# 1. *Magneto-hydrodynamic Oscillations in the Earth's Core.*

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**Summary**

Possibilities of magneto-hydrodynamic oscillations in the earth's core are discussed in relation to the stability of Elsasser-Bullard-Takeuchi's dynamo. If there are stable magnetic fields which are caused by the dynamo-action in the earth's core, there will be oscillations of magnetic field and fluid motion around the steady state. This sort of oscillation is here studied under some simplifications. Although the study is rather crude, it is tentatively concluded that the toroidal magnetic field is not so large, otherwise the small oscillations of dynamo would not be stable. Since the fundamental equations for magnetic changes in the earth's core are obtained, we may forecast the magnitude of magnetic field of certain type, say $S_1^0$-type, starting from suitable initial conditions. This is done in this paper on the basis of the spherical harmonic analyses for various epochs. Although no definite result is obtained because the observation period of the earth's magnetism seems too short for close comparison between the observation and theory, the conclusion concerning the internal magnetic field is compatible with that obtained in the stability problem.

The effect of the earth's rotation as well as that of the mutual dependence of various spherical harmonics are ignored throughout this study. That sort of study will be carried out later on.

## 1. Introduction

W. M. Elsasser[1]. E. C. Bullard[2], H. Takeuchi[3] and Y. Shimazu[3] have shown that the origin of the earth's main magnetic field seems

1) W. M. ELSASSER, *Phys. Rev.*, **69** (1946), 106, **70** (1946), 202.

2) E. C. BULLARD, *Proc. Roy. Soc. London A*, **197** (1949), 433, **199** (1949), 413.

3) H. TAKEUCHI and Y. SHIMAZU, *Journ. Phys. Earth.*, **1** (1952), 1. *Journ. Geophys. Res.* **58** (1953), 497.

to be explained by the self-exciting dynamo in the earth's core though further laborious works would be necessary in order to complete the theory. As far as the convection currents, magnetic fields and electric currents which are expressed by low degree spherical harmonics are concerned, however, it is likely that there will be a solution for the eigen-value problem which proves the existence of a self-exciting process. If this is to be taken for granted, we have steady motions which excite steady electric currents and consequently steady magnetic fields.

Since the earth's main magnetic field seems to be maintained over long periods, the dynamo must be fairly stable. Palaeomagnetic studies[4] suggest that the earth's magnetic field might have been reversed quite a long time ago, say $10^5$—$10^6$ *years*. But there is no evidence that the reversal of the earth's magnetic field occurred after this epoch. So the dynamo must have been maintained during this order of years. It is believed that the study on the stability is also quite important to establish the dynamo-theory in the core. Bullard[5] studied the oscillation of a homopolar disc dynamo as the first step to this problem. Although the disc dynamo is quite different from the dynamo in the earth's core, his study is important and interesting for the stability problem. The writer would here like to make another approach to the stability problem which is approximately applicable to the real earth.

Now let us consider an equilibrium of convection currents, magnetic fields and electric currents. We shall introduce small deviations of these quantities from their steady values. The behaviour of these small quantities will be studied here. If they remain small enough for a long time, the small oscillation of the dynamo may be regarded as stable. If they increase indefinitely, it will not be stable.

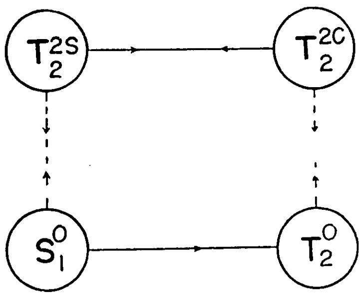

Fig. 1. The "A-approximation" of the self-exciting dynamo.

The writer takes the so-called "A-approximation" as the simplest model of the steady state. As shown in Fig. 1, it contains magnetic fields of $S_1^0$, $T_2^0$, $T_2^{2c}$ and $T_2^{2s}$ type and velocities of $T_1^0$ and $S_2^{2c}$ type. The fundamental equations for small deviations are constructed in Section 2. In order to solve these equations, only small deviation of the magnetic field of $S_1^0$ type is taken into account ignoring all magnetic

4) T. NAGATA, *Spec. Rep. A.T.M.E.I.U.G.G. Rome Meeting* (1954).

5) E. C. BULLARD, Personal Communication.

fields of other types which will come out during the course of magneto-hydrodynamic interactions. Thus as shown in Section 3, we can solve the pressure equation which is deduced from the fundamental equations. Then the velocity is calculated in Section 4, getting $T_1^0$, $T_3^0$, $S_2^0$, $S_2^{2c}$, $S_2^{2s}$, $T_3^{2c}$, and $T_3^{2s}$ motions. The couplings between these motions and the steady magnetic fields will give many types of magnetic field. But we pick up the couplings which give only the $S_1^0$-type magnetic field. So we can have an integro-differential equation for small deviation of $S_1^0$-type magnetic field as given in Section 5. In Section 6, this equation is solved assuming suitable distributions of the steady fields. Thus the periods of free magneto-hydrodynamic oscillations are approximately obtained. The above-stated procedure is similar to the studies of magnetic oscillations of highly conducting stars[6),7),8)] though there are some differences between both studies.

It is found out that the amplitude of the above-considered small deviations will increase tremendously unless the intensities of the steady magnetic fields satisfy certain conditions. In other words, this is a sort of stability condition from which we presume that the $T_2^{2c}$ and $T_2^{2s}$ magnetic fields would not be larger than the steady part of the $S_1^0$ field as will be discussed in Section 7. In that case the period of free oscillation of $S_1^0$ field will be of the order of scores or hundreds of *years*. However, as was suggested by Bullard[9)], the effect of Corioli's force due to the earth's rotation would be fairly large. If we take into account this effect, it is quite probable that the stability condition would become fairly modified. Strictly speaking, the error due to the present procedure in which we pick up only the $S_1^0$-type magnetic field is not clear though the effect of neglecting higher degree terms is thought to be not so serious from physical considerations. However, these points remain to be made clearer.

The integro-differential equation (45) which is obtained in Section 5 can be regarded as the fundamental equation which governs the small oscillations of the $S_1^0$ field superposed on the steady one. If we start from suitable initial conditions, we may forecast the behaviour of geomagnetic change which is thought to correspond to the secular variation. With the aid of the various spherical harmonic analyses we choose the initial condition at 1900. Then a forecasting is made in

6) M. SCHWARZSCHILD, *Ann. D'Astrophys.*, **12** (1949), 148.
7) V. C. A. FERRARO and D. J. MEMORY, *M.N.R.A.S.*, **112** (1952), 361.
8) C. PLUMPTON and V. C. A. FERRARO, *M.N.R.A.S.*, **113** (1953), 647.
9) E. C. BULLARD, Personal communication.

Section 8 and is also compared with the observed course of the secular change in the $S_1^0$-field. Since the observation does not cover a period sufficient for getting a definite result from the comparison between the forecasting and the observation, we failed to obtain any conclusive result. But the assumption that the toroidal field is not so large in the core as is discussed in Section 7 seems to hold good here also, otherwise the forecast curves would have become quite different from the observed one.

