5. *Elasticity of Marble with Special Reference to its Elastic Aeolotropy.*

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Introduction.

It is a well known fact that rock apparently of the same kind has considerably different velocities of propagation of elastic waves at the laboratory measurements. For example, the velocities of longitudinal waves in hard limestone are scattered within the range of 2.8 km/sec and 6.4 km/sec[1]. Rocks of other kinds also show the scattering of velocity within a considerable range. Several possible causes may be enumerated for this experimental facts. Among them, elastic aeolotropy is perhaps the most important one. If a rock specimen is aeolotropic, the velocity of elastic waves will naturally change according to the direction of propagation, and hence the fluctuation in the observed values of elastic wave velocity of the rock will occur, if the direction is not specified.

In this paper, the velocities of dilatational and rotational waves for three directions, perpendicular to each other, of cubic samples of the marble are compared and the aeolotropy is discussed taking into consideration the petrofabric analysis of the samples. Theoretical treatment on the elasticity of marble is also described.

Experimental Procedure.

Ultrasonic pulse apparatus were used for the determination of the velocities of elastic waves in the specimen. The equipment has been the same with that described in a previous paper[2] except for some little modifications. A block diagram of the apparatus is presented in Fig. 1. Trigger pulses are delayed about 10 microseconds by a variable delay line, then amplified by the pulse amplifier, and are impressed on the driving quartz crystal. The signal, transmitted through the specimen and received by a similar crystal, is amplified and displayed on the

1) F. BIRCH, *Handbook of Physical Constants*, 1942.
2) D. SHIMOZURU, *Bull. Earthq. Res. Inst.*, **32**, (1954), 271.

screen of a cathode ray oscilloscope. The sweep of oscilloscope, accurately synchronized with the trigger pulse, was adjusted to have a 100 microsecond duration. Time marks are displayed at every 2.5 microseconds.

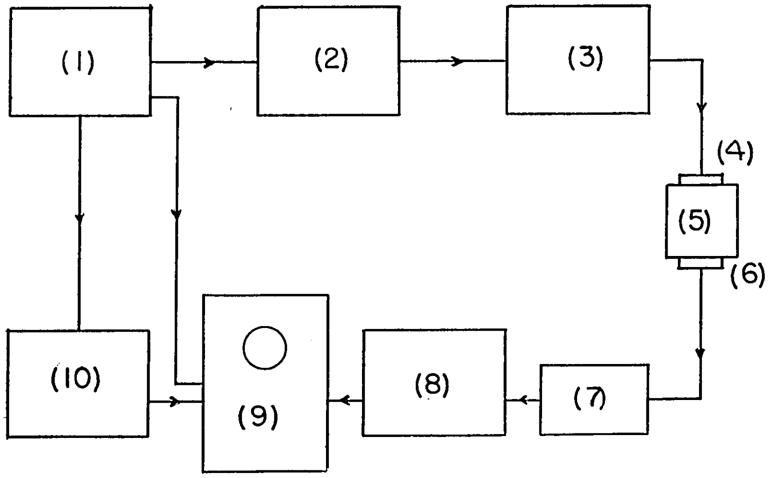

Fig. 1. Block diagram of the apparatus. (1) Main trigger pulse, (2) Variable delay line, (3) Pulse amplifier, (4) Driving crystal, (5) Secimen, (6) Receiving crystal, (7) Preamplifier, (8) Amplifier, (6) Cathode ray oscilloscope, (10) Time marker.

Each tested sample is cut into a cube of about 5 cm in one edge length. Velocities were measured along the three rectangular edges of each cubic specimen. These samples of marble were placed at the author's disposal by the kindness of Mr. T. Yabashi. These sapmles are tabulated in Table I.

Table I.

