2. *Study on Surface Waves X.* *Equivalency of SH-waves and Sound Waves in a Liquid.*

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(Read Dec. 16, 1952.—Received Jan. 4, 1954.)

1. Introduction.

The existence of two sorts of elastic waves in a homogeneous isotropic solid body has been a well-known fact ever since the late 19th century[1]. They are named as P- and S-waves, and their nature was investigated by many authors. It was found out By K. Sezawa[2] that the S-waves can be separated into two kinds, and in a two-dimensional problem with one or several parallel plane boundaries, it is classified in SV- and SH-waves. Although the former alone cannot satisfy boundary conditions, the latter can exist by itself. If we denote the displacement of SH-waves by v, it is the solution of a differential equation of the type

$$\frac{\partial^2 v}{\partial t^2}=V_S^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial z^2}\right), \qquad \ldots\ldots(1.1)$$

where

$V_S=\sqrt{(\mu/\rho)}$ is the propagation velocity of S-waves,
x, y, z are the Cartesian coordinates,
v is the displacement in y-direction.

On the other hand, the sound waves in an elastic liquid also satisfies the equation

$$\frac{\partial^2 \Delta}{\partial t^2}=V^{*2}\left(\frac{\partial^2 \Delta^*}{\partial x^2}+\frac{\partial^2 \Delta^*}{\partial z^2}\right) \quad ..(1.2)$$

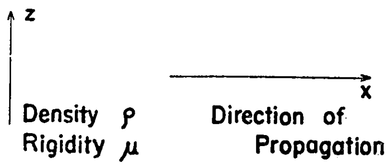

Fig. 1.

1) A historical sketch of our knowledge in this field of seismology is admirably given in H. KAWASUMI's elaborate review entitled "An Historical Sketch of the Development of Knowledge concerning the Initial Motion of an Earthquake," *Publication du Bureau central seismologique international, serie A: Travaux scientifiques, fascicule* **15** (1937).

2) K. SEZAWA, "Dilatational and Distorsional Waves generated from a Cylindrical or a Spherical Origin," *Bull. Earthq. Res. Inst.*, **2** (1927), 13. *Sindôron*, p. 652 (in Japanese.)

where $V^*=\sqrt{(k^*/\rho^*)}$ is the propagation velocity of sound waves, and Δ^* is the volume dilatation.

The two equations (1.1) and (1.2) have the same form, while the SH-waves in a layered medium forms the surface waves now called as Love-waves. Thus a doubt naturally occurs whether sound waves in a liquid might form a similar or same type of surface waves with Love-waves[3)].

In this paper, however, we will try a more general way of treatment and examine the equivalency of surface waves constituted by SH-waves and sound waves in an elastic liquid.

2. Boundary conditions.

Boundary conditions required at the plane of contact of different media or at the free surface are;

SH-waves;
$$v_n=v_{n+1}\,, \qquad \ldots\ldots(2.1a)$$

$$\widehat{yz}_n=\widehat{yz}_{n+1}\,, \qquad \ldots\ldots(2.1b)$$

where v_n implies a displacement of y-component in the n-th medium and the latter condition is modified to the next form

$$\mu_n\frac{\partial v_n}{\partial z}=\mu_{n+1}\frac{\partial v_{n+1}}{\partial z}\,. \qquad \ldots\ldots(2.1b)'$$

Sound waves;
$$w_n^*=w_{n+1}^*\,, \qquad \ldots\ldots(2.2a)$$

$$\widehat{zz}_n^*=\widehat{zz}_{n+1}^*\,, \qquad \ldots\ldots(2.2b)$$

where w_n implies a displacement of z-component in the n-th medium. If the conditions (2.2a) and (2.2b) are expressed in terms of Δ^*, they are modified, using the relation

$$\rho_n^*\frac{\partial^2 w_n^*}{\partial t^2}=k_n^*\frac{\partial\Delta_n^*}{\partial z}\,,$$

$$\widehat{zz}_n=k_n^*\Delta_n^*\,,$$

(where k_n^* is a bulk modulus) as

3) Relation between the velocity equation of the gravity wave in the water and that of Love-waves was pointed out more than ten years ago. K. SEZAWA and K. KANAI, "On Shallow Water Waves Transmitted in the Direction Parallel to a Sea Coast, with Special Reference to Love-waves in a Heterogeneous Media," *Bull. Earthq. Res. Inst.*, **17** (1939), 685.

$$\begin{cases} -\frac{1}{p^2} V_n^{*2} \frac{\partial \Delta_n^*}{\partial z} = -\frac{1}{p^2} V_{n+1}^{*2} \frac{\partial \Delta_{n+1}^*}{\partial z} & \qquad \text{......(2.3a)} \\ k_n^* \Delta_n^* = k_{n+1}^* \Delta_{n+1}^* & \qquad \text{......(2.3b)} \end{cases}$$

where Δ_n^* is assumed to be proportional to $\exp(ipt)$.