## 2. Fundamental equations

Assuming that the magnetic permeability is unity, the fundamental equations are given as

$$\vec{I}=\sigma(\vec{E}+\vec{V}\wedge\vec{H})\ , \tag{1}$$

$$\mathrm{curl}\,\vec{E}=-\partial\vec{H}/\partial t\ , \tag{2}$$

$$\mathrm{curl}\,\vec{H}=4\pi\vec{I}\ , \tag{3}$$

$$\rho d\vec{V}/dt=\vec{I}\wedge\vec{H}-\mathrm{grad}\,P+\vec{G}\ , \tag{4}$$

where $\vec{I}$, $\vec{E}$, $\vec{H}$, $\sigma$, $\rho$, $\vec{V}$, $P$ and $\vec{G}$ denote respectively the electric current density, electric field, magnetic field, electrical conductivity, density, velocity, pressure and non-electromagnetic force. We will write as

$$\left.\begin{aligned}&\vec{I}=\vec{I}_0+\vec{i}\ ,\quad \vec{E}=\vec{E}_0+\vec{e}\ ,\quad \vec{H}=\vec{H}_0+\vec{h}\ ,\\ &\vec{V}=\vec{V}_0+\vec{v}\ ,\quad P=P_0+p\ ,\end{aligned}\right\} \tag{5}$$

in which the quantities with suffix $o$ are not dependent on time, while those denoted with small letters are regarded as first order small quantities. Introducing (5) into the fundamental equations and ignoring the second order quantities, we have

$$(\mathrm{I})\left\{\begin{aligned}&\vec{I}_0=\sigma(\vec{E}_0+\vec{V}_0\wedge\vec{H}_0)\ ,\\ &\mathrm{curl}\,\vec{E}_0=0\ ,\\ &\mathrm{curl}\,\vec{H}_0=4\pi\vec{I}_0\ ,\\ &\vec{I}_0\wedge\vec{H}_0-\mathrm{grad}\,P_0+\vec{G}=0\ ,\end{aligned}\right.$$

$$(\mathrm{II})\left\{\begin{aligned}&\vec{i}=\sigma(\vec{e}+\vec{v}\wedge\vec{H}_0+\vec{V}_0\wedge\vec{h})\ , && (6)\\ &\mathrm{curl}\,\vec{e}=-\partial\vec{h}/\partial t\ , && (7)\\ &\mathrm{curl}\,\vec{h}=4\pi\vec{i}\ , && (8)\\ &\rho\partial\vec{v}/\partial t=\vec{i}\wedge\vec{H}_0+\vec{I}_0\wedge\vec{h}-\mathrm{grad}\,p\ . && (9)\end{aligned}\right.$$

It is assumed here that the liquid is incompressible. So the relation

$$\mathrm{div}\,\vec{v}=0 \tag{10}$$

is taken into account in obtaining the equation (6).

The first system of equations (I) is precisely the one from which the self-exciting dynamo in the earth's core might be deduced. It has been shown by Bullard, Takeuchi and others that the solution of (I) would be so complicated that one can hardly solve it completely. But it is likely that the main feature of a self-exciting dynamo will be given with certain magnetic fields and velocity fields which may be expressed by spherical harmonics of low degree. Therefore the writer assumes that the solution of (I) is given by the "A-approximation" as was called by Takeuchi and Shimazu. The solution contains the $T_1^0$ and $S_2^{2c}$ types of velocity and magnetic fields of the $S_1^0$, $T_2^0$, $T_2^{2c}$, and $T_2^{2s}$ types.

Now we are in a position to solve (II) in which $\vec{V}_0$, $\vec{H}_0$ and $\vec{I}_0$ are given as the solution of (I). By taking curl of (6) and eliminating $\vec{e}$ with the aid of (7), we have

$$\text{curl}\,\vec{i}=\sigma\{-D\vec{h}+\text{curl}(\vec{v}\wedge\vec{H}_0)+\text{curl}(\vec{V}_0\wedge\vec{h})\} \tag{11}$$

where we write $D$ in place of $\partial/\partial t$. On the other hand we have from (8)

$$4\pi\,\text{curl}\,\vec{i}=\text{curl}\,\text{curl}\,\vec{h}=-\nabla^2\vec{h}\ . \tag{12}$$

From (11) and (12), we have

$$\{D-(4\pi\sigma)^{-1}\nabla^2\}\vec{h}=\text{curl}\,(\vec{V}_0\wedge\vec{h})+\text{curl}\,(\vec{v}\wedge\vec{H}_0)\ . \tag{13}$$

We also have from (9)

$$\rho D\vec{v}=(4\pi)^{-1}\{(\text{curl}\,\vec{h}\wedge\vec{H}_0)+(\text{curl}\,\vec{H}_0\wedge\vec{h})\}-\text{grad}\,p\ . \tag{14}$$

If we make div. of (14), we obtain

$$\nabla^2 p=-(4\pi)^{-1}\{\vec{H}_0\cdot\nabla^2\vec{h}+\vec{h}\cdot\nabla^2\vec{H}_0+2\,\text{curl}\,\vec{h}\cdot\text{curl}\,\vec{H}_0\}\ . \tag{15}$$

## 3. The solution for pressure

Generally, $\vec{h}$ can be expressed as

$$\vec{h}=\sum_{n,m}\vec{h}_{s,n}^m+\sum_{n,m}\vec{h}_{t,n}^m \tag{16}$$

where $\vec{h}_{s,n}^m$ is of the poloidal type, the $r$, $\theta$ and $\phi$ components being written as

$$\vec{h}^m_{s,n}=\begin{cases} -n(n+1)s^m_n(r)r^{n-1}Y^m_n \\ -\left[r\dfrac{ds^m_n}{dr}+(n+1)s^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\partial\theta}\,, \\ -\left[r\dfrac{ds^m_n}{dr}+(n+1)s^m_n\right]r^{n-1}\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\,, \end{cases} \tag{17}$$

and $\vec{h}^m_{t,n}$ is of the toroidal type which is written as

$$\vec{h}^m_{t,n}=\begin{cases} 0 \\ -t^m_n(r)r^n\dfrac{\partial Y^m_n}{\sin\theta\partial\phi}\,, \\ t^m_n(r)r^n\dfrac{\partial Y^m_n}{\partial\theta}\,, \end{cases} \tag{18}$$

in which

$$Y^m_n=P^m_n(\cos\theta)\,{\cos \atop \sin}\,m\phi\,. \tag{19}$$

Introducing (16) into (15), we obtain a partial differential equation from which we can determine $p$ as a suitable solution with some constants. Since $p$ becomes known, $\vec{v}$ will be obtained from (14). The constants should be determined by the condition that the radial component of $\vec{v}$ vanishes at the surface of the core whose radius is denoted by $a$ in later statement. Putting both $\vec{h}$ and $\vec{v}$ in (13), we shall have an equation by which we may determine $s^m_n$ and $t^m_n$ under the condition that the magnetic field is continuous at the surface of the core. Theoretically speaking, we can thus obtain the magnetic field and velocity. However, as has been shown by Elsasser, Bullard, and others, an interaction between certain electric currents and magnetic fields produces many types of motions and that between motions and magnetic fields produces many types of electric currents. Thus it is almost impossible to treat the problem quite generally. As has been done in the study of self-exciting dynamo, we shall take an approximation.

As the simplest, we shall assume that the magnetic field is of the $S^0_1$-type which is the most predominant one in the earth's magnetic field. The magnetic field of other types which will appear in the righthand-side of (13) will be ignored.

The magnetic field $\vec{h}$, curl $\vec{h}$ and $\nabla^2\vec{h}$ are given as

$$\vec{h}=\begin{cases} -2s(r)P_1\ , \\ -\left(r\dfrac{ds}{dr}+2s\right)\dfrac{dP_1}{d\theta}\ , \\ 0\ , \end{cases} \qquad \operatorname{curl}\vec{h}=\begin{cases} 0\ , \\ 0\ , \\ -\left(r\dfrac{d^2s}{dr^2}+4\dfrac{ds}{dr}\right)\dfrac{dP_1}{d\theta}\ , \end{cases}$$

$$\nabla^2\vec{h}=\begin{cases} -2\left(\dfrac{d^2s}{dr^2}+\dfrac{4}{r}\dfrac{ds}{dr}\right)P_1\ , \\ -\left(r\dfrac{d^3s}{dr^3}+6\dfrac{d^2s}{dr^2}+\dfrac{4}{r}\dfrac{ds}{dr}\right)\dfrac{dP_1}{d\theta}\ , \\ 0\ , \end{cases} \tag{20}$$

while the stationary magnetic field and velocity are given as

$$\left.\begin{aligned} \vec{H}_0&=\vec{H}_1+\vec{H}_2+\vec{H}_3+\vec{H}_4\ , \\ \vec{V}_0&=\vec{V}_1+\vec{V}_2\ , \end{aligned}\right\} \tag{21}$$

where

$S_1^0$ magnetic field :

$$\vec{H}_1=\begin{cases} -2S_1(r)P_1\ , \\ -\left(r\dfrac{dS_1}{dr}+2S_1\right)\dfrac{dP_1}{d\theta}\ , \\ 0\ , \end{cases} \qquad \operatorname{curl}\vec{H}_1=\begin{cases} 0\ , \\ 0\ , \\ -\left(r\dfrac{d^2S_1}{dr^2}+4\dfrac{dS_1}{dr}\right)\dfrac{dP_1}{d\theta}\ , \end{cases}$$

$$\nabla^2\vec{H}_1=\begin{cases} -2\left(\dfrac{d^2S_1}{dr^2}+\dfrac{4}{r}\dfrac{dS_1}{dr}\right)P_1\ , \\ -\left(r\dfrac{d^3S_1}{dr^3}+6\dfrac{d^2S_1}{dr^2}+\dfrac{4}{r}\dfrac{dS_1}{dr}\right)\dfrac{dP_1}{d\theta}\ , \\ 0\ , \end{cases} \tag{22}$$

