_{n-1}}{\partial\theta}\,, \end{cases}$$

$$\bar{C}^m_{n-1}=H\frac{(n+1)(n+m)}{n(2n+1)}\left\{r\frac{dt^m_n}{dr}+(2n+1)t^m_n\right\}. \qquad (48)$$

Thus we have got the velocities of the $S^{m,s}_n$, $T^{m,s}_{n+1}$ and $T^{m,c}_{n-1}$ types for $\vec{h}^{m,c}_{t,n}$ and of the $S^{m,c}_n$, $T^{m,s}_{n+1}$ and $T^{m,s}_{n-1}$ types for $\vec{h}^{m,s}_{t,n}$ respectively.

We also obtain

$$4\pi\rho D\,\mathrm{curl}(\vec{v}_1\wedge\vec{H}_0)=\begin{cases} -(n+1)(n+2)\bar{D}^m_{n+1}r^n\tilde{Y}^m_{n+1} \\ -\left[r\dfrac{d\bar{D}^m_{n+1}}{dr}+(n+2)\bar{D}^m_{n+1}\right]r^n\dfrac{\partial\tilde{Y}^m_{n+1}}{\partial\theta} \\ -\left[r\dfrac{d\bar{D}^m_{n+1}}{dr}+(n+2)\bar{D}^m_{n+1}\right]r^n\dfrac{\partial\tilde{Y}^m_{n+1}}{\sin\theta\partial\phi} \end{cases}$$

$$\bar{D}^m_{n+1}=H\frac{n(n-m+1)}{(2n+1)(n+1)}r^{-1}\frac{d\bar{A}^m_n}{dr}$$

$$+\begin{cases} -(n-1)n\bar{E}^m_{n-1}r^{n-2}\tilde{Y}^m_{n-1} \\ -\left[r\dfrac{d\bar{E}^m_{n-1}}{dr}+n\bar{E}^m_{n-1}\right]r^{n-2}\dfrac{\partial\tilde{Y}^m_{n-1}}{\partial\theta} \\ -\left[r\dfrac{d\bar{E}^m_{n-1}}{dr}+n\bar{E}^m_{n-1}\right]r^{n-2}\dfrac{\partial\tilde{Y}^m_{n-1}}{\sin\theta\partial\phi} \end{cases}$$

$$\bar{E}^m_{n-1}=H\frac{(n+1)(n+m)}{(2n+1)n}\left\{r\frac{d\bar{A}^m_n}{dr}+(2n+1)\bar{A}^m_n\right\}$$

$$+\begin{cases} 0 \\ -\bar{F}^m_n r^n\dfrac{\partial Y^m_n}{\sin\theta\partial\phi} \\ \bar{F}^m_n r^n\dfrac{\partial Y^m_n}{\partial\theta} \end{cases}$$

$$\bar{F}^m_n=H\frac{m^2}{n(n+1)}\left\{\frac{d^2\bar{A}^m_n}{dr^2}+2(n+1)r^{-1}\frac{d\bar{A}^m_n}{dr}\right\} \qquad (49)$$

$$4\pi\rho D\,\mathrm{curl}(\vec{v}_2\wedge\vec{H}_0)=\begin{cases}-(n+1)(n+2)\bar{G}^m_{n+1}r^n\tilde{Y}^m_{n+1}\\[2mm] -\left[r\dfrac{d\bar{G}^m_{n+1}}{dr}+(n+2)\bar{G}^m_{n+1}\right]r^n\dfrac{\partial\tilde{Y}^m_{n+1}}{\partial\theta}\\[2mm] -\left[r\dfrac{d\bar{G}^m_{n+1}}{dr}+(n+2)\bar{G}^m_{n+1}\right]r^n\dfrac{\partial\tilde{Y}^m_{n+1}}{\sin\theta\partial\phi}\end{cases}$$

$$\bar{G}^m_{n+1}=\frac{H}{(n+1)(n+2)}\bar{B}^m_{n+1}$$

$$+\begin{cases}0\\[2mm] -\bar{H}^m_{n+2}r^{n+2}\dfrac{\partial Y^m_{n+2}}{\sin\theta\partial\phi}\\[2mm] \bar{H}^m_{n+2}r^{n+2}\dfrac{\partial Y^m_{n+2}}{\partial\theta}\end{cases}$$