Specimen No.	Description	Origin	Commercial Name
I.	Marble	Yamaguchi Pref.	Usukumo-shirate
II.	〃	〃	Kozakura
III.	〃	〃	Mizutani
IV.	〃	Kochi Pref.	Akebono
V.	Limestone	Gifu Pref.	
VI.	〃	〃	

Experimental Results.

For determining the velocities of dilatational and rotational elastic waves, both of the X-cut and Y-cat quartz crystals were used. Table II gives the respective values of the dilatational and rotational waves. Each specimen has six values corresponding to the combination of three directions and two sorts of elastic waves. The velocities given in the same row of the table correspond to the velocities for the same directions.

Table II.

Specimen No.	$V_p(m)$	$V_s(m)$
I	5530	3150
	5550	3200
	5390	3100
II	5450	3070
	5410	3050
	5350	3010
III	5070	2770
	5320	2810
	5390	2790
IV	5490	3250
	5470	3210
	5440	3180
V	5730	3270
	5730	3240
	5720	3250
VI	5870	3190
	5850	3200
	5860	3200

Theoretical Calculation of the Elasticity of Marble.

We will calculate the theoretically expectable value of the elasticity of marble on the assumption that marble is polycrystalline aggregate of calcite only and that the grain boundary of each crystal does not have any effect on the elasticity. Calcite is a trigonal group of the hexagonal system. So taking the axis of symmetry as z-axis, we find the strain energy function to be

$$2W=\frac{1}{2}C_{11}e_{xx}^2+C_{12}e_{xx}e_{yy}+C_{13}e_{xx}e_{zz}+C_{14}e_{xx}e_{yz}$$

$$+\frac{1}{2}C_{11}e_{yy}^2+C_{13}e_{yy}e_{zz}-C_{14}e_{yy}e_{yz}$$

$$+\frac{1}{2}C_{33}e_{zz}^2+\frac{1}{2}C_{44}e_{yz}^2+\frac{1}{2}e_{zx}^2+\frac{1}{4}(C_{11}-C_{12})e_{xy}^2\ .$$

The elastic moduli of a number of minerals have been determined by W. Voigt by the twisting and bending experiments of crystal rods. According to him[3], the elastic moduli of calcite are

$$S_{11}=11.141\times10^{-10}\,,\qquad S_{33}=17.131\times10^{-10}\,,\qquad S_{44}=39.521\times10^{-10}\,,$$

$$S_{12}=-3.671\times10^{-10}\,,\qquad S_{13}=-4.241\times10^{-10}\,,\qquad S_{14}=8.98\times10^{-10}\,.$$

These moduli are expressed in units of gram weight per square centimeter. Elastic constants and elastic moduli are connected with the following relations in the case of trigonal group.

3) W. VOIGT, *Abhandlungen über Elasticität.*
〃 , *Lehrbuch der Kristallphysik.* Leipzig, 1928.

$$C_{11}+C_{12}=\frac{S_{33}}{S}\,,\qquad C_{11}-C_{12}=\frac{S_{44}}{S'}\,,\qquad C_{13}=\frac{-S_{13}}{S'}\,,\qquad C_{14}=\frac{-S_{14}}{S'}\,,$$

$$C_{33}=\frac{S_{11}+S_{12}}{S}\,,\qquad C_{44}=\frac{S_{11}-S_{12}}{S'}\,;$$

where $$S=S_{33}(S_{11}+S_{12})-2S_{13}^2\,,$$
and $$S'=S_{44}(S_{11}-S_{12})-2S_{14}^2\,.$$

Therefore, the six elastic constants of calcite come to be

$C_{11}=13.97\times10^8$ gr., $C_{33}=8.12\times10^8$ gr., $C_{44}=3.49\times10^8$ gr.,
$C_{12}=4.65\times10^8$ gr., $C_{13}=4.60\times10^8$ gr., $C_{14}=-2.12\times10^8$ gr..

These elastic constants are isothermal values since they have been determined from the bending and twisting experiments of statical method of the crystal rods.

