25. *Transmission and Reflection of Seismic Waves through Multilayered Elastic Medium.*

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1. Introduction.

The problem of reflection and refraction of elastic wave has been studied by many authorities[1][2][3][4][5][6][7], but owing to the complicated nature of elastic wave which produces four secondary waves when primary wave meets a discontinuous plane (with the exeception of SH type wave), their studies had to be limited only to the case of one or two boundary planes.

When there exist n parallel discontinuities in elastic medium and the incident wave is the dilatational one, the calculation of the amplitude of transmitted waves as well as reflected waves is reduced to the problem of the simultaneous linear equation of $4n$-th order, and though it is a simple problem in principle, it is extremely subtle in actual treatment. However, in spite of this difficulty which has prevented further study of the problem, we must, in view of the importance of the problem, investigate how the multilayered medium affects the transmission and reflection of elastic wave.

Recently useful studies have been perfomed by Thomson[8] and Torikai[9]

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1) C. G. KNOTT, "Reflexion and Refraction of Elastic Waves, with Seismological Applications", *Phil. Mag.*, **48** (1899), 64.

2) G. W. WALKER, "Surface Reflection of Earthquake Waves", *Phil. Trans.*, *A* **218** (1819), 373.

3) H. JEFFREYS, "The Reflexion and Refraction of Elastic Waves", *M. N. R. A. S. Geophys. Suppl.*, **1** (1926), 321–334.

4) K. SEZAWA, "Possibility of the Free-oscillations of the Surface-layer Excited by the Seismic Waves", *Bull. Earthq. Res. Inst.*, **8** (1930), 1

5) T. MATUZAWA, "An Example of Surface Reflection of Elastic Plane Waves", *Zisin*, **4** (1932), 7. (in Japanese)

6) K. KAWASUMI and T. SUZUKI, "Reflexion and Refraction of Seismic Waves at Plane Interface of the Earth's Surface Crust", Zisin **4** (1932), 277. (in Japanese)

7) K. SEZAWA and K. KANAI, "Possibility of Free-oscillations of Strata Excited by Seismic Waves (Part 4)", *Bull. Earthq. Res. Inst.*, **10** (1932), 273.

8) W. T. THOMSON, "Transmission of Elastic Waves through a Stratified Solid", *J. Appl. Phys.*, **21** (1950), 89.

9) Y. TORIKAI, "Transmission of Acoustic Waves through a Stratified Solid Medium", *J. Acoust. Soc. Japan.*, **8** (1952), 21.

concerning the transmission of super-sonic wave through the multilayered elastic plate. They both utilize the matrix calculation and express the solution of that problem in a very compact form.

Following their mothod in this paper, we employ the boundary conditions applicable to the seismic wave, and obtain the amplitude of transmitted and reflected waves in the case of the incidence of the dilatational and distortional wave. Using this result, we calculated numerically the amplitudes for the case $n=2$. The results of these calculations may show some light on the discussion of wave amplitude transmitted through a layer.

2. Expression of Stress and Displacement Components.

In this paper, we will confine our problem to the two-dimensional waves for the sake of simplicity, and each elastic medium is assumed to be isotropic and homogeneous. We take a Cartesian coordinate system and assume that the incident waves as well as refracted and reflected waves are included in xy-plane.

Putting the density and Lamé's elastic constants as ρ and λ, μ respectively, the equations of motion in that medium are

$$\left.\begin{aligned} \rho\frac{\partial^2\Delta}{\partial t^2} &= (\lambda+2\mu)\left(\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial^2\Delta}{\partial y^2}\right) \\ \rho\frac{\partial^2\tilde{\omega}}{\partial t^2} &= \mu\left(\frac{\partial^2\tilde{\omega}}{\partial x^2}+\frac{\partial^2\tilde{\omega}}{\partial y^2}\right) \end{aligned}\right\} \quad (2.1)$$

where Δ is the dilatation, $\tilde{\omega}$ is the rotation, and they are both expressed by means of the displacement components u and v as follows;

$$\left.\begin{aligned} \Delta &= \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \\ 2\tilde{\omega} &= \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \end{aligned}\right\} \quad (2.2)$$