If the solutions of the type in the elastic body;

$$v_n = \{A_n \exp(\beta_n z) + B_n \exp(-\beta_n z)\} \exp(ifx + ipt),$$

$$\beta_n = \sqrt{(f^2 - k_n^2)},$$

$$k_n = p/V_{Sn}, \qquad \text{......(2.4)}$$

and in the liquid;

$$\Delta_n^* = \{C_n \exp(\alpha_n^* z) + D_n \exp(-\alpha_n^* z)\} \exp(ifx + ipt),$$

$$\alpha_n^* = \sqrt{(f^2 - h_n^{*2})},$$

$$h_n^* = p/V_n^*, \qquad \text{......(2.5)}$$

are employed, the above conditions become

for SH-waves;

$$A_n \exp(\beta_n z_n) + B_n \exp(-\beta_n z_n)$$
$$= A_{n+1} \exp(\beta_{n+1} z_n) + B_{n+1} \exp(-\beta_{n+1} z_n), \qquad \text{......(2.6a)}$$

$$\mu_n \beta_n \{A_n \exp(\beta_n z_n) - B_n \exp(-\beta_n z_n)\}$$
$$= \mu_{n+1} \beta_{n+1} \{A_{n+1} \exp(\beta_{n+1} z_n) - B_{n+1} \exp(-\beta_{n+1} z_n)\},$$
$$\text{......(2.6b)}$$

for Sound waves;

$$-V_n^{*2} \alpha_n^* \{C_n \exp(\alpha_n^* z_n) - D_n \exp(-\alpha_n^* z_n)\}$$
$$= -V_{n+1}^{*2} \alpha_{n+1}^* \{C_{n+1} \exp(\alpha_{n+1}^* z_n) - D_{n+1} \exp(-\alpha_{n+1}^* z_n)\},$$
$$\text{......(2.7a)}$$

$$k_n^* \{C_n \exp(\alpha_n^* z_n) + D_n \exp(-\alpha_n^* z_n)\}$$
$$= k_{n+1}^* \{C_{n+1} \exp(\alpha_{n+1}^* z_n) + D_{n+1} \exp(-\alpha_{n+1}^* z_n)\},$$
$$\text{......(2.7b)}$$

where $z = z_n$ implies the boundary surface of two media "n" and "$n+1$".

If we put

$$k_n^* C_n \equiv C_n^* \quad \text{and} \quad k_n^* D_n \equiv D_n^*, \qquad \text{......(2.8)}$$

we obtain, from the expressions (2.7a) and (2.7b),

$$-\frac{1}{\rho_n^*} \alpha_n^* \{C_n^* \exp(\alpha_n^* z_n) - D_n^* \exp(-\alpha_n^* z_n)\}$$

$$= -\frac{1}{\rho_{n+1}^*}\alpha_{n+1}^*\{C_{n+1}^*\exp(\alpha_{n+1}^* z_n) - D_{n+1}^*\exp(-\alpha_{n+1}^* z_n)\}\,, \qquad \ldots\ldots(2.9a)$$

$$C_n{}^*\exp(\alpha_n{}^* z_n) + D_n{}^*\exp(-\alpha_n{}^* z_n) = C_{n+1}^*\exp(\alpha_{n+1}^* z_n) + D_{n+1}^*\exp(-\alpha_{n+1}^* z_n)\,, \qquad \ldots\ldots(2.9b)$$

Comparing the boundary conditions of SH-waves (2.6a) and (2.6b) with those of sound waves in liquid (2.9a) and (2.9b), we shall at once notice that (2.6a) and (2.9b) are quite similar in the from of expressions. For, if the frequeney, wave-length and the velocity of bodily waves are respectively equal in both problems,

$$\alpha_n{}^* = \beta_n\,. \qquad \ldots\ldots(2.10)$$

Thus the expression (2.9b) becomes

$$C_n{}^*\exp(\beta_n z_n) + D_n{}^*\exp(-\beta_n z_n) = C_{n+1}^*\exp(\beta_{n+1} z_n) + D_{n+1}^*\exp(-\beta_{n+1} z_n)\,, \qquad \ldots\ldots(2.11)$$

which is identical with (2.6a), only the constants A_n and B_n are replaced by $C_n{}^*$ and $D_n{}^*$ respectively.