$$\bar{H}^m_{n+2}=-H\frac{n(n-m+2)}{2n+3}\left\{r^{-1}\frac{d\bar{B}^m_{n+1}}{dr}+(n+3)r^{-2}\bar{B}^m_{n+1}\right\}$$

$$+\begin{cases}0\\[2mm] -\bar{I}^m_n r^n\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\\[2mm] \bar{I}^m_n r^n\dfrac{\partial Y^m_n}{\partial\theta}\end{cases}$$

$$\bar{I}^m_n=-H\frac{(n+3)(n+m+1)}{2n+3}\left\{r\frac{d\bar{B}^m_{n+1}}{dr}+(n+3)\bar{B}^m_{n+1}\right\}\,, \tag{50}$$

$$4\pi\rho D\,\mathrm{curl}(\vec{v}_3\wedge\vec{H}_0)=\begin{cases}-(n-1)n\bar{J}^m_{n-1}r^{n-2}\tilde{Y}^m_{n-1}\\[2mm] -\left[r\dfrac{d\bar{J}^m_{n-1}}{dr}+n\bar{J}^m_{n-1}\right]r^{n-2}\dfrac{\partial\tilde{Y}^m_{n-1}}{\partial\theta}\\[2mm] -\left[r\dfrac{d\bar{J}^m_{n-1}}{dr}+n\bar{J}^m_{n-1}\right]r^{n-2}\dfrac{\partial\tilde{Y}^m_{n-1}}{\sin\theta\partial\phi}\end{cases}$$

$$\bar{J}^m_{n-1}=\frac{H}{(n-1)n}\bar{C}^m_{n-1}$$

$$+\begin{cases}0\\[2mm] -\bar{K}^m_n r^n\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\\[2mm] \bar{K}^m_n r^n\dfrac{\partial Y^m_n}{\partial\theta}\end{cases}$$

$$\bar{K}_n^m = -H\frac{(n-2)(n-m)}{2n-1}\left\{r^{-1}\frac{\bar{C}_{n-1}^m}{dr} + (n+1)r^{-2}\bar{C}_{n-1}^m\right\}$$

$$+\begin{cases} 0 \\ -\bar{L}_{n-2}^m r^{n-2}\dfrac{\partial Y_{n-2}^m}{\sin\theta\partial\phi} \\ \bar{L}_{n-2}^m r^{n-2}\dfrac{\partial Y_{n-2}^m}{\partial\theta} \end{cases}$$

$$\bar{L}_{n-2}^m = -H\frac{(n+1)(n+m-1)}{2n-1}\left\{r\frac{d\bar{C}_{n-1}^m}{dr} + (n+1)\bar{C}_{n-1}^m\right\}\ . \quad (51)$$

In this case, the induced electric currents produce magnetic fields of the $S_{n+1}^{m,s}$, $S_{n-1}^{m,s}$, $T_{n+2}^{m,c}$, $T_n^{m,c}$ and $T_{n-2}^{m,c}$ types when we take $\vec{h}_{t,n}^{m,c}$. We also have $S_{n+1}^{m,c}$, $S_{n-1}^{m,c}$, $T_{n+2}^{m,s}$, $T_n^{m,s}$ and $T_{n-2}^{m,s}$ types for $\vec{h}_{t,n}^{m,s}$.

The left-hand side of (8) becomes

$$\{D-(4\pi\sigma)^{-1}\nabla^2\}\vec{h}_{t,n}^m = \begin{cases} 0\ , \\ -\bar{\gamma}_n^m r^n\dfrac{\partial Y_n^m}{\sin\theta\partial\phi}\ , \\ \bar{\gamma}_n^m r^n\dfrac{\partial Y_n^m}{\partial\theta}\ , \end{cases}$$

$$\bar{\gamma}_n^m = Dt_n^m - (4\pi\sigma)^{-1}\left\{\frac{d^2t_n^m}{dr^2} + 2(n+1)r^{-1}\frac{dt_n^m}{dr}\right\}\ . \quad (52)$$

### 5. The differential equations for the radial parts

The results in the sections 4 and 5 can be summarized as shown in Table I.