The fundamental solution of equation (2.1) which represents the progressive waves towards the positive side of x-axis is

$$\left.\begin{aligned} \Delta &= \begin{Bmatrix} A \\ C \end{Bmatrix} \exp\{ih(V_p t - x\sin\theta \mp y\cos\theta)\} \\ 2\tilde{\omega} &= \begin{Bmatrix} B \\ D \end{Bmatrix} \exp\{ik(V_s t - x\sin\omega \mp y\cos\omega)\} \end{aligned}\right\} \quad (2.3)$$

where

$$\left.\begin{aligned} h^2 &= \rho p^2/(\lambda+2\mu) = (p/V_p)^2 \\ k^2 &= \rho p^2/\quad \mu \quad = (p/V_s)^2 \end{aligned}\right\} \qquad (2.4)$$

and θ and ω are the angles measured from y-axis to the normal of wave front of the dilatational and equivoluminal wave respectively. From Snell's law the next relation is easily found:

$$h\sin\theta = k\sin\omega \qquad (\equiv \xi;\ \text{constant}) \qquad (2.5)$$

If the elastic medium is bounded by two planes of separation perpendicular to the y-axis, the waves in the layered medium, are wholly expressed by the two upward-going waves and the two downward-going waves. (Here the consideration of the z component of displacement is not necessary.) In the following calculations, we will denote the amplitudes of upward-going P- and S-wave as A and B, and downward-going P- and S-wave as C and D. The displacement components due to the above waves A and B are denoted as u_A, v_A and u_B, v_B respectively. Then the displacement components are calculated from (2.3) and (2.2) as follows:

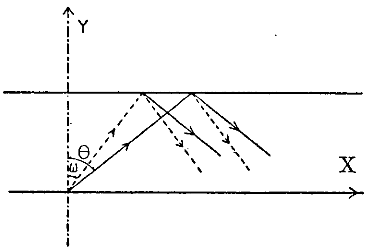

Fig. 1.

$$\left.\begin{aligned} u_A &= ip^{-1}V_pA\sin\theta\cdot\exp\{ih(V_pt-x\sin\theta-y\cos\theta)\} \\ v_A &= ip^{-1}V_pA\cos\theta\cdot\exp\{ih(V_pt-x\sin\theta-y\cos\theta)\} \\ u_B &= -ip^{-1}V_sB\cos\omega\cdot\exp\{ik(V_st-x\sin\omega-y\cos\omega)\} \\ v_B &= ip^{-1}V_sB\sin\omega\cdot\exp\{ik(V_st-x\sin\omega-y\cos\omega)\} \\ u_C &= ip^{-1}V_pC\sin\theta\cdot\exp\{ih(V_pt-x\sin\theta+y\cos\theta)\} \\ v_C &= -ip^{-1}V_pC\cos\theta\cdot\exp\{ih(V_pt-x\sin\theta+y\cos\theta)\} \\ u_D &= ip^{-1}V_sD\cos\omega\cdot\exp\{ik(V_st-x\sin\omega+y\cos\omega)\} \\ v_D &= ip^{-1}V_sD\sin\omega\cdot\exp\{ik(V_st-x\sin\omega+y\cos\omega)\} \end{aligned}\right\} \qquad (2.6)$$

The displacement at an arbitrary point in that medium is expressed as the sum of each displacement component in (2.6).

From these expressions we can easily calculate the velocity components of a particle $\dot{u}$ and $\dot{v}$ and stress components X_y and Y_y. Using the matrix expression and omitting the factor $\exp\{i(pt-\xi x)\}$ for convenience, we get

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix} = \begin{pmatrix} -V_p \sin\theta, & 0, & V_s \cos\omega, & 0, \\ 0, & -V_p \cos\theta, & 0, & -V_s \sin\omega \\ \rho V_p^2 \cos 2\omega, & 0, & \rho V_s^2 \sin 2\omega, & 0, \\ 0, & \rho V_s^2 \sin 2\theta, & 0, & -\rho V_s^2 \cos 2\omega \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha y} + Ce^{i\alpha y} \\ Ae^{-i\alpha y} - Ce^{i\alpha y} \\ Be^{-i\beta y} - De^{i\beta y} \\ Be^{-i\beta y} + De^{i\beta y} \end{pmatrix} \quad (2.7)$$

in which

$$h \cos\theta = \alpha \qquad k \cos\omega = \beta \quad (2.8)$$

In the right side of (2.7), we can find that the first matrix involves only the material constants and the angles of incidence, and is dependent neither on the period nor the amplitude of waves. Therefore we will call this matrix the characteristic matrix of that medium and express it by $[M]$, then

$$\begin{pmatrix} \\ M \\ \\ \end{pmatrix} = \begin{pmatrix} -V_p \sin\theta, & 0, & V_s \cos\omega, & 0 \\ 0, & -V_p \cos\theta, & 0, & -V_s \sin\omega \\ \rho V_{p1}^3 \cos 2\omega, & 0, & \rho V_s^2 \sin 2\omega, & 0 \\ 0, & \rho V_s^2 \sin 2\theta, & 0, & -\rho V_s^2 \cos 2\omega \end{pmatrix} \quad (2.10)$$

Modifying the equation (2.7), we have

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix} = \begin{pmatrix} \\ M \\ \\ \end{pmatrix} \begin{pmatrix} \cos\alpha y, & -i\sin\alpha y, & 0, & 0 \\ -i\sin\alpha y, & \cos\alpha y, & 0, & 0 \\ 0, & 0, & \cos\beta y, & -i\sin\beta y \\ 0, & 0, & -i\sin\beta y, & \cos\beta y \end{pmatrix} \begin{pmatrix} A+C \\ A-C \\ B-D \\ B+D \end{pmatrix} \quad (2.10)$$