On the other hand, introducing (2.10) into the expression (2.9a), we have at once

$$\frac{1}{\rho_n{}^*}\beta_n\{C_n{}^*\exp(\beta_n z_n) - D_n{}^*\exp(-\beta_n z_n)\} = \frac{1}{\rho_{n+1}^*}\beta_{n+1}\{C_{n+1}^*\exp(\beta_{n+1} z_n) - D_{n+1}^*\exp(-\beta_{n+1} z_n)\}\,, \qquad \ldots\ldots(2.12)$$

which becomes identical with (2.6b) if we may put

$$1/\rho_n{}^* = K\mu_n\,, \qquad \ldots\ldots(2.13)$$

using a common constant K independent upon n.

Summarizing the above discussion, we can conclude as follows:

"The elastic surface waves constituted by SH-waves and sound waves in a liquid are equivalent if

1) the frequency, wave-length and the velocity of bodily waves are respectively equal in both problems,

and 2) $\rho_n{}^*$ is inversely proportional to μ_n."

However, it must be added that the displacement in one problem corresponds to the stress in the other, hence the condition of the continuity of displacement of SH-waves corresponds to the continuity of stress of sound waves in a liquid, while the continuity of stress of the former corresponds to that of the displacement of the latter.

Therefore, if a free surface exists (in the case of an elastic plate two free surfaces exist), the two kinds of problems are not quite equivalent. For the condition of the free surface of SH-waves corresponds to that of the fixed surface of sound waves in a liquid, and *vice versa*. Thus the phenomenon of the reflection of SH-waves at the free boundary corresponds to that of the reflection of sound waves at a rigid wall. We will explain the details by employing some appropriate examples in the following sections.

The next table briefly shows the correspondency of the various elements in both problems.

Table I. Correspondency of the elements in the problems of SH-waves and sound waves in a liquid.

SH-waves.	Sound waves in a liquid.
H_n (Thickness of the layer.)	H_n (Thickness of the layer.)
$z=z_n$ (Boundary surface.)	$z=z_n$ (Boundary surface.)
V_{sn} (Velocity of S-waves.)	V_n^* (Velocity of sound waves.)
$k_n=p/V_{sn}$	$h_n^*=p/V_n^*$
$\beta_n=\sqrt{(f^2-k_n^2)}$	$\alpha_n^*=\sqrt{(f^2-h_n^{*2})}$
μ_n (Rigidity.)	$1/\rho_n^*$ (Density.)
χ $(=\mu'/\mu)$	$1/\Gamma^*$ $\left(=1\Big/\left(\frac{\rho^{*\prime}}{\rho^*}\right).\right)$
v_n (Horizontal displacement.)	$\widehat{zz}_n^*$ (Stress component.)
$\widehat{yz}_n$ (Stress component.)	$-w_n^*$ (Vertical displacement.)
Condition of free surface.	Condition of fixed surface.
Condition of fixed surface.	Condition of free surface.

3. Elastic liquid surface waves corresponding to Love-waves.

As is well-known, Love-waves are a very conspicuous type of surface waves constituted by SH-waves, and an example of the above theory may readily be given. We will obtain a corresponding one in the field of elastic liquid waves.

Let us suppose an elastic solid medium specified by Fig. 2, in which the Love-type of waves can naturally exist and propagate. If the condition

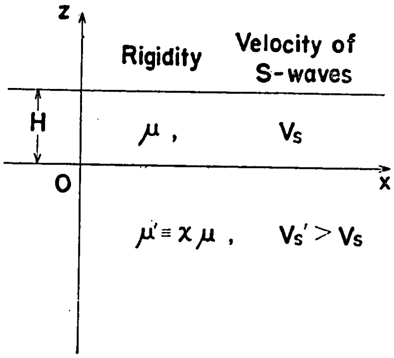

Fig. 2. Elastic media in which Love-waves can exist.

$$V_s < V_s' \qquad \ldots\ldots(3.1)$$

is satisfied, the distribution of amplitude of v and $\widehat{yz}$ becomes as illustrated in Figs. 3a and 3b.

The corresponding liquid medium is shown in Fig. 4. If the condition $\rho^{*\prime} = \rho^*/\chi$ holds, as is in the figure,

$$\mu : \mu' = \frac{1}{\rho^*} : \frac{1}{\rho^{*\prime}}, \qquad \ldots\ldots(3.2)$$

and the correspondency explained in the previous section comes into existence. In this case the distribution of the amplitude of displacement and stress becomes as are shown in the Figs. 5a and 5b. Fig. 3a corresponds to Fig. 5b and Fig. 3b to Fig. 5a.