Now we go back to the general case (12). Putting the relations from (32) to (35) and those from (49) to (52) into (8) and equating the coefficients of the corresponding surface harmonics, we obtain the following relations for respective types of the magnetic field;

$$\left.\begin{aligned}
4\pi\rho D\gamma_n^{m,c} &= E_n^{m,c} + m(\bar{D}_n^{m,c} + \bar{G}_n^{m,c}) + F_n^{m,c} + H_n^{m,c} + K_n^{m,c} + m(\bar{E}_n^{m,c} + \bar{J}_n^{m,c}) + I_n^{m,c}\ , \\
4\pi\rho D\gamma_n^{m,s} &= E_n^{m,s} - m(\bar{D}_n^{m,s} + \bar{G}_n^{m,s}) + F_n^{m,s} + H_n^{m,s} + K_n^{m,s} - m(\bar{E}_n^{m,s} + \bar{J}_n^{m,s}) + I_n^{m,s}\ , \\
4\pi\rho D\bar{\gamma}_n^{m,c} &= \bar{H}_n^{m,c} + m(G_n^{m,c} + L_n^{m,c}) + \bar{F}_n^{m,c} + \bar{I}_n^{m,c} + \bar{K}_n^{m,c} + m(J_n^{m,c} + M_n^{m,c}) + \bar{L}_n^{m,c}\ , \\
4\pi\rho D\bar{\gamma}_n^{m,s} &= \bar{H}_n^{m,s} - m(G_n^{m,s} + L_n^{m,s}) + \bar{F}_n^{m,s} + \bar{I}_n^{m,s} + \bar{K}_n^{m,s} - m(J_n^{m,s} + M_n^{m,s}) + \bar{L}_n^{m,s}\ .
\end{aligned}\right\} \quad (53)$$

Table I.

| Magnetic field. | Velocity (Interaction with $\vec{H}_0$) | Magnetic field caused by the induced currents (Interaction with $\vec{H}_0$) |
|---|---|---|
| | | $S_{n+2}^{m,c}$ |
| | $S_{n+1}^{m,c}$ | $S_{n}^{m,c}$ |
| | | $T_{n+1}^{m,s}$ |
| | | $S_{n}^{m,c}$ |
| $S_n^{m,c}$ | $S_{n-1}^{m,c}$ | $S_{n-2}^{m,c}$ |
| | | $T_{n-1}^{m,s}$ |
| | | $S_{n}^{m,c}$ |
| | $T_{n}^{m,s}$ | $T_{n+1}^{m,s}$ |
| | | $T_{n-1}^{m,s}$ |
| | | $S_{n+2}^{m,s}$ |
| | $S_{n+1}^{m,s}$ | $S_{n}^{m,s}$ |
| | | $T_{n+1}^{m,c}$ |
| | | $S_{n}^{m,s}$ |
| $S_n^{m,s}$ | $S_{n-1}^{m,s}$ | $S_{n-2}^{m,s}$ |
| | | $T_{n-1}^{m,c}$ |
| | | $S_{n}^{m,s}$ |
| | $T_{n}^{m,c}$ | $T_{n+1}^{m,c}$ |
| | | $T_{n-1}^{m,c}$ |
| | | $S_{n+1}^{m,s}$ |
| | $S_{n}^{m,s}$ | $S_{n-1}^{m,s}$ |
| | | $T_{n}^{m,c}$ |
| | | $S_{n+1}^{m,s}$ |
| $T_n^{m,c}$ | $T_{n+1}^{m,c}$ | $T_{n+2}^{m,c}$ |
| | | $T_{n}^{m,c}$ |
| | | $S_{n-1}^{m,s}$ |
| | $T_{n-1}^{m,c}$ | $T_{n}^{m,c}$ |
| | | $T_{n-2}^{m,c}$ |

(to be continued.)