$T_2^0$ magnetic field :

$$\vec{H}_2=\begin{cases} 0\ , \\ 0\ , \\ a^{-2}r^2T_2(r)\dfrac{dP_2}{d\theta}\ , \end{cases} \qquad \operatorname{curl}\vec{H}_2=\begin{cases} -6a^{-2}rT_2P_2\ , \\ -a^{-2}\left(r\dfrac{dT_2}{dr}+3T_2\right)r\dfrac{dP_2}{d\theta}\ , \\ 0\ , \end{cases}$$

$$\nabla^2\vec{H}_2=\begin{cases} 0\,, \\ 0\,, \\ a^{-2}\left(\dfrac{d^2T_2}{dr^2}+\dfrac{6}{r}\dfrac{dT_2}{dr}\right)r^2\dfrac{dP_2}{d\theta}\,. \end{cases} \tag{23}$$

$T_2^{2c}$ magnetic field :

$$\vec{H}_3=\begin{cases} 0\,, \\ -a^{-2}r^2T_3(r)\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\,, \\ a^{-2}r^2T_3(r)\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\,, \end{cases}$$

$$\text{curl}\,\vec{H}_3=\begin{cases} -6a^{-2}rT_3P_2^2\cos 2\phi\,, \\ -a^{-2}\left(r\dfrac{dT_3}{dr}+3T_3\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\,, \\ -a^{-2}\left(r\dfrac{dT_3}{dr}+3T_3\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\,, \end{cases}$$

$$\nabla^2\vec{H}_3=\begin{cases} 0\,, \\ -a^{-2}\left(\dfrac{d^2T_3}{dr^2}+\dfrac{6}{r}\dfrac{dT_3}{dr}\right)r^2\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\,, \\ a^{-2}\left(\dfrac{d^2T_3}{dr^2}+\dfrac{6}{r}\dfrac{dT_3}{dr}\right)r^2\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\,, \end{cases} \tag{24}$$

$T_2^{2s}$ magnetic field :

$$\vec{H}_4=\begin{cases} 0\,, \\ -a^{-2}r^2T_4(r)\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi}\,, \\ a^{-2}r^2T_4(r)\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta}\,, \end{cases}$$

$$\text{curl}\,\vec{H}_4=\begin{cases} -6a^{-2}rT_4P_2^2\sin 2\phi\,, \\ -a^{-2}\left(r\dfrac{dT_4}{dr}+3T_4\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta}\,, \\ -a^{-2}\left(r\dfrac{dT_4}{dr}+3T_4\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi}\,, \end{cases}$$

$$\nabla^2\vec{H}_4=\begin{cases} 0\ , \\ -a^{-2}\left(\dfrac{d^2T_4}{dr^2}+\dfrac{6}{r}\dfrac{dT_4}{dr}\right)r^2\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi}\ , \\ a^{-2}\left(\dfrac{d^2T_4}{dr^2}+\dfrac{6}{r}\dfrac{dT_4}{dr}\right)r^2\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta}\ . \end{cases} \tag{25}$$

$T_1^0$ velocity :

$$\vec{V}_1=\begin{cases} 0\ , \\ 0\ , \\ a^{-1}r\,V_1(r)\dfrac{dP_1}{d\theta}\ . \end{cases} \tag{26a}$$

$S_2^{2c}$ velocity :

$$\vec{V}_2=\begin{cases} -6a^{-1}r\,V_2P_2^2\cos 2\phi\ , \\ -a^{-1}\left(r\dfrac{dV_2}{dr}+3V_2\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\ , \\ -a^{-1}\left(r\dfrac{dV_2}{dr}+3V_2\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\ . \end{cases} \tag{26b}$$

From (20), (21), (22), (23), (24) and (25), we can calculate the righthand members of (15) as follows ;

$$\begin{aligned}\vec{H}_0\cdot\nabla^2\vec{h}=&\frac{2}{3}\left\{2S_1\left(\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)+\left(r\frac{dS_1}{dr}+2S_1\right)\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)\right\}\\ &+\frac{2}{3}\left\{4S_1\left(\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)-\left(r\frac{dS_1}{dr}+2S_1\right)\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)\right\}P_2\\ &+2a^{-2}T_3r^2\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)P_2^2\sin 2\phi\\ &-2a^{-2}T_4r^2\left(r\frac{d^3s}{dr^3}+6\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)P_2^2\cos 2\phi\ ,\end{aligned}$$

$$\begin{aligned}\vec{h}\cdot\nabla^2\vec{H}_0=&\frac{2}{3}\left\{2\left(\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)s+\left(r\frac{d^3S_1}{dr^3}+6\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)\left(r\frac{ds}{dr}+2s\right)\right\}\\ &+\frac{2}{3}\left\{4\left(\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)s-\left(r\frac{d^3S_1}{dr^3}+6\frac{d^2S_1}{dr^2}+\frac{4}{r}\frac{dS_1}{dr}\right)\left(r\frac{ds}{dr}+2s\right)\right\}P_2\\ &+2a^{-2}\left(r^2\frac{d^2T_3}{dr^2}+6r\frac{dT_3}{dr}\right)\left(r\frac{ds}{dr}+2s\right)P_2^2\sin 2\phi\\ &-2a^{-2}\left(r^2\frac{d^2T_4}{dr^2}+6r\frac{dT_4}{dr}\right)\left(r\frac{ds}{dr}+2s\right)P_2^2\cos 2\phi\end{aligned}$$

$$\mathrm{curl}\,\vec{h}\cdot\mathrm{curl}\,\vec{H}_0=\frac{2}{3}\left(r\frac{d^2S_1}{dr^2}+4\frac{dS_1}{dr}\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)$$
$$-\frac{2}{3}\left(r\frac{d^2S_1}{dr^2}+4\frac{dS_1}{dr}\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2$$
$$+2a^{-2}\left(r^2\frac{dT_3}{dr}+3rT_3\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2^2\sin 2\phi$$
$$-2a^{-2}\left(r^2\frac{dT_4}{dr}+3rT_4\right)\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2^2\cos 2\phi$$

For the sake of simplicity, we may assume that $S_1$, $T_2$, $T_3$ and $T_4$ are independent of $r$. We shall be able to obtain a rough idea of the magneto-hydrodynamic behaviour in the earth's core by this simplification. In that case (15) becomes

$$\nabla^2 p=f_0+f_2P_2+f_2^{2c}P_2^2\cos 2\phi+f_2^{2s}P_2^2\sin 2\phi \tag{27}$$

where

$$-4\pi f_0=\frac{4}{3}S_1\left(r\frac{d^3s}{dr^3}+7\frac{d^2s}{dr^2}+\frac{8}{r}\frac{ds}{dr}\right),$$
$$-4\pi f_2=-\frac{4}{3}S_1\left(r\frac{d^3s}{dr^3}+4\frac{d^2s}{dr^2}-\frac{4}{r}\frac{ds}{dr}\right),$$
$$-4\pi\begin{pmatrix}f_2^{2c}\\ f_2^{2s}\end{pmatrix}=\mp 2a^{-2}\begin{pmatrix}T_4\\ T_3\end{pmatrix}r^2\left(r\frac{d^3s}{dr^3}+12\frac{d^2s}{dr^2}+\frac{28}{r}\frac{ds}{dr}\right). \tag{28}$$

The solution of (27) may be expressed as

$$p=\sum_{n,m}q_n^m(r)Y_n^m\,. \tag{29}$$

Putting (29) in $\nabla^2 p$, we obtain

$$\nabla^2 p=\sum_{n,m}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{n(n+1)}{r^2}\right\}q_n^m(r)Y_n^m\,. \tag{30}$$

Comparing (27) with (30), we obtain a series of ordinary differential equations such as

$$\left.\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)q_0^0=f_0\,,\\ &\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)-\frac{6}{r^2}\right\}\begin{pmatrix}q_2^0\\ q_2^{2c}\\ q_2^{2s}\end{pmatrix}=\begin{pmatrix}f_2\\ f_2^{2c}\\ f_2^{2s}\end{pmatrix}.\end{aligned}\right\} \tag{31}$$

$q_n^m$ for the other combinations of $n$, $m$ are disregarded without loss of generality.