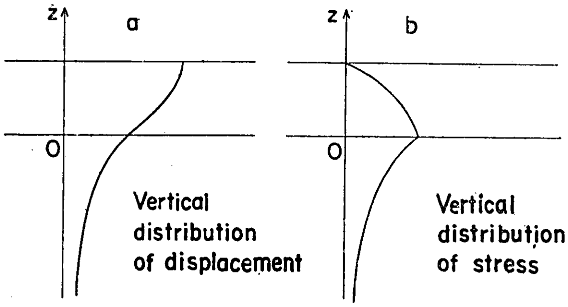

Fig. 3. Vertical distribution of displacement and stress in Love-waves. (not to scale)

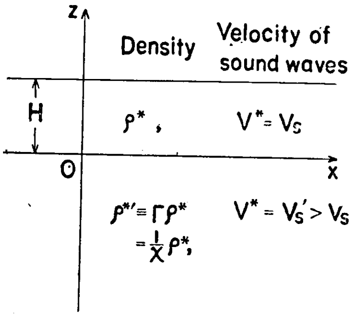

Fig. 4. Elastic liquid media in which surface waves of Love-type can exist. Correspondency of elements in this and Love-type of waves is shown in Table I.

Therefore, if we solve some problem of Love-waves, giving some material constants, the solution at once will be applicable to the problem of elastic surface waves in a layered liquid medium[4]. Since, in ordinary cases, the rigidity of the surface layer is smaller than that

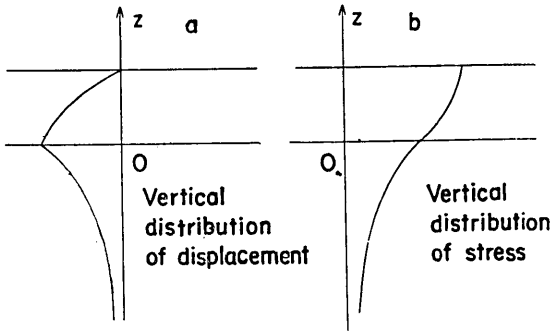

Fig. 5. Elastic liquid media in which surface waves of Love-type can exist. Correspondency of elements in this and Love-type of waves is shown in Table I. In this case upper surface is a fixed boundary. (not to scale)

of the substratum, we must naturally assume that the density of the upper liquid is larger than that of the lower. However, if we put the distribution of the substance upside down, the new arrangement of the medium has a rigid bottom (free boundary of Love-waves corresponds to the fixed boundary, cf. Table I.), and the heavy liquid layer lies between this rigid bottom and the other light liquid medium. This structure of medium is not unlike to that of oceans suggested by a number of authors.[5]

4. Love-waves with a fixed boundary. Elastic liquid surface waves with a free surface.

In the previous section we have treated the ordinary Love-waves and the corresponding ones existent in an elastic liquid medium. As a matter of course, ordinary Love-waves have a free surface, consequently the corresponding liquid waves assume a fixed boundary. On the other hand, in this section, we will solve the problem of Love-waves with a fixed boundary. Or one may suppose a layered elastic liquid medium with a free surface, which may be more natural than the above model.

Assuming (cf. Fig. 2)

$$v=\{A\exp(\beta z)+B\exp(-\beta z)\}\exp(ifx+ipt)$$

in the part of the surface layer, and

4) *cf.*, T. SAKAI, SINDÔ, **1** (1947), 124, (in Japanese.).
5) *e.g.*, M. EWING and F. PRESS, *Bull. Seism. Soc. Amer.*, **40** (1950), 271.

$$v'=A'\exp(\beta' z)\cdot\exp(ifx+ipt) \qquad \ldots\ldots(4.1)$$

in the subjacent medium, we put the boundary conditions

$$\text{at} \quad z=H\,; \quad v=0\,, \qquad \ldots\ldots(4.2)$$

$$\text{and at} \quad z=0\,; \quad v=v'\,,$$

$$\widehat{yz}=\widehat{yz}'\,, \qquad \ldots\ldots(4.3)$$

we thus obtain the expressions

$$\left\{\begin{array}{lll} A\exp(\beta H)+B\exp(-\beta H)=0\,, & & \\ A & +B & =A'\,, \\ A\mu\beta & -B\mu\beta & =A'\mu'\beta'\,, \end{array}\right. \qquad \ldots\ldots(4.4)$$

from which we have

$$\beta\cosh\beta H+\chi\beta'\sinh\beta H=0\,, \qquad \ldots\ldots(4.5)$$

and by somewhat modifying it

$$\tilde{\beta}H=\mathrm{Arctan}\left(\chi\frac{\beta'}{\tilde{\beta}}\right)+\frac{\pi}{2}\,. \qquad \ldots\ldots(4.6)$$