(continued.)

| | | |
|---|---|---|
| $T_n^{m,s}$ | $S_n^{m,c}$ | $S_{n+1}^{m,c}$ |
| | | $S_{n-1}^{m,c}$ |
| | | $T_n^{m,s}$ |
| | $T_{n+1}^{m,s}$ | $S_{n+1}^{m,c}$ |
| | | $T_{n+2}^{m,s}$ |
| | | $T_n^{m,s}$ |
| | $T_{n-1}^{m,s}$ | $S_{n-1}^{m,c}$ |
| | | $T_n^{m,s}$ |
| | | $T_{n-2}^{m,s}$ |

These equations can be regarded as the simultaneous differential equations for $s_n^{m,c}$, $s_n^{m,s}$, $t_n^{m,c}$ and $t_n^{m,s}$. It should be noticed that five terms having respectively $n+2$, $n+1$, $n$, $n-1$ and $n-2$ as their degrees come out at the same time. Therefore those terms are coupling each other. This feature is quite different from electromagnetic, elastic or hydrodynamic oscillation problems of a spherical body. If we can solve (53) somehow, we would be able to make clear the magneto-hydrodynamic oscillation problem. But it seems quite difficult to solve the general case as given by (53) because of mathematical complexity.

## 6. Zonal harmonic oscillations

Taking the simplest case, the writer would here like to study only the zonal oscillations because we have, as can be seen in (53), no coupling between zonal and tesseral harmonic constituents.

In that case, (53) becomes

$$\left.\begin{aligned} 4\pi\rho D\gamma_n &= E_n+F_n+H_n+I_n\ , \\ 4\pi\rho D\overline{\gamma}_n &= \overline{H}_n+\overline{I}_n+\overline{K}_n+\overline{L}_n\ , \end{aligned}\right\} \tag{54}$$

from which we find out that there is no coupling between poloidal and toroidal modes. As the toroidal field does not appear outside the sphere, we are in the main interested in the poloidal field. Putting the relations given by (32), (33), (34) and (35) into (54) and taking into account the relations (28), (29) and (30), the first equation of (54) can be written as

$$4\pi\rho DH^{-2}\left[Ds_n-(4\pi\sigma)^{-1}\left\{\frac{d^2s_n}{dr^2}+\frac{2(n+1)}{r}\frac{ds_n}{dr}\right\}\right]$$
$$=\frac{(n-1)(n-2)}{(2n-1)(2n-3)}\left(r\frac{d^2s_{n-2}}{dr^2}+\frac{ds_{n-2}}{dr}\right)r^{-3}$$

$$+\frac{1}{2n+1}\left(\frac{n(n+2)}{2n+3}+\frac{(n-1)(n+1)}{2n-1}\right)\left(\frac{d^2s_n}{dr^2}+\frac{2(n+1)}{r}\frac{ds_n}{dr}\right)$$

$$+\frac{(n+2)(n+3)}{(2n+3)(2n+5)}\left(r^2\frac{d^2s_{n+2}}{dr^2}+(2n+7)r\frac{ds_{n+2}}{dr}+(2n+5)s_{n+2}\right)$$

$$+H^{-1}a^{-(n+1)}(\alpha_{n+1}+\beta_{n+1}) . \tag{55}$$

(55) suggests that the coupling occurs between constituents of even or odd degrees, no coupling being possible between constitutents of even degrees and those of odd ones.

As (55) is true for any $n$, we have to solve the simultaneous differential equations of infinite numbers with infinite unknowns.

It seems to be almost impossible to obtain rigorous solutions of (55). So the writer would here like to obtain the solutions for the $S_1^0$ and $S_3^0$ type magnetic fields in an approximate way neglecting the $S_5^0$, $S_7^0$, $\cdots\cdot$ type fields. From (55), we have

$$K^2s_1=\frac{1}{5}\left(\frac{d^2s_1}{dr^2}+4r^{-1}\frac{ds_1}{dr}\right)+\frac{12}{35}\left(\frac{d^2\bar{s}_3}{dr^2}+5r^{-1}\frac{d\bar{s}_3}{dr}-5r^{-2}\bar{s}_3\right)+a^{-2}H^{-1}(\alpha_2+\beta_2) , \tag{56}$$