Further, solving this equation with respect to A, B, C, and D

$$\begin{pmatrix} A+C \\ A-C \\ B-D \\ B+D \end{pmatrix} = \begin{pmatrix} \cos\alpha y, & i\sin\alpha y, & 0, & 0 \\ i\sin\alpha y, & \cos\alpha y, & 0, & 0 \\ 0, & 0, & \cos\beta y, & i\sin\beta y \\ 0, & 0, & i\sin\beta y, & \cos\beta y \end{pmatrix} \begin{pmatrix} \\ M \\ \\ \end{pmatrix}^{-1} \begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix} \quad (2.11)$$

where $[M]^{-1}$ is the reciprocal matrix of the characteristic matrix $[M]$, and its elements all take finite value except on the occasion when the determinant $|M|$ vanishes.

Calculating this reciprocal matrix actually, we get

$$\begin{pmatrix} \\ \boldsymbol{M} \\ \\ \end{pmatrix}^{-1} = \begin{pmatrix} -\dfrac{V_s \sin 2\omega}{V_p^2 \cos\omega}, & 0, & \dfrac{1}{\rho V_p^2}, & 0 \\ 0, & -\dfrac{\cos 2\omega}{V_p \cos\theta}, & 0, & \dfrac{\sin\theta}{\rho V_p^2 \cos\theta} \\ \dfrac{\cos 2\omega}{V_s \cos\omega}, & 0, & \dfrac{\sin\omega}{\rho V_s^2 \cos\omega}, & 0 \\ 0, & -\dfrac{\sin 2\theta}{V_p \cos\theta}, & 0, & -\dfrac{1}{\rho V_s^2} \end{pmatrix} \tag{2.12}$$

Now, if we substitute $y+d$ for y in the equation (2.10), and put the equation (2.11) into the relation just obtained

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y+d} &= \begin{pmatrix} \\ \boldsymbol{M} \\ \\ \end{pmatrix} \begin{pmatrix} \cos\alpha(y+d), & -i\sin\alpha(y+d), & 0, & 0 \\ -i\sin\alpha(y+d), & \cos\alpha(y+d), & 0, & 0 \\ 0, & 0, & \cos\beta(y+d), & -i\sin\beta(y+d) \\ 0, & 0, & -i\sin\beta(y+d), & \cos\beta(y+d) \end{pmatrix} \\ &\times \begin{pmatrix} \cos\alpha y, & i\sin\alpha y, & 0, & 0 \\ i\sin\alpha y, & \cos\alpha y, & 0, & 0 \\ 0, & 0, & \cos\beta y, & i\sin\beta y \\ 0, & 0, & i\sin\beta y, & \cos\beta y \end{pmatrix} \begin{pmatrix} \\ \boldsymbol{M} \\ \\ \end{pmatrix}^{-1} \begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y} \\ &= \begin{pmatrix} \\ \boldsymbol{M} \\ \\ \end{pmatrix} \begin{pmatrix} \cos\alpha d, & -i\sin\alpha d, & 0, & 0 \\ -i\sin\alpha d, & \cos\alpha d, & 0, & 0 \\ 0, & 0, & \cos\beta d, & -i\sin\beta d \\ 0, & 0, & -i\sin\beta d, & \cos\beta d \end{pmatrix} \begin{pmatrix} \\ \boldsymbol{M} \\ \\ \end{pmatrix}^{-1} \begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y} \end{aligned} \tag{2.13}$$

This is the very relation connecting the elements on the plane $y=y$ and that on $y=y+d$, which is invariable even if the conditions of the outer part of the plenes $y=y$ and $y=y+d$ are replaced by different ones.

Now we assume that there exist n discontinuous boundary planes perpendicular to y-axis and lying at $y=y_1, y_2, y_3, \ldots . y_n$. Expressing the elements on the boundary plane $y=y_k$ by the suffix k, we get the relation generally

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_k} = \begin{pmatrix} \\ \boldsymbol{K}_{k-1} \\ \\ \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_{k-1}} \tag{2.14}$$

where

$$\begin{bmatrix} \boldsymbol{K}_{k-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_{k-1} \end{bmatrix} \begin{pmatrix} \cos\alpha_{k-1}(y_k-y_{k-1}), & -i\sin\alpha_{k-1}(y_k-y_{k-1}), & 0, & 0 \\ -i\sin\alpha_{k-1}(y_k-y_{k-1}), & \cos\alpha_{k-1}(y_k-y_{k-1}), & 0, & 0 \\ 0, & 0, & \cos\beta_{k-1}(y_k-y_{k-1}), & -i\sin\beta_{k-1}(y_k-y_{k-1}) \\ 0, & 0, & -i\sin\beta_{k-1}(y_k-y_{k-1}), & \cos\beta_{k-1}(y_k-y_{k-1}) \end{pmatrix} \begin{bmatrix} \boldsymbol{M}_{k-1} \end{bmatrix}^{-1} \tag{2.15}$$

The matrix $[\boldsymbol{M}_{k-1}]$ is the characteristic matrix defined by the medium between $y=y_{k-1}$ and $y=y_k$.

