1. *Velocity of Elastic Waves Propagated in Media with Small Obstacles.*

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1. Introduction.

In the previous paper entitled "Velocity of Elastic Waves Propagated in Media with Small Holes"[1] we have treated the substance with small holes which are filled up with some kind of liquid. As a natural sequel to this work, we will take up a substance with small solid obstacles, in this paper. Since it is very difficult to treat obstacles of irregular shape and magnitude, we are obliged to confine our attention to the case of small spherical bodies of the same size. We will also assume some postulates which are necessary to develop our theory.

Of course we must strictly examine the validity of our model when we apply the following theory to some actual problems, e. g. the velocity of elastic waves in gravel which is constituted by spherical pebbles and homogeneous matrix.

The calculations in this paper are performed under the following assumption:

1) We can replace any volume of actual material containing large number of obstacles by equivalent homogeneous continuum.

2) The obstacles are spherical bodies of the same size.

3) The proportion of the volume of obstcles is small compared with the whole medium, and its square terms can be neglected without causing fatal error.

4) The continuity of displacement and stress components strictly holds at the boundary surface.

5) When the velocity of wave propagation comes into question, the wave-length is assumed to be far larger than the distance or the diameter of the obstacles.

These conditions are the same as those of our previous paper, and so long as they are admitted, we believe that our theory can give a ready solution of first approximation.

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1) Y. SATÔ, *Bull. Earthq. Res. Inst.*, **30** (1952), 179.

2. General type of the solution.[2)]

Now, there are six independent solutions of the elastic equations which can be derived from solid harmonics of order n, that is[3)]

$$\text{Type } \phi; \quad \mathfrak{d}=\operatorname{grad}\phi_n$$

$$\text{Type } \omega; \quad \mathfrak{d}=r^2\operatorname{grad}\omega_n+\alpha_n r\mathfrak{r}\omega_n \left(\alpha_n=-2\frac{n\lambda+(3n+1)\mu}{(n+3)\lambda+(n+5)\mu}\right) \tag{2.1}$$

$$\text{Type } \chi; \quad \mathfrak{d}=[r\mathfrak{r}, \operatorname{grad}\chi_n]$$

where $\mathfrak{d}$ implies displacement, $\mathfrak{r}$ is a unit vector of radial direction and ϕ_n, ω_n, and χ_n are the spherical solid harmonics of order n. Therefore, using a spherical surface harmonics of order n, we can express as

$$\phi_n,\ \omega_n,\ \chi_n=r^nS_n,\ r^{-n-1}S_n \quad \dots\dots\dots\dots\dots\dots\dots\dots (2.2)$$

Introducing these relations into the above expressions

$$\begin{aligned}
\text{Type } \phi; \quad & \mathfrak{d}_{\phi n+}=\operatorname{grad}(r^nS_n)=\{n, 1\}_n r^{n-1}\\
& \mathfrak{d}_{\phi n-}=\operatorname{grad}(r^{-n-1}S_n)=\{-n-1, 1\}_n r^{-n-2}\\
\text{Type } \omega; \quad & \mathfrak{d}_{\omega n+}=r^2\operatorname{grad}(r^nS_n)+\alpha_n\mathfrak{r}r^{n+1}S_n=\{n+\alpha_n, 1\}_n r^{n+1}\\
& \mathfrak{d}_{\omega n-}=r^2\operatorname{grad}(r^{-n-1}S_n)+\alpha_{-n-1}\mathfrak{r}r^{-n}S_n\\
& \qquad =\{-n-1+\alpha_{-n-1}, 1\}_n r^{-n} \qquad \dots\dots\dots\dots (2.3)\\
\text{Type } \chi; \quad & \mathfrak{d}_{\chi n+}=[r\mathfrak{r}, \operatorname{grad}(r^nS_n)]=-r^nS_n'\mathfrak{s}\\
& \mathfrak{d}_{\chi n-}=[r\mathfrak{r}, \operatorname{grad}(r^{-n-1}S_n)]=-r^{-n-1}S_n'\mathfrak{s} \qquad (n\neq 0)
\end{aligned}$$

in which

$$\{p, q\}_n\equiv pS_n\mathfrak{r}+qS_n'\mathfrak{t}$$

$\mathfrak{s}$ and $\mathfrak{t}$ are unit vectors of azimuthal and colatitudinal direction.