$$K^2\bar{s}_3=\frac{2}{15}\left(\frac{d^2s_1}{dr^2}+r^{-1}\frac{ds_1}{dr}\right)+\frac{7}{15}\left(\frac{d^2\bar{s}_3}{dr^2}+4r^{-1}\frac{d\bar{s}_3}{dr}-10r^{-2}\bar{s}_3\right)+a^{-4}H^{-1}(\alpha_4+\beta_4)r^2 , \tag{57}$$

where

$$K^2=4\pi\rho D^2H^{-2} , \qquad \bar{s}_3=r^2s_3$$

in which $\sigma$ is taken to be infinity for the sake of simplicity. If we take a rigid boundary at $r=a$, $\alpha_2$, $\beta_2$, $\alpha_4$ and $\beta_4$ are determined by (28) and (29) as

$$\left.\begin{aligned}
&a^{-2}H^{-1}\alpha_2=-a^{-1}\left(\frac{ds_1}{dr}\right)_{r=a} ,\\
&a^{-4}H^{-1}\alpha_4=-\frac{15}{7}a^{-1}\left(\frac{ds_3}{dr}\right)_{r=a}=-\frac{15}{7}a^{-3}\left(\frac{d\bar{s}_3}{dr}-2r^{-1}\bar{s}_3\right)_{r=a} ,\\
&a^{-2}H^{-1}\beta_2=-\frac{12}{7}\left(r\frac{ds_3}{dr}+7s_3\right)_{r=a}=-\frac{12}{7}\left(r^{-1}\frac{d\bar{s}_3}{dr}+5r^{-2}\bar{s}_3\right)_{r=a} ,\\
&\beta_4=0 ,
\end{aligned}\right\} \tag{58}$$

because $v_r=0$ at $r=a$.

As was done in the writer's previous paper, (56) and (57) may be approximately solved by putting

$$\left.\begin{aligned} s_1 &= \sum_{s=0}^{4} a_s(r/a)^s \ , \\ \bar{s}_3 &= \sum_{s=0}^{4} b_s(r/a)^s \ , \end{aligned}\right\} \tag{59}$$

in which the terms with powers higher than 4 are ignored. Putting (59) into (56) and (57), we obtain the following relations:

$$\begin{aligned} K^2 \sum_{s=0}^{4} a_s(r/a)^s &= \frac{1}{5}a^{-2}\sum_{s=2}^{4} s(s-1)a_s(r/a)^{s-2} + \frac{4}{5}a^{-2}\sum_{s=1}^{4} sa_s(r/a)^{s-2} \\ &+ \frac{12}{35}a^{-2}\sum_{s=2}^{4} s(s-1)b_s(r/a)^{s-2} + \frac{12}{7}a^{-2}\sum_{s=1}^{4} sb_s(r/a)^{s-2} - \frac{12}{7}a^{-2}\sum_{s=0}^{4} b_s(r/a)^{s-2} \\ &- a^{-2}\sum_{s=1}^{4} sa_s - \frac{12}{7}a^{-2}\sum_{s=1}^{4} sb_s - \frac{60}{7}a^{-2}\sum_{s=0}^{4} b_s \ , \end{aligned} \tag{60}$$

$$\begin{aligned} K^2 \sum_{s=0}^{4} b_s(r/a)^s &= \frac{2}{15}a^{-2}\sum_{s=2}^{4} s(s-1)a_s(r/a)^{s-2} + \frac{2}{15}a^{-2}\sum_{s=1}^{4} sa_s(r/a)^{s-2} \\ &+ \frac{7}{15}a^{-2}\sum_{s=2}^{4} s(s-1)b_s(r/a)^{s-2} + \frac{28}{15}a^{-2}\sum_{s=1}^{4} sb_s(r/a)^{s-2} - \frac{14}{3}a^{-2}\sum_{s=0}^{4} b_s(r/a)^{s-2} \\ &- \frac{15}{7}a^{-2}\sum_{s=1}^{4} sb_s(r/a)^2 + \frac{30}{7}a^{-2}\sum_{s=0}^{4} b_s(r/a)^2 \ . \end{aligned} \tag{61}$$