The solutions of (31) which remain finite at $r=0$ are given as

$$\left.\begin{aligned} q_0^0&=K_0^0+\int_0^r rf_0dr-r^{-1}\int_0^r r^2f_0dr\ ,\\ \begin{pmatrix} q_2^0\\ q_2^{2c}\\ q_2^{2s}\end{pmatrix}&=a^{-2}r^2\begin{pmatrix} K_2^0\\ K_2^{2c}\\ K_2^{2s}\end{pmatrix}+\frac{1}{5}\left\{r^2\int_0^r r^{-1}\begin{pmatrix} f_2\\ f_2^{2c}\\ f_2^{2s}\end{pmatrix}dr-r^{-3}\int_0^r r^4\begin{pmatrix} f_2\\ f_2^{2c}\\ f_2^{2s}\end{pmatrix}dr\right\}\ ,\end{aligned}\right\} \quad (32)$$

where $K_0^0$, $K_2^0$, $K_2^{2c}$ and $K_2^{2s}$ are constants which shall be determined later. Putting (28) in (32), we have, after some calculations, the following expressions:

$$\left.\begin{aligned} 4\pi q_0^0&=L_0^0-\frac{4}{3}S_1\left(r\frac{ds}{dr}+3s\right)\ ,\\ 4\pi q_2^0&=a^{-2}r^2L_2^0+\frac{4}{3}S_1r\frac{ds}{dr}\ ,\\ 4\pi q_2^{2c}&=a^{-2}r^2L_2^{2c}+2a^{-2}T_4\left(r^3\frac{ds}{dr}+4r^2s-2r^{-3}\int_0^r r^4sdr\right)\ ,\\ 4\pi q_2^{2s}&=a^{-2}r^2L_2^{2s}-2a^{-2}T_3\left(r^3\frac{ds}{dr}+4r^2s-2r^{-3}\int_0^r r^4sdr\right)\ ,\end{aligned}\right\} \quad (33)$$

where

$$L_0^0=4\pi K_0^0\ ,\quad L_2^0=4\pi K_2^0\ ,\quad L_2^{2c}=4\pi K_2^{2c}\ ,\quad L_2^{2s}=4\pi K_2^{2s}\ .$$

Thus the solution for the pressure is given by (29) together with (33).

## 4. The solution for velocity

The velocity can be calculated from (14). We have

$$\operatorname{curl}\vec{h}\wedge\vec{H}_1=\begin{cases} -\frac{4}{3}S_1\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)\\ 0\\ 0\end{cases}+\begin{cases}\frac{4}{3}S_1\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2\\ \frac{2}{3}S_1\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)\frac{dP_2}{d\theta}\\ 0\end{cases}\ ,$$

$$\operatorname{curl}\vec{h}\wedge\vec{H}_2=\begin{cases}0\\ 0\\ 0\end{cases}\ ,$$

$$\operatorname{curl}\vec{h}\wedge\vec{H}_3=\begin{cases}-2a^{-2}T_3r^2\left(r\frac{d^2s}{dr^2}+4\frac{ds}{dr}\right)P_2^2\sin 2\phi\\ 0\\ 0\end{cases}\ ,$$

$$\operatorname{curl}\vec{h}\wedge\vec{H}_4=\begin{cases}2a^{-2}T_4r^2\left(r\dfrac{d^2s}{dr^2}+4\dfrac{ds}{dr}\right)P_2^2\cos 2\phi\\0\\0\end{cases},$$

$$\operatorname{curl}\vec{H}_1\wedge\vec{h}=\begin{cases}0\\0\\0\end{cases},$$

$$\operatorname{curl}\vec{H}_2\wedge\vec{h}=\begin{cases}0\\0\\-\dfrac{6}{5}a^{-2}T_2r\left(r\dfrac{ds}{dr}+5s\right)\dfrac{dP_1}{d\theta}\end{cases}+\begin{cases}0\\0\\\dfrac{6}{5}a^{-2}T_2r^2\dfrac{ds}{dr}\dfrac{dP_3}{d\theta}\end{cases},$$

$$\operatorname{curl}\vec{H}_3\wedge\vec{h}=\begin{cases}-6a^{-2}T_3r\left(r\dfrac{ds}{dr}+2s\right)P_2^2\sin 2\phi\\-6a^{-2}T_3rs\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta}\\-2a^{-2}T_3r\left(r\dfrac{ds}{dr}+3s\right)\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi}\end{cases}+\begin{cases}0\\0\\\dfrac{2}{5}a^{-2}T_3r^2\dfrac{ds}{dr}\dfrac{\partial(P_3^2\cos 2\phi)}{\partial\theta}\end{cases},$$

$$\operatorname{curl}\vec{H}_4\wedge\vec{h}=\begin{cases}6a^{-2}T_4r\left(r\dfrac{ds}{dr}+2s\right)P_2^2\cos 2\phi\\6a^{-2}T_4rs\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\\2a^{-2}T_4r\left(r\dfrac{ds}{dr}+3s\right)\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\end{cases}+\begin{cases}0\\0\\\dfrac{2}{5}a^{-2}T_4r^2\dfrac{ds}{dr}\dfrac{\partial(P_3^2\sin 2\phi)}{\partial\theta}\end{cases}. \quad (34)$$

With these expressions we can calculate $\vec{v}$ from (14) in which $p$ is given by (19) and (33). Hence we have

$$4\pi\rho Dv_r = -2\left(a^{-2}rL_2^0 - 2S_1\frac{ds}{dr}\right)P_2$$
$$-2a^{-2}\left(rL_2^{2c} + 6T_4 r^{-4}\int_0^r r^4 s dr\right)P_2^2 \cos 2\phi$$
$$-2a^{-2}\left(rL_2^{2s} - 6T_3 r^{-4}\int_0^r r^4 s dr\right)P_2^2 \sin 2\phi\ ,$$

$$4\pi\rho Dv_\theta = \left\{-a^{-2}rL_2^0 + \frac{2}{3}S_1\left(r\frac{d^2s}{dr^2} + 2\frac{ds}{dr}\right)\right\}\frac{dP_2}{d\theta}$$
$$-a^{-2}\left\{rL_2^{2c} + 2T_4\left(rs - 2r^{-4}\int_0^r r^4 s dr\right)\right\}\frac{\partial(P_2^2 \cos 2\phi)}{\partial\theta}$$
$$-a^{-2}\left\{rL_2^{2s} - 2T_3\left(rs - 2r^{-4}\int_0^r r^4 s dr\right)\right\}\frac{\partial(P_2^2 \sin 2\phi)}{\partial\theta}$$
$$-\frac{2}{5}a^{-2}T_3 r^2\frac{ds}{dr}\frac{\partial(P_3^2 \cos 2\phi)}{\sin\theta\partial\phi}$$
$$-\frac{2}{5}a^{-2}T_4 r^2\frac{ds}{dr}\frac{\partial(P_3^2 \sin 2\phi)}{\sin\theta\partial\phi}\ ,$$