We must therefore derive the adiabatic elastic constants as we aim at the comparison of the theoretical value with our experimentally obtained results.

In general, the relation between the adiabatic elastic constants and the isothermal one is expressed as follows;

$$\hat{C}_{hk}=C_{hk}+\frac{\theta q_h q_k}{\rho J C_v}\,,\qquad h=1,\ 2,\ 3, \tag{1}$$

where, $\hat{C}_{hk}$ ...... adiabatic elastic constant,
C_{hk} ...... isothermal elastic constant,
θ ...... absolute temperature,
q_k ...... $\sum_h a_h C_{hk}$,
a_{kh} ...... coefficient of linear thermal expansion,
ρ ...... density,
J ...... mechanical equivalent of heat,
C_v ...... specific heat of constant volume.

In the case of hexagonal system, the following relations hold,

$$\left.\begin{aligned} q_1=q_2&=a_1(C_{11}+C_{12})+a_3C_{13}\\ q_3&=2a_1C_{13}+a_3C_{33}\end{aligned}\right\}, \tag{2}$$

and

$$\left.\begin{aligned} &\hat{C}_{11}-C_{11}=\hat{C}_{12}-C_{12}=\frac{\theta q_1^2}{\rho JC_v}\,,\quad \hat{C}_{44}-C_{44}=0\,,\quad \hat{C}_{13}-C_{13}=\frac{\theta q_1q_3}{\rho JC_v}\,,\\ &\hat{C}_{33}-C_{33}=\frac{\theta q_3^2}{\rho JC_v}\,,\qquad \hat{C}_{14}-C_{14}=0\,.\end{aligned}\right\} \tag{3}$$

In the present case, the following values are assumed,

$$\rho=2.715\ \mathrm{gr/cm^3}\,,\qquad \theta=293^\circ\,,\qquad a_1=-5.65\times10^{-6}\,,$$
$$a_3=25.26\times10^{-6}\,,\qquad J=4.19\times10^7\ \mathrm{erg/cal}\,,\qquad C_p=0.175\,.$$

As the difference between the values of C_v and C_p is small, the value of C_p is used instead of C_v. Inserting the above quantities into equations (2) and (3), we obtain the adiabatic elastic constants of calcite as follows:

$$\left.\begin{aligned}&\hat{C}_{11}=13.69\times10^{11}\,,\quad \hat{C}_{33}=7.99\times10^{11},\quad \hat{C}_{44}=3.42\times10^{11}\,,\\ &\hat{C}_{12}=4.56\times10^{11}\,,\quad \hat{C}_{13}=4.51\times10^{11},\quad \hat{C}_{14}=-2.08\times10^{11}\,.\end{aligned}\right\}\qquad(4)$$

These constants are expressed in units of dynes per square centimetre.

Next, we consider the longitudinal and transversal elastic constants of calcite for any direction. Taking $l_1, m_1, n_1, \ldots, n_3$ to be the direction consines between the arbitrary directed coordinate axes $x'y'z'$ and the x-y-z-axis fixed to the crystal, as given below, the longitudinal elastic constant in the direction of x' which corresponds to $(\lambda+2\mu)$ for an isotropic medium comes to be

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3 ,

$$\begin{aligned}\tilde{C}_{11}=\hat{C}_{11}(l_1^2+m_1^2)^2+\hat{C}_{33}n_1^4+2(\hat{C}_{13}+2\hat{C}_{44})n_1^2(l_1^2+m_1^2)\\ +4C_{14}m_1n_1(3l_1^2-m_1^2)\,.\end{aligned}\qquad(5)$$