Equating the coefficients of the corresponding terms of the two sides of (60) and (61), we obtain the relations between $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $b_0$, $b_1$, $b_2$, $b_3$ and $b_4$, the relations being simplified as follows:

$$\left.\begin{aligned} & a_1 = a_3 = b_1 = b_3 = 0 \ , \\ & a^2K^2a_0 = -4a_4 - \frac{60}{7}b_0 - \frac{48}{5}b_2 - \frac{108}{7}b_4 \ , \\ & a^2K^2a_2 = \frac{28}{5}a_4 + \frac{324}{35}b_4 \ , \\ & a^2K^2b_0 = \frac{8}{15}a_2 \ , \\ & a^2K^2b_2 = \frac{32}{15}a_4 + \frac{30}{7}b_0 + \frac{144}{35}b_4 \ . \end{aligned}\right\} \tag{62}$$

However, it is easily seen from (29) and (30) that we should make $b_0=0$ in order to have finite velocities at $r=0$. As can be obtained from (62), this condition also gives $a_2=0$.

Further, the solutions must satisfy the boundary condition that the magnetic field is continuous at $r=a$. The magnetic field outside the sphere is generally given as

$$\vec{h}=\begin{cases} (n+1)(r/a)^{-n-2}h_nP_n \ , \\ -(r/a)^{-n-2}h_n\dfrac{dP_n}{d\theta} \ , \\ 0 \ . \end{cases} \tag{63}$$

The continuity at $r=a$ gives

$$-n(\bar{s}_n)_{r=a}=h_n$$

$$\left(r\frac{ds_n}{dr}+\overline{n+1}\,s_n\right)_{r=a}=\left(r\frac{d\bar{s}_n}{dr}+2\bar{s}_n\right)_{r=a}=h_n$$

where

$$\bar{s}_n=r^{n-1}s_n \ .$$

Eliminating $h_n$, we obtain

$$\left(r\frac{d\bar{s}_n}{dr}+\overline{n+2}\,\bar{s}_n\right)_{r=a}=0 \ . \tag{64}$$

For the present case, (64) can be written as

$$\left.\begin{aligned} 3a_0+5a_2+7a_4=0 \ , \\ 5b_0+7b_2+9b_4=0 \ . \end{aligned}\right\} \tag{65}$$

In order to have non-vanishing solutions for $a_0$, $a_4$, $b_2$ and $b_4$ which satisfy the equations (62) and (65), we obtain

$$\begin{vmatrix} 3 & 7 & 0 & 0 \\ 0 & 0 & 7 & 9 \\ a^2K^2 & 4 & \dfrac{48}{5} & \dfrac{108}{7} \\ 0 & -\dfrac{32}{15} & a^2K^2 & -\dfrac{144}{35} \end{vmatrix}=0 \ , \tag{66}$$

the condition $b_0=a_2=0$ being taken into account here. (66) is the equation from which we can determine the period of the oscillation considered here. The stable oscillation is specified by one of the roots of (66) which is calculated as

$$(aK)^2=-2.70 \ . \tag{67}$$

The other roots do not give oscillatory motions. As $K^2=4\pi\rho D^2H^{-2}$, by taking into account the relation $D^2=-\left(\frac{2\pi}{T}\right)^2$ for periodic cases, the period of the oscillation $T$ becomes

$$T=2.4\pi^{3/2}\rho^{1/2}aH^{-1} . \tag{68}$$

Thus we have got an oscillating magnetic field with a period given by (68). The mode of the oscillation is a superposition of the $S_1^0$ and $S_3^0$ type fields. Outside the sphere, the magnetic field has a potential composed of the two spherical harmonic constituents, $(r/a)^{-2}P_1$ and $(r/a)^{-4}P_3$, the proportion of both constituents being obtained by determining the coefficients $a_4$, $b_2$, $\cdots$ from the simultaneous equations in (62).