$$4\pi\rho Dv_\phi = -\frac{6}{5}a^{-2}T_2 r\left(r\frac{ds}{dr} + 5s\right)\frac{dP_1}{d\theta}$$
$$+\frac{6}{5}a^{-2}T_2 r^2\frac{ds}{dr}\frac{dP_3}{d\theta}$$
$$-a^{-2}\left\{rL_2^{2c} + 2T_4\left(rs - 2r^{-4}\int_0^r r^4 s dr\right)\right\}\frac{\partial(P_2^2 \cos 2\phi)}{\sin\theta\partial\phi}$$
$$-a^{-2}\left\{rL_2^{2s} - 2T_3\left(rs - 2r^{-4}\int_0^r r^4 s dr\right)\right\}\frac{\partial(P_2^2 \sin 2\phi)}{\sin\theta\partial\phi}$$
$$+\frac{2}{5}a^{-2}T_3 r^2\frac{ds}{dr}\frac{\partial(P_3^2 \cos 2\phi)}{\partial\theta}$$
$$+\frac{2}{5}a^{-2}T_4 r^2\frac{ds}{dr}\frac{\partial(P_3^2 \sin 2\phi)}{\partial\theta}\ .$$

But it is more convenient to express velocity with a sum of poloidal and toroidal velocity fields. We can write as follows;

$$\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 + \vec{v}_5 + \vec{v}_6 + \vec{v}_7\ , \qquad (35)$$

where

$T_1^0$-motion :

$$4\pi\rho D\vec{v}_1=\begin{cases}0\\0\\t_1 r\dfrac{dP_1}{d\theta}\,,\end{cases}\qquad t_1=-\frac{6}{5}a^{-2}T_2\left(r\frac{ds}{dr}+5s\right), \tag{36}$$

$T_3^0$-motion :

$$4\pi\rho D\vec{v}_2=\begin{cases}0\\0\\t_3 r^3\dfrac{dP_3}{d\theta}\,,\end{cases}\qquad t_3=\frac{6}{5}a^{-2}T_2 r^{-1}\frac{ds}{dr}\,, \tag{37}$$

$S_2^0$-motion :

$$4\pi\rho D\vec{v}_3=\begin{cases}-6s_2 rP_2\\-\left(r\dfrac{ds_2}{dr}+3s_2\right)r\dfrac{dP_2}{d\theta}\,,\\0\end{cases}\qquad s_2=\frac{1}{3}\left(a^{-2}L_2^0-2S_1 r^{-1}\frac{ds}{dr}\right), \tag{38}$$

$S_2^{2c}$-motion :

$$4\pi\rho D\vec{v}_4=\begin{cases}-6s_2^{2c}rP_2^2\cos 2\phi\\-\left(r\dfrac{ds_2^{2c}}{dr}+3s_2^{2c}\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\partial\theta}\\-\left(r\dfrac{ds_2^{2c}}{dr}+3s_2^{2c}\right)r\dfrac{\partial(P_2^2\cos 2\phi)}{\sin\theta\partial\phi}\,,\end{cases}\qquad s_2^{2c}=\frac{1}{3}a^{-2}L_2^{2c}+2T_4 a^{-2}r^{-5}\int_0^r r^4 s dr\,, \tag{39}$$

$S_2^{2s}$-motion :

$$4\pi\rho D\vec{v}_5=\begin{cases}-6s_2^{2s}rP_2^2\sin 2\phi\\-\left(r\dfrac{ds_2^{2s}}{dr}+3s_2^{2s}\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\partial\theta}\\-\left(r\dfrac{ds_2^{2s}}{dr}+3s_2^{2s}\right)r\dfrac{\partial(P_2^2\sin 2\phi)}{\sin\theta\partial\phi}\,,\end{cases}\qquad s_2^{2s}=\frac{1}{3}a^{-2}L_2^{2s}-2T_3 a^{-2}r^{-5}\int_0^r r^4 s dr\,, \tag{40}$$

$T_3^{2c}$-motion :

$$4\pi\rho D\vec{v}_6=\begin{cases}0\\-t_3^{2c}r^3\dfrac{\partial(P_3^2\cos 2\phi)}{\sin\theta\partial\phi}\\t_3^{2c}r^3\dfrac{\partial(P_3^2\cos 2\phi)}{\partial\theta}\,,\end{cases}\qquad t_3^{2c}=\frac{2}{5}a^{-2}T_3 r^{-1}\frac{ds}{dr}\,, \tag{41}$$

$T_3^{2s}$-motion :

$$4\pi\rho D\vec{v}_7=\begin{cases}0\\ -t_3^{2s}r^3\dfrac{\partial(P_3^2\sin 2\phi)}{\sin\theta\partial\phi}\\ t_3^{2s}r^3\dfrac{\partial(P_3^2\sin 2\phi)}{\partial\theta}\,.\end{cases}\qquad t_3^{2s}=\frac{2}{5}a^{-2}T_4r^{-1}\frac{ds}{dr}\,, \tag{42}$$

The constants are determined by the condition $v_r=0$ at $r=a$. They are given as

$$L_2^0=2aS_1\left(\frac{ds}{dr}\right)_{r=a},\quad L_2^{2c}=-6T_4a^{-5}\int_0^a r^4 s dr\,,\quad L_2^{2s}=6T_3a^{-5}\int_0^a r^4 s dr\,. \tag{43}$$

## 5. The integro-differential equation for magnetic field

We introduce, then, $\vec{v}$ thus obtained into (13) together with $\vec{V}_0$ of (26). Although we obtain many types of magnetic field from curl $(\vec{V}_0\wedge\vec{h})$ and curl $(\vec{v}\wedge\vec{H}_0)$, we only pick up the components of the $S_1^0$ type. This can be done with the aid of the selection rules for electromagnetic couplings. The components which contribute to the $S_1^0$ type magnetic field are found to be as

$$\left.\begin{aligned}
&4\pi\rho D(\vec{v}_3\wedge\vec{H}_1)\to\begin{cases}0\\ 0\\ -\dfrac{6}{5}S_1r\left(r\dfrac{ds_2}{dr}+5s_2\right)\dfrac{dP_1}{d\theta}\,,\end{cases}\\
&4\pi\rho D\,\mathrm{curl}\,(\vec{v}_3\wedge\vec{H}_1)\to\begin{cases}\dfrac{12}{5}S_1\left(r\dfrac{ds_2}{dr}+5s_2\right)P_1\\ \dfrac{6}{5}S_1\left(r\dfrac{d}{dr}+2\right)\left(r\dfrac{ds_2}{dr}+5s_2\right)\dfrac{dP_1}{d\theta}\\ 0\end{cases}\,,\\
&4\pi\rho D(\vec{v}_4\wedge\vec{H}_4)\to\begin{cases}0\\ 0\\ -\dfrac{216}{5}T_4a^{-2}s_2^{2c}r^3\dfrac{dP_1}{d\theta}\,,\end{cases}\\
&4\pi\rho D\,\mathrm{curl}\,(\vec{v}_4\wedge\vec{H}_4)\to\begin{cases}\dfrac{432}{5}T_4a^{-2}r^2s_2^{2c}P_1\\ \dfrac{216}{5}T_4a^{-2}\left(r\dfrac{d}{dr}+2\right)(r^2s_2^{2c})\dfrac{dP_1}{d\theta}\\ 0\end{cases}\,,
\end{aligned}\right\} \tag{44}$$

$$4\pi\rho D(\vec{v}_5 \wedge \vec{H}_3) \to \begin{cases} 0 \\ 0 \\ \frac{216}{5} T_3 a^{-2} s_2^{2s} r^3 \frac{dP_1}{d\theta} \,, \end{cases}$$

$$4\pi\rho D \operatorname{curl}(\vec{v}_5 \wedge \vec{H}_3) \to \begin{cases} -\frac{432}{5} T_3 a^{-2} r^2 s_2^{2s} P_1 \\ -\frac{216}{5} T_3 a^{-2}\left(r\frac{d}{dr}+2\right)(r^2 s_2^{2s}) \frac{dP_1}{d\theta} \\ 0 \end{cases} .$$

Hence (13) is satisfied if we have the following relation:

$$4\pi\rho D\left\{Ds-(4\pi\sigma)^{-1}\left(\frac{d^2s}{dr^2}+\frac{4}{r}\frac{ds}{dr}\right)\right\}$$
$$=-\frac{6}{5}S_1\left(r\frac{ds_2}{dr}+5s_2\right)+\frac{216}{5}a^{-2}r^2(T_3 s_2^{2s}-T_4 s_2^{2c}) \,. \tag{45}$$

## 6. An approximate solution for the magnetic field

(45) is an integro-differential equation for $s$. It will be of great difficulty to obtain rigorous solutions for it. However, we may be able to find a solution for a simple radial distribution in the following way.