If we connect the boundary conditions by the relation (2.14) from y_1-plane to y_n-plane, in the order of $k=2, 3, 4, \ldots\ldots n$, we reach the following result;

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_n} = \begin{bmatrix} \boldsymbol{K}_{n-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_{n-2} \end{bmatrix} \cdots\cdots \begin{bmatrix} \boldsymbol{K}_1 \end{bmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_1} = \begin{bmatrix} \boldsymbol{L} \end{bmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_1} \tag{2.16}$$

where

$$\begin{bmatrix} \boldsymbol{L} \end{bmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{34} & l_{35} \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} = \begin{bmatrix} \boldsymbol{K}_{n-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_{n-2} \end{bmatrix} \cdots\cdots \begin{bmatrix} \boldsymbol{K}_1 \end{bmatrix} \tag{2-17}$$

Equation (2.16) is the relation connecting the elements on the plane of discontinuity $y=y_n$ and that on $y=y_1$. Here $[\boldsymbol{L}]$ is the product of $3(n-1)$ matrices shown in (2.17) and (2.15). The equation (2.16) indicates that the velocity and stress component at y_n-plane can be calculated by operating $3n$ matrices to the velocity and stress components at y_1-plane.

3. The Solution when the Multilayered Medium Exists between Two Semi-Infinite Elastic Media.

First we assume that n discontinuities exist at $y=y_1, y_2, y_3, \ldots, y_n$ and dilatational wave is incident upon y_1 plane as shown in Fig. 2. Further, we denote the amplitudes of waves as follows;

Incident P-wave: $\mathfrak{A}$
Transmitted P-wave: A^*
Transmitted S-wave: B^*

Reflected P-wave: C^*
Reflected S-wave: D^*

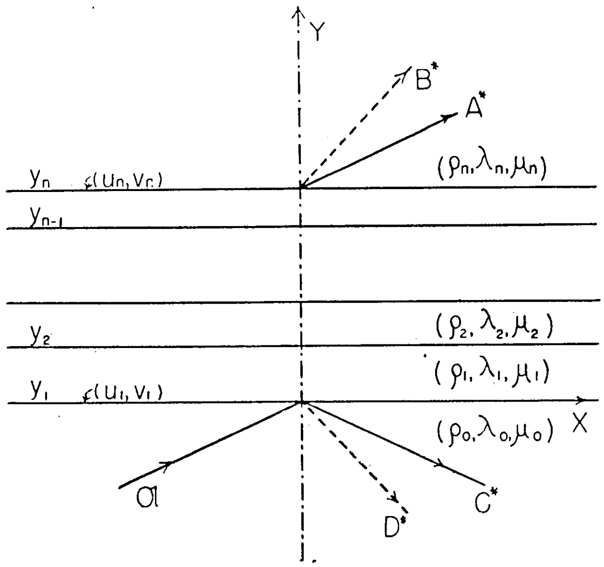

Fig. 2.

From the equation (2.7), the velocity and stress components which are produced on y_1 plane by the waves $\mathfrak{A}$, C^* and D^* are expressed as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_1} = \begin{pmatrix} M_0 \end{pmatrix} \begin{pmatrix} \mathfrak{A}\ e^{-i\alpha_0 y_1} + C^*\ e^{i\alpha_0 y_1} \\ \mathfrak{A}\ e^{-i\alpha_0 y_1} - C^*\ e^{i\alpha_0 y_1} \\ -D^*\ e^{i\beta_0 y_1} \\ D^*\ e^{i\beta_0 y_1} \end{pmatrix} \tag{3.1}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_n} = \begin{pmatrix} M_n \end{pmatrix} \begin{pmatrix} A^*\ e^{-i\alpha_n y_n} \\ A^*\ e^{-i\alpha_n y_n} \\ B^*\ e^{-i\beta_n y_n} \\ B^*\ e^{-i\beta_n y_n} \end{pmatrix} \tag{3.2}$$