If we ignore all the couplings between each constituent and pick up only the $S_1^0$ type magnetic field, the characteristic equation becomes

$$\begin{vmatrix} 3 & 5 & 7 \\ a^2K^2 & 0 & 4 \\ 0 & a^2K^2 & -\frac{28}{5} \end{vmatrix} =0 \tag{69}$$

which gives

$$(aK)^2=-\frac{16}{7} \tag{70}$$

for the oscillating case. Hence, the period is given as

$$T_0=2.6\pi^{3/2}\rho^{1/2}aH^{-1} . \tag{71}$$

Comparing (68) with (71), it is seen that the period of the oscillating magnetic field becomes a bit shorter by the coupling between the two spherical harmonic constituents. In any case, the period is proportional to the radius of the sphere, the square root of the density and the inverse of the intensity of the external magnetic field. Suppose we take a sphere, the density of which is assumed to be $10\,g/cm^3$, the radius $10\,cm$, and the applied magnetic field 10 *gauss*, the period is found to be 42 *sec.* from (68). However, an experiment for such a small sphere would be almost impossible because of low conductivity of fluid even if we use mercury.

Although the writer can not obtain rigorous solutions because of the mathematical complexity, the general differential equation as given in (55) seems to be useful for studying the couplings between the spherical harmonic constituents of the magnetic field. An approximate solution for the magnetic fields composed of the $S_1^0$ and $S_3^0$ type ones tells us that the period of the oscillation is estimated at about 90 per cent of that for the $S_1^0$ field when we ignore the coupling of the two fields. But it is not clear how far the period will be affected when we take into account the couplings with the other higher harmonics.

The relative magnitudes of the coefficients $a_0$, $a_4$, $b_2$ and $b_4$ can be obtained from (62) into which the eigen-value of $aK$ obtained in (67) should be put. We obtain

$$a_4=-0.428a_0\,,\quad b_2=-1.83a_0\,,\quad b_4=1.43a_0\,.$$

With these coefficients, the magnetic fields can be calculated as follows:

$$\vec{h}=\begin{cases} -\{2.00-0.856(r/a)^4\}a_0P_1 \\ -\{2.00-2.57(r/a)^4\}\,a_0\dfrac{dP_1}{d\theta} \\ 0 \end{cases}+\begin{cases} \{22.0-17.2(r/a)^2\}(r/a)^2a_0P_3 \\ \{7.32-8.52(r/a)^2\}(r/a)^2a_0\dfrac{dP_3}{d\theta} \\ 0 \end{cases}\quad \text{for } r<a\,,$$

$$\vec{h}=\begin{cases} -1.14(r/a)^{-3}a_0P_1 \\ 0.572(r/a)^{-3}a_0\dfrac{dP_1}{d\theta} \\ 0 \end{cases}+\begin{cases} 4.80(r/a)^{-5}a_0P_3 \\ -1.20(r/a)^{-5}a_0\dfrac{dP_3}{d\theta} \\ 0 \end{cases}\quad \text{for } r>a\,.$$

We can also obtain the velocity. From (28), (29) and (30), it is found that

$$A_2=0.571\{1-(r/a)^2\}Ha^{-2}a_0\,,\quad B_2=A_4=0\,,$$

so that the velocity is of the $S_2^0$ type and it is given as

$$4\pi\rho DH^{-1}a\vec{v}=\begin{cases} -3.43\{1-(r/a)^2\}(r/a)P_2 \\ -\{1.71-2.86(r/a)^2\}(r/a)\dfrac{dP_2}{d\theta} \\ 0\,. \end{cases}$$

The distribution of the magnetic field and the velocity are respectively shown in Figs. 1 and 2.