Let us assume that

$$s=\sum_{n=0}^{\infty} a_n (r/a)^n \tag{46}$$

in which we do not take into account the negative powers of $r$ because the magnetic field should remain finite at the centre of the earth. Introducing (46) into (45), we have the following relation

$$4\pi\rho D^2 \sum_n a_n (r/a)^n - \rho D\sigma^{-1} a^{-2} \sum_n a_n n(n+3)(r/a)^{n-2}$$
$$=-\frac{4}{5}S_1^2 a^{-2}\sum_n a_n\{5-(n+3)(r/a)^{n-2}\}$$
$$+\frac{432}{5}\{(T_3)^2+(T_4)^2\}a^{-2}\sum_n \frac{a_n}{n+5}(r/a)^2\{1-(r/a)^n\} \,. \tag{47}$$

Equating the corresponding terms of both the sides of (47), we can obtain the relations between the coefficients $a_0$, $a_1$, $a_2 \cdots$ . If we ignore the terms for $n>4$, the relations become

$$4\left(\rho D\sigma^{-1}+\frac{4}{5}S_1^2\right)a_1=0\ , \tag{48}$$

$$4\pi\rho D^2a_0-10\rho D\sigma^{-1}a^{-2}a_2+16S_1^2a^{-2}a_4=0\ , \tag{49}$$

$$4\pi\rho D^2a_1-18a^{-2}\left(\rho D\sigma^{-1}+\frac{4}{5}S_1^2\right)a_3=0\ , \tag{50}$$

$$\left\{4\pi\rho D^2-\frac{432}{35}\overline{(T_3)^2+(T_4)^2}a^{-2}\right\}a_2-a^{-2}\left\{28\rho D\sigma^{-1}+\frac{48}{5}\overline{(T_3)^2+(T_4)^2}+\frac{112}{5}S_1^2\right\}a_4=0\ . \tag{51}$$

It can be seen from (48) and (50) that $a_1=a_3=0$.

Further, the solution must satisfy the boundary condition that the magnetic field is continuous at $r=a$. The magnetic field outside the core is given as

$$\vec{h}=\begin{cases} 2a^2r^{-3}h_1P_1 \\ -a^2r^{-3}h_1\dfrac{dP_1}{d\theta} \\ 0 \end{cases} \tag{52}$$

which tends to zero at infinity. The continuity at $r=a$ gives

$$-s_{r=a}=a^{-1}h_1\ ,\qquad \left(r\frac{ds}{dr}+2s\right)_{r=a}=a^{-1}h_1\ ,$$

from which we have

$$\left(r\frac{ds}{dr}+3s\right)_{r=a}=0\ . \tag{53}$$

This in turn may be written as

$$3a_0+5a_2+7a_4=0\ . \tag{54}$$

In order to have a non-vanishing solution for $a_0$, $a_2$ and $a_4$ which satisfies the equations (49), (51) and (54), we have

$$\begin{vmatrix} 3 & 5 & 7 \\ 4\pi\rho D^2 & -10\rho D\sigma^{-1}a^{-2} & 16S_1^2a^{-2} \\ 0 & 4\pi\rho D^2-\dfrac{432}{35}\overline{(T_3)^2+(T_4)^2}a^{-2} & -a^{-2}\left\{28\rho D\sigma^{-1}+\dfrac{112}{5}S_1^2+\dfrac{48}{5}\overline{(T_3)^2+(T_4)^2}\right\} \end{vmatrix}=0. \tag{55}$$

If we assume that the substance in the core is perfectly conductive, (55) can be simplified as

$$D^4+4(\pi\rho)^{-1}a^{-2}\left(\frac{4}{7}S_1^2-T^2\right)D^2+\frac{1296}{245}(\pi\rho)^{-2}a^{-4}S_1^2T^2=0 \tag{56}$$

where

$$T^2=(T_3)^2+(T_4)^2 .$$

The assumption may be safely taken because the coefficients for $D^3$ and $D$ are very small provided we adopt $\sigma \simeq 10^{-6}$ *emu* which has been accepted from various reasons.

For the magnetic field to be oscillatory, $D^2$ must be negative. This condition is satisfied if we have

$$\left(\frac{4}{7}S_1^2-T^2\right)^2>\frac{324}{245}S_1^2T^2 . \tag{57}$$

On the contrary, we have exponentially increasing solutions provided $\left(\frac{4}{7}S_1^2-T^2\right)^2\leq\frac{324}{245}S_1^2T^2$. In this case, the small oscillations of the dynamo is regarded to be unstable because a slight deviation from the stationary state will increase exponentially.

Under the condition (57), we can expect simple harmonic oscillations whose periods are given as

$$2^{\frac{1}{2}}\pi^{\frac{3}{2}}\rho^{\frac{1}{2}}a\left\{\left(\frac{4}{7}S_1^2-T^2\right)\pm\sqrt{\left(\frac{4}{7}S_1^2-T^2\right)^2-\frac{324}{245}S_1^2T^2}\right\}^{-\frac{1}{2}} . \tag{58}$$

## 7. Discussions

Thus far we see that the A-approximation of the self-exciting dynamo seems to be stable under the condition (57), otherwise its small oscillations will be unstable. In order to maintain the earth's magnetic field over long periods, the dynamo mechanism should be highly stable. Although the condition (57) is derived from a crude calculation, we may assume that such a condition approximately holds in the earth's core. It should be noticed that (57) implies smaller $T_2^{2c}$ and $T_2^{2s}$ magnetic field than $S_1^0$ field. This result does not seem to agree with Bullard's view in which he considered that the $T_2^0$ field is quite large and $T_2^{2c}$ and $T_2^{2s}$ field would be of intermediate intensity. As for the dynamo itself, however, we may take a considerable variety of velocity distribution and magnetic field distribution. It would perhaps be possible to construct a dynamo under (57). To get (57), however, we made assumptions that the stationary magnetic fields are distributed in a simple way throughout the core and the effect of the magnetic field and velocity which are expressed with higher harmonics can be ignored. The influence of the earth's rotation is also negleted in the fundamental equation. So we do not say that the condition (57) is definitely right. It is a rough approximation for the stability condition.

It is likely, however, that $S_1^0$ magnetic field amounts to a few *gauss* at the surface of the core as calculated from the value at the earth's surface. Then $S_1$ becomes of the order of a few *gauss*. So the values in the bracket of (58) would be of the same order or a little smaller. Adopting $\rho=10\ gm/cm^3$ and $a=3.5\times10^8\ cm$, the periods amount to scores or hundreds of *years*. Therefore we can have magnetic oscillations with period of scores or hundreds of *years* which are superposed on the stationary fields of the dynamo. The secular variation in the earth's magnetic field might be interpreted in this way. The damping of these oscillations are not considered here. But it is likely that the relaxation time would be of the order of $10^4$–$10^5$ *years* provided the conductivity is of the order of $10^{-6}$ *emu*.

In the above calculations we did not take into account higher powers of $r^n$ than $n=4$. If we take them into account, we shall have many other solutions of more complicated radial distribution and of different periods. Since the distribution considered here is an approximation for the so-called ground tone of oscillation mode, the other oscillations may be considered to have shorter periods than that studied here. For shorter period oscillations the shielding due to the conducting mantle will become considerable.