Put these equations into (2.16), and we have

$$\begin{pmatrix} M_n \end{pmatrix} \begin{pmatrix} A^*\ e^{-i\alpha_n y_n} \\ A^*\ e^{-i\alpha_n y_n} \\ B^*\ e^{-i\beta_n y_n} \\ B^*\ e^{-i\beta_n y_n} \end{pmatrix} = \begin{pmatrix} L \end{pmatrix} \begin{pmatrix} M_0 \end{pmatrix} \begin{pmatrix} \mathfrak{A}\ e^{-i\alpha_0 y_1} + C^*\ e^{i\alpha_0 y_1} \\ \mathfrak{A}\ e^{-i\alpha_0 y_1} - C^*\ e^{i\alpha_0 y_1} \\ -D^*\ e^{i\beta_0 y_1} \\ D^*\ e^{i\beta_0 y_1} \end{pmatrix} \tag{3.3}$$

where $[\boldsymbol{M}_n]$ and $[\boldsymbol{M}_0]$ are the characteristic matrices which are determined by the medium $y>y_n$ and the medium $y<y_1$ respectively. When there exists reciprocal matrix of $[\boldsymbol{M}_n]$, we get the next relation by modifying the above equation

$$\begin{pmatrix} A^* & e^{-i\alpha_n y_n} \\ A^* & e^{-i\alpha_n y_n} \\ B^* & e^{-i\beta_n y_n} \\ B^* & e^{-i\beta_n y_n} \end{pmatrix} = \begin{pmatrix} \\ \boldsymbol{N} \\ \\ \end{pmatrix} \begin{pmatrix} \mathfrak{A} \; e^{-i\alpha_0 y_1} + C^* \; e^{i\alpha_0 y_1} \\ \mathfrak{A} \; e^{-i\alpha_0 y_1} - C^* \; e^{i\alpha_0 y_1} \\ -D^* \; e^{i\beta_0 y_1} \\ D^* \; e^{i\beta_0 y_1} \end{pmatrix} \tag{3.4}$$

where

$$\begin{pmatrix} \\ \boldsymbol{N} \\ \\ \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{pmatrix} = \begin{pmatrix} \\ \boldsymbol{M}_n \\ \\ \end{pmatrix}^{-1} \begin{pmatrix} \\ \boldsymbol{L} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \boldsymbol{M}_0 \\ \\ \end{pmatrix} \tag{3.5}$$

When $\mathfrak{A}$ is given, the unknowns of this problem are the amplitudes of transmitted and reflected P- and S-waves; viz., A^*, B^*, C^*, and D^*. Therefore solving the equation (3.4) with respect to A^*, B^*, C^* and D^*, we get

$$\begin{pmatrix} A^* \\ B^* \\ C^* \\ D^* \end{pmatrix} = \begin{pmatrix} e^{-i\alpha_n y_n} & 0 & (-n_{11}+n_{12})e^{i\alpha_0 y_1} & (n_{13}-n_{14})e^{i\beta_0 y_1} \\ e^{-i\alpha_n y_n} & 0 & (-n_{21}+n_{22})e^{i\alpha_0 y_1} & (n_{23}-n_{24})e^{i\beta_0 y_1} \\ 0 & e^{-i\beta_n y_n} & (-n_{31}+n_{32})e^{i\alpha_0 y_1} & (n_{33}-n_{34})e^{i\beta_0 y_1} \\ 0 & e^{-i\beta_n y_n} & (-n_{41}+n_{42})e^{i\alpha_0 y_1} & (n_{43}-n_{44})e^{i\beta_0 y_1} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} n_{11}+n_{12} \\ n_{21}+n_{22} \\ n_{31}+n_{32} \\ n_{41}+n_{42} \end{pmatrix} \mathfrak{A} \; e^{-i\alpha_0 y_1} \tag{3.6}$$

This is the solution we seek for, and it gives the amplitudes of reflected and transmitted waves. Here, $[\boldsymbol{N}]$ is the product of $(3n-1)$ matrices of the 4th order shown in the equations (3.5), (2.17) and (2.1). In actual treatment, the next formula is more convenient for numerical calculation. That is,

$$A^* = -\frac{2\, e^{-i(\alpha_0 y_1 - \alpha_n y_n)}\, \mathfrak{A}}{\Gamma} \begin{vmatrix} 0 & n_{11} & n_{12} & (n_{13}-n_{14}) \\ 0 & n_{21} & n_{22} & (n_{23}-n_{24}) \\ 1 & n_{31} & n_{32} & (n_{32}-u_{34}) \\ 1 & n_{41} & n_{42} & (n_{43}-n_{44}) \end{vmatrix}$$