Transversal elastic constant, wich corresponds to μ for an isotropic medium, is expressed as follows,

$$\begin{aligned}\tilde{C}_{44}=\hat{C}_{11}\left[(l_1l_2+m_1m_2)^2+\left(\frac{l_1m_2-l_2m_1}{2}\right)^2\right]-\frac{\hat{C}_{12}}{2}(l_1m_2-m_1l_2)^2\\ +2\hat{C}_{13}n_1n_2(l_1l_2+m_1m_2)+\hat{C}_{33}n_1^2n_2^2+\hat{C}_{44}[(m_1n_2+n_1m_2)^2\\ +(n_1l_2+l_1n_2)^2]-4C_{14}m_1n_1(3l_1l_2-m_1m_2)\,.\end{aligned}\qquad(6)$$

Equations (5) and (6) give the apparent elastic constants of marble for any orientation if all composing crystals are pointed to one and the same direction. When the crystal orientation of calcite is equally probable for all directions, the elastic constants of marble will be expressed taking a space mean of equations (5) and (6) for all directions. Taking the new axis as shown in Fig. (2), direction cosines come to be

$$\left\{\begin{aligned}l_1&=\sin\theta\cdot\cos\varphi\\ m_1&=\sin\theta\cdot\sin\varphi\\ n_1&=\cos\theta\end{aligned}\right.\quad.$$

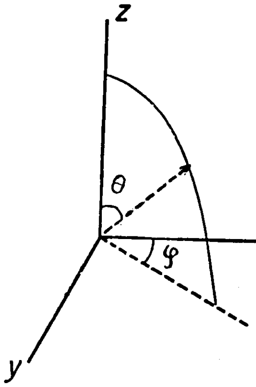

Fig. 2.

Therefore, from equation (5), the mean longitudinal elastic constant, $\overline{C}_{11}$, will be

$$\overline{C}_{11}=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\tilde{C}_{11}\sin\theta\cdot d\theta d\varphi$$
$$=\frac{1}{15}\left\{8\hat{C}_{11}+3\hat{C}_{33}+4(\hat{C}_{13}+2\hat{C}_{44})+\frac{1}{2}C_{14}\right\}. \qquad (7)$$

As for the calculation of the mean transversal elastic constant, sets of direction cosines are expressed as follows, taking the parameters θ, φ as shown in Fig. 3.

$$\begin{cases} l_1=\cos\theta\cdot\cos\varphi, & m_1=\cos\theta\cdot\sin\varphi, & n_1=-\sin\theta, \\ l_2=-\sin\varphi, & m_2=\cos\varphi, & n_2=0, \\ l_3=\cos\varphi\sin\theta, & m_3=\sin\varphi\sin\theta, & n_3=\mathrm{con}\,\theta. \end{cases}$$

Inserting these direction cosines into equation (6), the mean transversal elastic constant $\overline{C}_{44}$ is calculated similarly as follows,

$$\overline{C}_{44}=\frac{1}{6}\hat{C}_{11}-\frac{1}{6}\hat{C}_{12}+\frac{2}{3}\hat{C}_{44}-\frac{1}{6}\hat{C}_{14}. \qquad (8)$$

From equations (7) and (8), the apparent velocities of elastic waves is found to be

$$\bar{v}_p=(\overline{C}_{11}/\rho)^{\frac{1}{2}}=6609 \text{ m/sec},$$

$$\bar{v}_s=(\overline{C}_{44}/\rho)^{\frac{1}{2}}=3904 \text{ m/sec}.$$

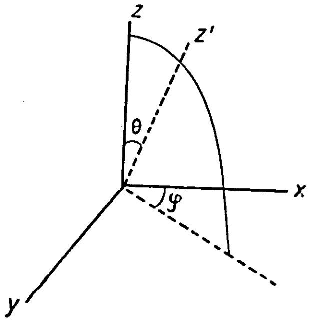

Fig. 3.

where, $\bar{v}_p$, $\bar{v}_s$ are the velocities of dilatational and rotational elastic waves respectively. This is the case of macroscopically isotropic marble in which the crystal orientation is equally probable for all directions.

Discussion of the Results.