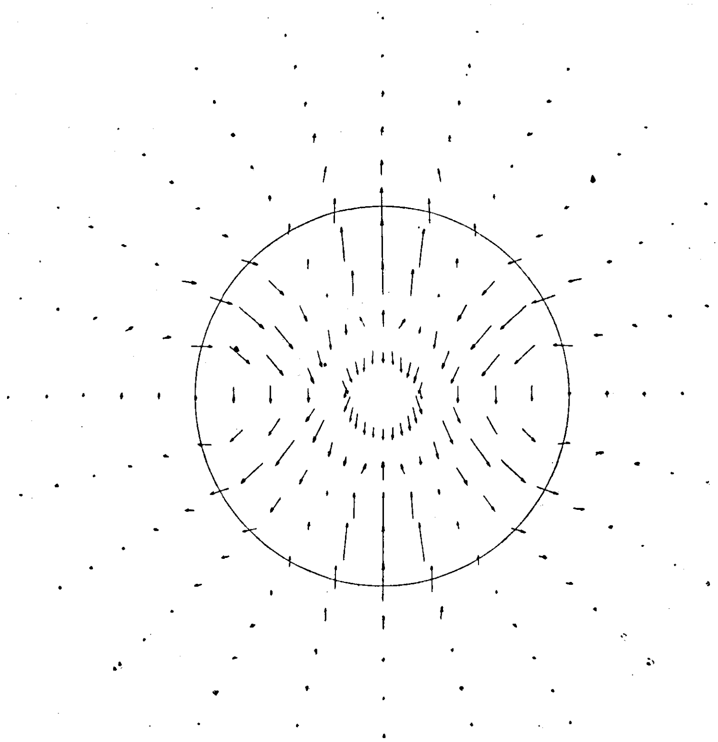

Fig. 1. The distribution of the oscillatory magnetic field.

## 7. Conclusion

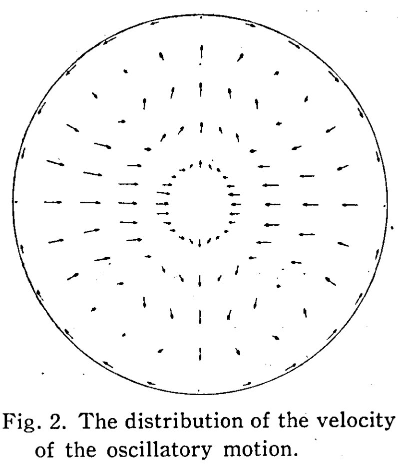

Fig. 2. The distribution of the velocity of the oscillatory motion.

For the purpose of examination of the mutual coupling between spherical harmonic constituents of different degrees, the general theory of the magneto-hydrodynamic oscillation of a liquid sphere is developed in this paper. The general differential equation is obtained as shown in (53) from which we can deduce some useful results concerning the coupling. It is made clear that there are couplings between poloidal and toroidal magnetic fields except for the case of zonal oscillations. As is closely examined in the case of zonal oscillation, it is no easy matter to determine the free oscillations because we have to solve a system of simultaneous differential equations of an infinite number with unknowns of an infinite number. In order to make the problem tractable, only the coupling between $S_1^0$ and $S_3^0$ type fields is studied and the other constituents are ignored. An approximate normal mode is obtained by expanding the solution in terms of a power series of the radial distance. But the oscillation mode thus obtained does not seem to be accurate.

The magneto-hydrodynamic oscillations of a liquid sphere are quite different from the electromagnetic, hydrodynamic or elastic oscillations which have hitherto been examined. In those oscillations, normal modes are usually obtained as spherical harmonics. However, the normal modes for the magneto-hydrodynamic oscillation are given as certain combinations of every spherical harmonic. If we assume that the geomagnetic secular variation is caused by the magneto-hydrodynamic effect in the earth's core, we may expect complicated magnetic fields which contain many harmonics.

Although the writer can obtain the general equation, which governs the magneto-hydrodynamic oscillations of a fluid sphere under the influence of a uniform permanent magnetic field and consequently some qualitative results, it turns out that the equation is not adequate for studying the normal modes of the oscillation because it contains unknowns of an infinite number. In order to study the free or forced oscillations, it is desirable to develop a new method which would be quite different from usual treatment with spherical harmonics. But it is not certain whether we can construct such a method or not.

## 12. 一様磁場内の導電流体球の電磁流体振動

地震研究所　力　武　常　次

前論文に於て地磁気永年変化に対応すると考えられる地球核内の電磁流体振動を論じる際に，ことなつたデグリーのポロイダルまたはトロイダル磁場間の相互の影響を無視した．この影響を見るために，もつとも簡単な場合として外部から一様な磁場を与えた時の導電流体球の電磁流体振動を論じた．その結果一般的方程式を求め得たが，各デグリーの成分が互に関係しあう連成振動となり，その固有値を求めることは容易でないことがわかつた．そこで比較的簡単な帯状振動について特に考察したところ，定性的には有意義な結果を得た．また第5次以上の振動を省略した場合について，近似的に基準振動を求めた．