The present study has nothing to do with the explanation of the reversal of the geomagnetic field which is supposed to occur during the geological time as suggested from the studies on some rocks' remanent magnetization. However, the stability condition might be broken provided something happens. For instance, the velocity distribution would change if we had changes in the rotation velocity of the earth—not necessarily the reversal of its rotation. In that case, we might have strong toroidal field which violates the stability condition. Once the stability breaks out, the field would change fairly rapidly until the system reaches another stable state. As the opposite-sign dynamo seems to be also capable of existence, we might get a reversal of the magnetic field. But we must pay attention to the fact that something must happen in this speculation. As long as the dynamo is considered stable, it is difficult to consider the earth's magnetic field to be oscillating with a period of $10^5$ *years* or so. We can only expect stable oscillations whose amplitude is smaller than the stationary field, the periods of these oscillations being presumably of the order of several hundred *years*.

## 8. Secular change in the $S_1^0$ field

As far as we assume the A-approximation of the self-exciting dynamo, the equation (45) is the one which governs changes in the $S_1^0$ magnetic field. As for the observational side of geomagnetism, we have a number of spherical harmonic analyses of the earth's magnetic field, the change in the $S_1^0$ field being traced through these analyses. The coefficients for the dipole field are reproduced in the following table.

Table I. The change in the $S_1^0$ field

| Source | Epoch | $g_1^0$ | $g_1^1$ | $h_1^1$ | $\sqrt{(g_1^0)^2+(g_1^1)^2+(h_1^1)^2}$ |
|---|---|---|---|---|---|
| Erman-Petersen | 1829 | 0.3201 | 0.0284 | −0.0601 | 0.3269 |
| Gauss | 1835 | 3235 | 311 | 625 | 3310 |
| Adams | 1845 | 3219 | 278 | 578 | 3282 |
| Adams | 1880 | 3168 | 243 | 603 | 3234 |
| Fritsche | 1885 | 3164 | 241 | 591 | 3228 |
| Schmidt | 1885 | 3168 | 222 | 595 | 3231 |
| Dyson-Furner | 1922 | 3095 | 226 | 592 | 3159 |
| Bartels | 1922 | 3090 | 227 | 586 | 3153 |
| Jones-Melotte | 1942 | 3039 | 218 | 555 | 3097 |
| Afanasieva | 1945 | 3032 | 229 | 590 | 3097 |
| Vestine-Lange | 1945 | 3057 | 211 | 581 | 3119 |

(Unit : *c.g.s.e.m.u.*)

The moment of the magnetic dipole may be used in expressing the intensity of the $S_1^0$ field. The moment is proportional to the root square sum of the coefficients of the first degree constituents, say $\sqrt{(g_1^0)^2+(g_1^1)^2+(h_1^1)^2}$, which is also given in Table I.

If we take the time origin at a certain epoch, say 1900, we can get the initial conditions for the change in the $S_1^0$ field. Starting from the initial conditions, we can obtain the change in the $S_1^0$ field by solving the equation (45). Although the stationary field $S_1$, $T_3$ and $T_4$ are not known, we may be able to choose the most probable values for these fields by comparing the calculated change with the observed one. This is what the writer would here like to do.

Here we again take the approximation

$$s=a_0+a_2(r/a)^2+a_4(r/a)^4$$

as was studied in the last section. We see that the time-change of the coefficients is given by an operational equation such as

$$\Phi(D)a_n=0 \qquad n=0,\ 2,\ 4$$

where $\Phi(D)$ denotes the operator expressed by the determinant in (55). So far as we consider the change to be within a period of some hundred *years*, we may safely assume that the electrical conductivity is infinite. In that case the intensity of the $S_1^0$ field at $r=a$ changes according to the following operational equation

$$\left\{D^4+4(\pi\rho)^{-1}a^{-2}\left(\frac{4}{7}S_1^2-T^2\right)D^2+\frac{1296}{245}(\pi\rho)^{-2}a^{-4}S_1^2T^2\right\}s=0$$

as was shown in the last section.

In order to solve this equation, we shall take the following initial condition

at $t=0$

$$s=s_0\,,\qquad \frac{ds}{dt}=-6.35\times10^{-12}emu\ sec^{-1} \tag{59a}$$

which is obtained from the change in the magnetic moment of the main dipole at 1900, $s$ denoting here $\sqrt{(g_1^0)^2+(g_1^1)^2+(h_1^1)^2}$ which is proportional to the intensity of the $S_1^0$ field, while $\frac{d^2s}{dt^2}$ and $\frac{d^3s}{dt^3}$ are to be ignored because the moment was changing almost linearly at this epoch.

Hence we have

at $t=0$

$$\frac{d^2s}{dt^2}=\frac{d^3s}{dt^3}=0\ . \tag{59b}$$

The solution which satisfies these initial conditions is given by interpreting the following operational equation

$$\left\{D^4+4(\pi\rho)^{-1}a^{-2}\left(\frac{4}{7}S_1^2-T^2\right)D^2+\frac{1296}{245}(\pi\rho)^{-2}a^{-4}S_1^2T^2\right\}s$$

$$=D^4S_0+D^3s_0'+4(\pi\rho)^{-1}a^{-2}\left(\frac{4}{7}S_1^2-T^2\right)(D^2s_0+Ds_0')\,, \tag{60}$$

where $s_0'$ stands for $\frac{ds}{dt}$ at $t=0$.

If we write

$$\left.\begin{matrix}\omega_1^2\\ \omega_2^2\end{matrix}\right\}=2(\pi\rho)^{-1}a^{-2}\left\{-\left(\frac{4}{7}S_1^2-T^2\right)\pm\sqrt{\left(\frac{4}{7}S_1^2-T^2\right)^2-\frac{324}{245}S_1^2T^2}\right\}\,,$$

(60) becomes

$$s=\frac{1}{\omega_1^2-\omega_2^2}\left\{(K+\omega_1^2)\frac{s_0D^2-s_0'D}{D^2-\omega_1^2}-(K+\omega_2^2)\frac{s_0D^2-s_0'D}{D^2-\omega_2^2}\right\} \tag{61}$$

where

$$K=4(\pi\rho)^{-1}a^{-2}\left(\frac{4}{7}S_1^2-T^2\right)$$

On interpreting (61) we have

$$s=\frac{1}{\omega_1^2-\omega_2^2}\left\{(K+\omega_1^2)\left(s_0\cosh\omega_1 t+\frac{s_0'}{\omega_1}\sinh\omega_1 t\right)\right.$$

$$\left.-(K+\omega_2^2)\left(s_0\cosh\omega_2 t+\frac{s_0'}{\omega_2}\sinh\omega_2 t\right)\right\}\quad\text{for}\quad \omega_1\neq\omega_2\ ,$$

$$s=s_0\cosh\omega t+\frac{s_0'}{\omega}\sinh\omega t+\frac{K+\omega^2}{2\omega^2}(s_0\omega\sinh\omega t+s_0'\cosh\omega t)t$$

$$\text{for}\quad \omega_1=\omega_2=\omega\ .$$

As already mentioned in the previous section, the solutions become infinite with $t\to\infty$ unless $\omega_1$ and $\omega_2$ are purely imaginary.

We see that the solutions contain an unknown constant $s_0$ which cannot be determined from the observed change in the $S_1^0$ field. However we may consider that $s_0$ is nearly naught provided we take the time origin at 1900. At this epoch the change is approximately symmetrical with respect to $t=0$, this condition being statisfied by taking $s_0=0$ in the above solutions which are composed of symmetrical and anti-symmetrical components. If this may be taken for granted, the change in the magnetic moment can be calculated for various combinations of $S_1$ and $T$. $S_1$ would be nearly 4 *gauss* as is estimated from the value at the earth's surface. Taking various values for $T$, we obtain

i) $T=0$ *gauss* $\quad s=-0.000635\times10^{-8}t$

ii) $T=0.5$ $\quad s=-0.0159\sin\dfrac{2\pi}{\Pi_1}t+0.000038\sin\dfrac{2\pi}{\Pi_2}t$

$(\Pi_1=492$ *years*, $\Pi_2=66.5$ *years*$)$ ,

iii) $T=1.0$ $\quad s=-0.00815\sin\dfrac{2\pi}{\Pi_1}t+0.000235\sin\dfrac{2\pi}{\Pi_2}t$

$(\Pi_1=233$ *years*, $\Pi_2=71.8$ *years*$)$ ,

iv) $T=1.5$ $\quad s=-0.00336\sin\dfrac{2\pi}{\Pi}t+0.000321\,t\cos\dfrac{2\pi}{\Pi}t$

$(\Pi=105$ *years*$)$ .