Velocities of elastic waves may be equal for all directions if the specimen is isotropic, whereas specimens I—IV show more or less aeolotropic character where the velocity of dilatational wave is scattered within significant ranges. For specimen III, the velocity is varied by such a large per cent as 6% according to the propagational direction of the wave. In order to make clear the aeolotropic character of the specimen, the fabric analysis was made by one of the authors (I. Murai). Fig. 4 shows an example of the fabric analysis for the specimen III. Measurements were made on 211 grains in a section.

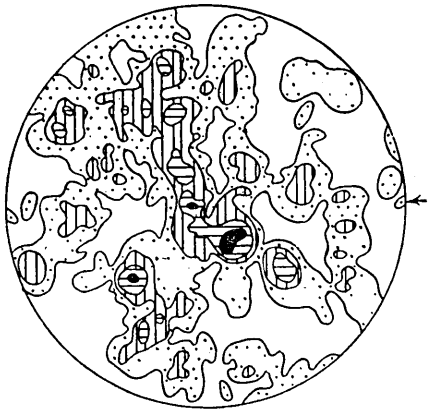

Fig. 4, Orientation diagram for c axes of calcite in specimen III. 211 grains per diagram. Contours, 0~1, 1~2, 2~4, 4~5%, per 1% area.

In the figure, we can see C axes of calcite is oriented somewhat predominantly to the direction almost normal to the plane of the section. This cause presumably gives rise to the said difference in velocities of elastic waves according to the propagational direction. In other words, the velocity difference can reach almost several percent according to the propagation direction, notwithstanding the slight aeolotropic behavior as has been analyzed from petrofablics. Specimens V, VI, the so-called limestone, do not show any elastic aeolotropy since the velocities for the three directions are almost equal. From the mathematical treatment given in the preceding paragraph, the velocity of dilatational and rotational waves for isotropic marble are expected to be 6609 m/sec and 3904 m/sec respectively. But, in the limestone, in which the crystal orientation is considered to be quite at random two velocities were measured and found by us to be 5860 m/sec and 3200 m/sec respectively. These discrepancies between the theory and the experiment may be considered to be the consequence of the effect of void space between each crystal boundary.

Acknowledgment.

We wish to express our sincere thanks to Prof. R. Takahasi who has kindly offered his some criticisms on the present study, with his deep interest in our work. Our thanks are also due to the Grant in Aid for the Miscellaneous Scientific Researches of the Department of Education for the expense of the present work.

5. 大理石の弾性，特にその異方性に就いて

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村 井 勇

我々が実験室で岩石中を伝はる弾性波の速度を測定すると，所謂，花崗岩とか石灰岩とかの同じ種類の岩石でも速度は非常にばらつく事を経験する．この原因として，色々あると思はれるが，此の論文では，弾性の異方性の問題を取りあげた．最も取り扱いやすい試料として，大理石を選び，一辺の長さがほぼ 5 cm の長さの立方体に作り，三方向の縦波及び横波の速度を X-cut, Y-cut の水晶振動子を用いて測定した．その結果に依ると，一つの試料でも方向に依つて，少しづつ速度が違うことがわかつた．その差は数%に及ぶ．この原因は結局，大理石を構成している方解石の結晶の軸があらゆる方向に等しく向いていないからである．その有様は Fabric analysis に依つて示されてある．又，方解石の単結晶の弾性率から，その集合体としての大理石の弾性常数を，2 つの場合に就いて計算して，実験の結果と比較して見た．即ち，一つは，一斉にある方向を向いている場合の任意の方向の弾性常数と，他は，方解石があらゆる方向に等しく向いている場合の見かけ上の平均の弾性常数とである．2 つの石灰岩の試料については，三方向の速度はほぼ等しく，一応巨視的に等方性であるといへるが，その場合，計算値は実測値より，かなり大きな値を示し，これは結局，実際の大理石では結晶個体の同志間の Void space がある為めに，弾性率が小さく出るのであらうと思はれる．