$$
\left.\begin{aligned}
B^* &= \frac{2\,e^{-i(\alpha_0 y_1-\beta_n y_n)}\,\mathfrak{A}}{\Gamma}\begin{vmatrix} 1 & n_{11} & n_{21} & (n_{13}-n_{14}) \\ 1 & n_{21} & n_{22} & (n_{23}-n_{24}) \\ 0 & n_{31} & n_{32} & (n_{33}-n_{34}) \\ 0 & n_{41} & n_{42} & (n_{43}-n_{44}) \end{vmatrix} \\
C^* &= \frac{e^{-2i\alpha_0 y_1}\,\mathfrak{A}}{\Gamma}\begin{vmatrix} 1 & 0 & (n_{11}+n_{12}) & (n_{13}-n_{14}) \\ 1 & 0 & (n_{21}+n_{22}) & (n_{23}-n_{24}) \\ 0 & 1 & (n_{31}+n_{32}) & (n_{33}-n_{34}) \\ 0 & 1 & (n_{41}+n_{42}) & (n_{43}-n_{44}) \end{vmatrix} \\
D^* &= \frac{2\,e^{-i(\alpha_0-\beta_0)y_1}\,\mathfrak{A}}{\Gamma}\begin{vmatrix} 1 & 0 & n_{11} & n_{12} \\ 1 & 0 & n_{21} & n_{22} \\ 0 & 1 & n_{31} & n_{32} \\ 0 & 1 & n_{41} & n_{42} \end{vmatrix} \\
&\text{where} \\
\Gamma &= \begin{vmatrix} 1 & 0 & (-n_{11}+n_{12}) & (n_{13}-n_{14}) \\ 1 & 0 & (-n_{21}+n_{22}) & (n_{23}-n_{24}) \\ 0 & 1 & (-n_{31}+n_{32}) & (n_{33}-n_{34}) \\ 0 & 1 & (-n_{41}+n_{42}) & (n_{43}-n_{44}) \end{vmatrix}
\end{aligned}\right\} \quad (3.7)
$$

4. The Reflection at the Surface of the Layered Crust.

In the foregoing paragraph, we have treated the case in which the multilayered medium is placed between two semi-infinite elastic media. But in actual cases, our observation of seismic wave is performed at the free surface of the earth crust which are known to have layered structure. Therefore the problem must be treated in which one boundary plane is free from any traction.

In order to solve this problem in this section, first we assume the boundary plane at $y=y_n$ to be free, and denote the amplitudes of incident *P*-, reflected *P*- and reflected *S*-wave as $\mathfrak{A}$, C^* and D^* respectively.

From (2.16) and (3.1), we get

$$
\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_n} = \begin{pmatrix} \boldsymbol{L} \end{pmatrix}\begin{pmatrix} \boldsymbol{M}_0 \end{pmatrix}\begin{pmatrix} \mathfrak{A}\,e^{-i\alpha_0 y_1}+C^*\,e^{i\alpha_0 y_1} \\ \mathfrak{A}\,e^{-i\alpha_0 y_1}-C^*\,e^{i\alpha_0 y_1} \\ -D^*\,e^{i\beta_0 y_1} \\ D^*\,e^{i\beta_0 y_1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \end{pmatrix}\begin{pmatrix} \mathfrak{A}\,e^{-i\alpha_0 y_1}+C^*\,e^{i\alpha_0 y_1} \\ \mathfrak{A}\,e^{-i\alpha_0 y_1}-C^*\,e^{i\alpha_0 y_1} \\ -D^*\,e^{i\beta_0 y_1} \\ D^*\,e^{i\beta_0 y_1} \end{pmatrix} \quad (4.1)
$$

where

$$\begin{pmatrix} \\ \boldsymbol{F} \\ \\ \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix} = \begin{pmatrix} \\ \boldsymbol{L} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \boldsymbol{M}_0 \\ \\ \end{pmatrix} \tag{4.2}$$

From the condition of free surface, normal stress Y_y and tangential stress X_y vanish at $y=y_n$. Putting this condition into (4.1), we get

$$\begin{pmatrix} C^* \\ D^* \end{pmatrix} = \begin{pmatrix} (-f_{31}+f_{32})e^{i\alpha_0 y_1} & (f_{33}-f_{34})e^{i\beta_0 y_1} \\ (-f_{41}+f_{42})e^{i\alpha_0 y_1} & (f_{43}-f_{44})e^{i\beta_0 y_1} \end{pmatrix}^{-1} \begin{pmatrix} f_{31}+f_{32} \\ f_{41}+f_{42} \end{pmatrix} \mathfrak{A} e^{-i\alpha_0 y_1} \tag{4.3}$$

or employing the expression of another form

$$\left.\begin{aligned} C^* &= \frac{e^{-2i\alpha_0 y_1}\ \mathfrak{A}}{\Gamma} \begin{vmatrix} (f_{31}+f_{32}) & (f_{33}-f_{34}) \\ (f_{41}+f_{42}) & (f_{43}-f_{44}) \end{vmatrix} \\ D^* &= \frac{-2\, e^{-i(\alpha_0+\beta_0)y_1}\ \mathfrak{A}}{\Gamma} \begin{vmatrix} f_{31} & f_{32} \\ f_{41} & f_{42} \end{vmatrix} \\ \text{where} \\ \Gamma &= \begin{vmatrix} (-f_{31}+f_{32}) & (f_{33}-f_{34}) \\ (-f_{41}+f_{42}) & (f_{43}-f_{44}) \end{vmatrix} \end{aligned}\right\} \tag{4.4}$$