In the above expressions we have employed ordinary notations, viz.,

$$\left\{\begin{array}{ll} \beta=i\tilde{\beta} & =\sqrt{(f^2-k^2)}\,, \\ \beta' & =\sqrt{(f^2-k'^2)}\,, \\ k & =p/V_s\,, \\ k' & =p/V_s'\,, \\ \chi & =\mu'/\mu\,. \end{array}\right. \qquad \ldots\ldots(4.7)$$

(4.6) is a formula with which we can calculate the velocity of this case, while the corresponding velocity equation of liquid medium is at once obtained from Table I.

$$\tilde{\alpha}^*H=\mathrm{Arctan}\left(\frac{1}{\Gamma^*}\frac{\alpha^{*\prime}}{\tilde{\alpha}^*}\right)+\frac{\pi}{2}\,, \qquad \ldots\ldots(4.8)$$

where

$$\left\{\begin{array}{ll} \alpha^* =i\tilde{\alpha}^* & =\sqrt{(f^2-h^{*2})}\,, \\ \alpha^{*\prime} & =\sqrt{(f^2-h^{*\prime 2})}\,, \\ h^* & =p/V^*\,, \\ h^{*\prime} & =p/V^{*\prime}\,, \\ \Gamma^* & =\rho^{*\prime}/\rho^*\,. \end{array}\right. \qquad \ldots\ldots(4.9)$$

6) *e.g.*, Y. SATÔ, *Bull. Earthq. Res. Inst.*, **29** (1951), 1.

When remember the characteristic equation of Love-waves[6], the phase velocity of the case without node in the layer is expressed by

$$\tilde{\beta}H = \mathrm{Arctan}\left(\chi\frac{\beta'}{\tilde{\beta}}\right), \qquad \ldots\ldots(4.10)$$

while that with a node is

$$\tilde{\beta}H = \mathrm{Arctan}\left(\chi\frac{\beta'}{\tilde{\beta}}\right) + \pi . \qquad \ldots\ldots(4.11)$$

(4.6) is just the arithmetic mean of the above two expressions, and may be described as the Love-waves with half a node. This is a somewhat interesting property for us.

5. Conclusion.

Correspondency of the surface waves formed by SH-waves propagated in an elastic solid medium and that in an elastic liquid medium was discussed and the following results were obtained.

If some kind of surface waves can exist in an elastic solid medium, then also the corresponding type of waves can exist always in an elastic liquid medium. In this case, the correspondency of the elements in both problems becomes as shown in Table I. The remarkable fact is that the condition of a free surface in one problem corresponds to that of a fixed surface in the other problem, and the displacement corresponds to the stress compoment. This condition was explained in § 2 and in the later sections some examples were given.

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2. 表面波の研究 X.
流体中の音波と SH−波のつくる表面波の対応性

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流体中の音波が満たす運動方程式と SH− 波のそれとは全く同じ形を持つてゐる．従つて，それらが作る表面波の間にも或る対応が存在し，一方の解は又他方の問題に於いても意味を持つた内容を表現する事が期待される (§ 1)．本文においては，この点を明らかにする事を目標とし，あはせて一二の例をも示した．

基礎をなす波動方程式が同形であることは上述の通りであるが，境界条件について見る時，両者は完全に同形とはならない．即ち一方の問題で変位について与へられる条件は，他方の問題では応力

に課せられる条件と同じであるし，又その逆の関係も成立つて居る．從つて一方の問題について得られた解を他方にあてはめようとする時には，一方の変位は他方の応力を，応力は逆に変位を表現して居る事になる．從つて又，一方の問題の自由表面は他方の固定表面と対応する事になる．この間の事情を第1表に要約して示してある（§ 2).

上に述べた如くであるから，SH-波の作る代表的な表面波である Love 波を表はす解を求めれば，これは，その各種の要素が表はす内容に関して，第1表に示したやうな適当な解釈を与へるならば，流体中の音波が作る表面波をも表現する事になる．それは，剛体の底の上に密度が大きく音波速度のおそい流体の層が横たはり，その上に十分に深い流体が存在する時に，主として層内を伝はつてゆく所の表面波である（§ 3).

一方では又，表面が自由との条件にかへて，表面で変位 0 との条件を課したところの，Love 波類似の表面波も存在するわけであるが，これは，流体中の波におきかへて考へるならば，表面が自由な一つの流体層があり，その下に音波速度のはやい流体が存在する場合の問題に相当する事になる．その速度方程式を求めてみると，層内に一つの節を持つた Love 波と，節を持たない Love 波の丁度中間の性質をもつたものである事がわかる．