The cases for $T>1.5$ *gauss* are not taken into account because the oscillation is not stable as well as for $T=1.5$ *gauss*.

Fig. 2 shows the changes in the $S_1^0$ field as calculated for the respective values of $T$. Any $T$ less than 1 *gauss* seems to be acceptable judging from the values obtained from the spherical harmonic analyses which are also shown in the figure. If we can trace the change in the future, a more conclusive result will be obtainable. As was suggested by Bullard,[10] if the moment attained its minimum at about 1935 and has been recovering since then, we should be able to gain some clue for determining the magnitude of the stationary magnetic field in the earth's core sooner or later.

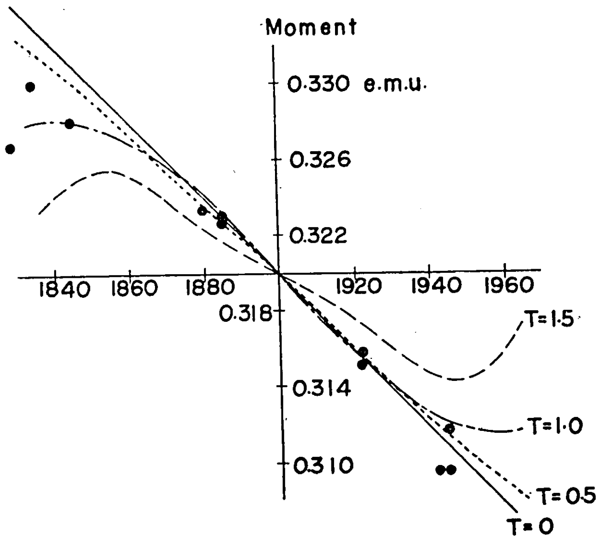


Fig. 2. Calculated and observed changes in the $S_1^0$ magnetic field.

## 9. Conclusion

The possible magneto-hydrodynamic oscillations in the earth's core are examined with their applications to the stability problem of the self-exciting dynamo and the prediction of the secular variation. In order to avoid mathematical difficulties, the writer was obliged to make some simplifications. Firstly, the earth's rotation is not taken into account. As is suggested by Bullard, the influence of Corioli's force might be appreciable. So the stability condition in this paper is not so reliable. Some further studies should be made upon this point as soon as possible. Secondly, the mutual coupling of various harmonic oscillations is ignored. In order to see to what extent one spherical harmonic component will be influenced by other components, a study on magneto-hydrodynamic oscillations of a spherical liquid body is now going on without ignoring high degree components. Although no definite result has been obtained yet, the oscillation is interpreted as

10) E. C. BULLARD, *Journ. Geophys. Res.*, **58** (1952), 277.

a coupled-oscillation of every harmonic constituent. Until we finish this sort of study the exact extent of the influence of higher degree components will not be clear though it is presumed to be not so great as is supposed from physical considerations.

In spite of those difficulties, investigations in this line would be of some use in the study of the stability problem. As is also shown in the last section, the fundamental equation for geomagnetic secular variation can be obtained provided we had sufficient knowledge on the stationary magnetic field in the core. The solutions of this equation will in turn be used for determining the internal field by comparing them with the observed secular variations. This would also be the only way to get at the fluid motion which is supposed to exist in the core. In conclusion, it may be said that a sort of magneto-hydrodynamic oscillation possibly exists in the earth's core. The oscillation seems to be closely connected with the secular variations. The examination of the nature of the oscillation is quite important for the discussion of the stability of the self-exciting dynamo.

The oscillations treated here, however, are of small amplitude. If we consider a disturbance of finite amplitude, there might be stable oscillations even though they violate the stability condition discussed here. We understand that the system of the fundamental equations, as given by (II), is not held for disturbances of finite amplitude. In that case we have to solve a system of non-linear differential equations as is done in Bullard's study on a homopolar dynamo. It should be added here that Bullard is going to mention in his unpublished study that there would be cases in which the oscillations will not depart indefinitely from their steady state even if the small oscillations are unstable.

The writer had the pleasure to talk with Sir Edward Bullard about the magneto-hydrodynamic problems while the writer was in England. The writer is very grateful to him for his helpful suggestions. The writer was encouraged by Professor A. T. Price to whom the writer's hearty thanks are also due. Professor T. Nagata and Dr. H. Takeuchi were also interested in the present study. The writer wishes to thank them for their discussions.

# 1. 地球核内の電磁流体振動

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Elsasser, Bullard, 竹内等によつて研究されたように，地球核内のダイナモ作用は，現在考え得られる地磁気原因としてもつとも有力である．地球主磁場がダイナモ作用によつて維持されているという事実は，数学的表現を用いる時には，核内に於て電磁場および流体の基礎方程式が定常な解を有するということに対応する．この定常解の存在は高次の流体運動および電磁場を省略する限り確立された．

本論文に於ては定常解の存在を認めた上でそれに重なつて起り得ると考えられる電磁流体振動 (Magneto-hydrodynamic oscillation) を考察し，その地球物理学的意味を論じる．今定常解よりの小偏差を考える時は，基礎方程式を分離して，定常なものと小偏差に対応するものとの二組にすることが出来る．高次項を省略すれば小偏差に対する方程式は線形に保たれる．この方程式は適当な手続きをほどこすことにより，例えば磁場のみに関するものに帰着さすことが出来るのであるが，球函数表示をもちいる時には，あらゆる次数の振動様式の連成振動となる．

これはある型の磁場内のある型の流体運動を考える時，多くの型の電流を発生し，その電流と磁場との相互作用はさらに多くの型の流体運動をひきおこすからである．これを厳密に取扱うことは困難であるが，ここでは特に双極子磁場のみに注目して，他の型の磁場をもたらすような相互作用はすべて省略してある．

定常解として竹内一島津によつて A 近似と称されたもつとも簡単なものを採用して議論を進めた結果，小偏差を支配する積分微分方程式を求めることが出来た．定常磁場の半径方向の分布を適当に与えて，この方程式の基本振動的解を近似的に求めたところ，定常 toroidal 磁場 $T_2^{2c}$, $T_2^{2s}$ があまり大きくない時には，振動性解を得た．さもない時には時間とともに極めて大きくなる解が得られ，この場合ダイナモは安定であるといえない．この toroidal 磁場の臨界値は 1.5 ガウスぐらいであり，Bullard の予想するような大きな toroidal 磁場は期待出来ないことになるが，Bullard が指摘したようにこの計算には地球廻転の影響が考慮されておらず，各種振動の相互依存性も無視されているから決定的結果であるとはいえない．しかし適当な条件のもとで核内に或種の電磁流体振動が起り得ることがあきらかになつたわけである．

また上に得た積分微分方程式に基ずいて，初期条件を与える時には，地磁気変化の予想をすることが出来る．従来の球函数分析結果をもととして 1900 年を $t=0$ とし，定常磁場をパラメーターとして，地磁気変化を計算してみたが，観測期間が短いために決定的なことはいえないけれども，上に求めた toroidal 磁場の臨界値以下なればもつともらしい様子を示すことが見られる．このような方法によつて観測の集積を待つて，内部定常磁場の大きさを推定することが可能のように思われる．ここに取扱つたことは，従来ダイナモ理論によつて地磁気永年変化が説明されるとしていた議論を定量化したものであり，今後さらに議論を精密に行うことにより核内の流体運動を永年変化からやや具体的に推察する手がかりとすることも可能であろう．