5. When the Incident Wave is Distortional.

In the foregoing problems, we have treated chiefly the case in which the incident wave is dilatational. Here we start under the condition that the incident wave is distortional and the displacement is involved in xy-plane (SV type wave). If we put the amplitudes of distortional wave as $\mathfrak{B}$, we get the expression corresponding to (3.1),

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ Y_y \\ X_y \end{pmatrix}_{y_1} = \begin{pmatrix} \\ \boldsymbol{M}_0 \\ \\ \end{pmatrix} \begin{pmatrix} C^*\ e^{i\alpha_0 y_1} \\ -C^*\ e^{i\alpha_0 y_1} \\ \mathfrak{B}\, e^{-i\beta_0 y_1} - D^*\ e^{i\beta_0 y_1} \\ \mathfrak{B}\, e^{-i\beta_0 y_1} + D^*\ e^{i\beta_0 y_1} \end{pmatrix} \tag{5.1}$$

Further calculation is quite similar to the foregoing one. Resulting equation corresponding to (3.6) is as follows:

$$\begin{pmatrix} A^* \\ B^* \\ C^* \\ D^* \end{pmatrix} = \begin{pmatrix} e^{-i\alpha_n y_n} & 0 & (-n_{11}+n_{12})e^{i\alpha_0 y_1} & (n_{13}-n_{14})e^{i\beta_0 y_1} \\ e^{-i\alpha_n y_n} & 0 & (-n_{21}+n_{22})e^{i\alpha_0 y_1} & (n_{23}-n_{24})e^{i\beta_0 y_1} \\ 0 & e^{-i\beta_n y_n} & (-n_{31}+n_{32})e^{i\alpha_0 y_1} & (n_{33}-n_{34})e^{i\beta_0 y_1} \\ 0 & e^{-i\beta_n y_n} & (-n_{41}+n_{42})e^{i\alpha_0 y_1} & (n_{43}-n_{44})e^{i\beta_0 y_1} \end{pmatrix}^{-1} \begin{pmatrix} n_{13}+n_{14} \\ n_{23}+n_{24} \\ n_{33}+n_{34} \\ n_{43}+n_{44} \end{pmatrix} \mathfrak{B}\, e^{-i\beta_0 y_1} \tag{5.2}$$

6. Numerical Examples.

We treat here the case $n=2$, that is, the case of a single layer existing. In this circumstance, the matrix $[N]$ in (3.5) becomes $[\boldsymbol{M}_2]^{-1} \times [\boldsymbol{K}_1] [\boldsymbol{M}_0]$.

We assume that Poisson's ratio σ is 1/4 in each medium and the incident angle of P-wave θ_0 is $\pi/6$. We take two models of the layered medium, one containing the low velocity layer among the high velocity media, and the other containing the high velocity layer.

In the former case, we assume that the ratio of the P-wave velocity and that of the density in each medium are as follows:

$$V_{p0} : V_{p1} : V_{p2} = 1 : 1/2 : 1$$
$$\rho_0 : \rho_1 : \rho_2 = 1 : 3/4 : 1$$

The results of numerical calculation are shown in Table I and Fig. 3.

In the latter case, the ratio of the P-wave velocity and that of the density in each medium are as follows:

$$V_{p0} : V_{p1} : V_{p2} = 1 : \sqrt{2} : 1$$
$$\rho_0 : \rho_1 : \rho_2 = 1 : 4/3 : 1$$

The results of numerical calculation are shown in Table II and Fig. 4.

In both figures, we take the ordinate as the ratio of the absolute value of the amplitude of transmitted and reflected P-wave A^* and C^* to that of incident wave $\mathfrak{A}$, and the abscissa as the ratio of thickness of the layer d to the wave length L of P-wave in that layer.

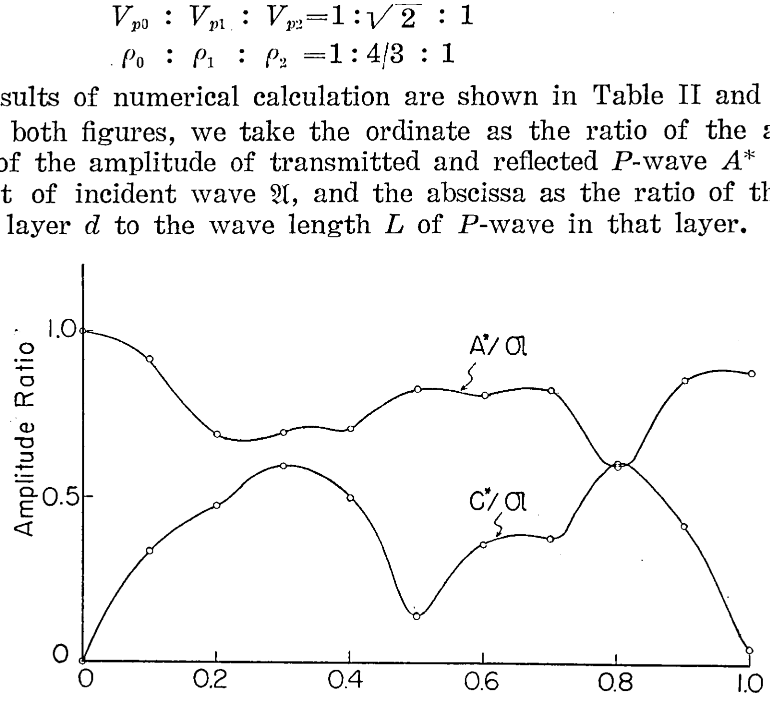

Fig. 3. Amplitude Ratio (low velocity layer)
$(\theta_0 = \frac{\pi}{6},\ V_{p0} : V_{p1} : V_{p2} = 1 : {}^1/_2 : 1,\quad \rho_0 : \rho_1 : \rho_2 = 1 : {}^3/_4 : 1)$

Table I. Amplitude Ratio (low velocity layer).

d/L	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Transmitted *P*-wave	1.0000	0.9180	0.6893	0.6928	0.7053	0.8251	0.8071	0.8206	0.5932	0.8533	0.8753
Transmitted *S*-wave	0.0000	0.1520	0.1807	0.7192	0.8556	0.6251	0.6944	0.9196	0.8648	0.4805	0.5023
Reflected *P*-wave	0.0000	0.3373	0.4748	0.5929	0.4992	0.1407	0.3568	0.3774	0.6001	0.4165	0.0459
Reflected *S*-wave	0.0000	1.0250	1.1660	0.6052	0.6425	0.6608	0.7354	0.6191	0.7988	0.4638	0.8632

Table II. Amplitude Ratio (high velocity layer).

d/L	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Transmitted *P*-wave	1.0000	0.9248	0.8048	0.8345	0.8878	0.8865	0.8671	0.8268	0.9147	0.6932	0.8668
Transmitted *S*-wave	0.0000	0.2519	0.4693	0.5866	0.6389	0.8508	0.8993	0.7699	0.6772	0.6336	0.8837
Reflected *P*-wave	0.0000	0.2380	0.6662	0.6785	0.5861	0.5812	0.7529	0.3176	0.0855	0.0977	0.3607
Reflected *S*-wave	0.0000	0.5895	0.6804	0.7372	0.4192	0.4738	0.9521	0.8977	0.5280	0.4107	1.4150

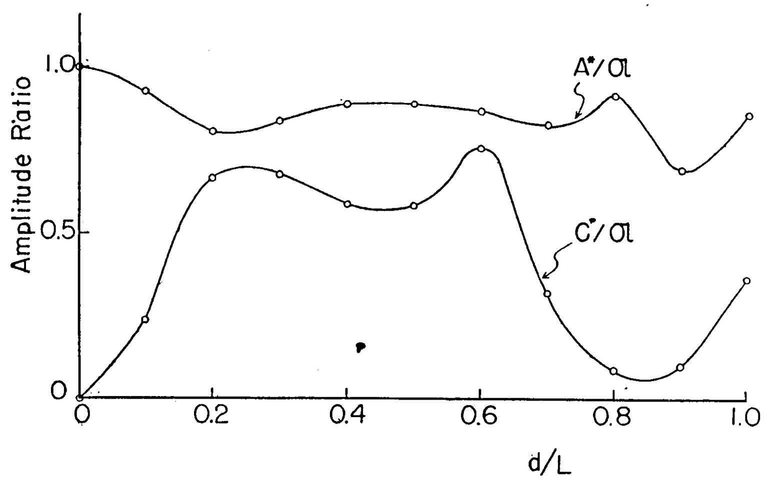

Fig. 4. Amplitude Ratio (high velocity layer)
$(\theta_0=\frac{\pi}{6},\ V_{p0}:V_{p1}:V_{p2}=1:\sqrt{2}:1,\ \rho_0:\rho_1:\rho_2=1:{}^4/_3:1)$

7. Acknowledgement.

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25. いくつかの平行な不連続面による地震波の反射及び透過

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平行な不連続面が一般に n 個ある場合に弾性波の反射・屈折の問題を解くことは原理的には簡単であるが，実際の取扱では非常に繁雑になる．ここではマトリツクスによる表現を用いて P 波及び SV 波入射に対する反射波及び透過波の振幅を求めた．

単一層の介在する場合についての数値計算の結果は，波長に対してかなりに薄い層でも相当に大きな反射波の振幅を与えることがわかつた．