Here, we must notice that when $n=0$, these expressions must be somewhat modified.

$$\left\{\begin{aligned}
& \mathfrak{d}_{\phi 0+}=0\\
& \mathfrak{d}_{\phi 0-}=\{-1, 0\}_0 r^{-2}\\
& \mathfrak{d}_{\omega 0+}=\{\alpha_0, 0\}_0 r\\
& \mathfrak{d}_{\omega 0-}=\{-1+\alpha_{-1}, 0\}_0 \qquad \dots\dots\dots\dots\dots\dots\dots\dots (2.3a)\\
& \qquad =0, \quad (\because\ \alpha_{-1}=1)\\
& \mathfrak{d}_{\chi 0+}=0\\
& \mathfrak{d}_{\chi 0-}=0
\end{aligned}\right.$$

2) Although we use the term "general", the following calculations are still restricted, for they employ only zonal harmonics and do not treat the type of solutions which involves tesseral harmonics. However, in view of the results obtained, the results will not be altered even if we use a solution containing tesseral harmonics.

3) LOVE, *Theory of Elasticity*. pp. 251, 252.

Corresponding to these fundamental types of displacement, we have the fundamental stresses which after some complicated calculations[4] turn to be

$$\text{Type } \phi\,; \quad \mathfrak{F}_{\phi n+}=\mu\, 2(n-1)\{n,\, 1\}_n\, r^{n-2}$$
$$\mathfrak{F}_{\phi n-}=-\mu\, 2(n+2)\{-n-1,\, 1\}_n\, r^{-n-3}$$
$$\text{Type } \omega\,; \quad \mathfrak{F}_{\omega n+}=\mu\{(n-3)\alpha_n+2(n^2-2n-1),\, 2n+\alpha_n\}_n\, r^n$$
$$\mathfrak{F}_{\omega n-}=\mu\{-(n+4)\alpha_{-n-1}+2(n^2+4n+2),\ -2(n+1)+\alpha_{-n-1}\}_n\, r^{-n-1} \quad \ldots.(2.4)$$
$$\text{Type } \chi: \quad \mathfrak{F}_{\chi n+}=-\mu(n-1)r^{n-1}S_n{}'\mathfrak{s}$$
$$\mathfrak{F}_{\chi n-}=\mu(n+2)r^{-n-2}S_n{}'\mathfrak{s}$$

When $n=0$, these expressions must of course be modified as before.

$$\begin{cases} \mathfrak{F}_{\phi 0+}=0 \\ \mathfrak{F}_{\phi 0-}=\mu\, 4\{1,\, 0\}_0\, r^{-3} \\ \mathfrak{F}_{\omega 0+}=\mu\{3\alpha_0 k/\mu,\, 0\}_0 \\ \mathfrak{F}_{\omega 0-}=\mu\{-4\alpha_{-1}+4,\, 0\}_0=0 \\ \mathfrak{F}_{\chi 0+}=0 \\ \mathfrak{F}_{\chi 0-}=0 \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.4\text{a})$$

If an arbitrary stress is applied to the surface of the equivalent homogeneous sphere (radius R), it can be expressed by

$$[K_n\mathfrak{F}_{\phi n+}+L_n\mathfrak{F}_{\omega n+}+M_n\mathfrak{F}_{\chi n+}]_{r=R} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.5)$$

and the displacement at the same surface corresponding to this stress is

$$[K_n\mathfrak{d}_{\phi n+}+L_n\mathfrak{d}_{\omega n+}+M_n\mathfrak{d}_{\chi n+}]_{r=R} \quad \ldots\ldots\ldots\ldots(2.6)$$

If we imply by the notation $\mathfrak{d}[\phi_n]$ the displacement in the part of the equivalent homogeneous continuum of the model in Fig. 1, when the stress $\mathfrak{F}_{\phi n+}$ is applied to the outer surface $r=R$, it can be expressed by the next formula

$$\mathfrak{d}[\phi_n]=A_n\mathfrak{d}_{\phi n+}+B_n\mathfrak{d}_{\phi n-}+C_n\mathfrak{d}_{\omega n+}+D_n\mathfrak{d}_{\omega n-}. \quad \ldots\ldots\ldots\ldots(2.7)$$

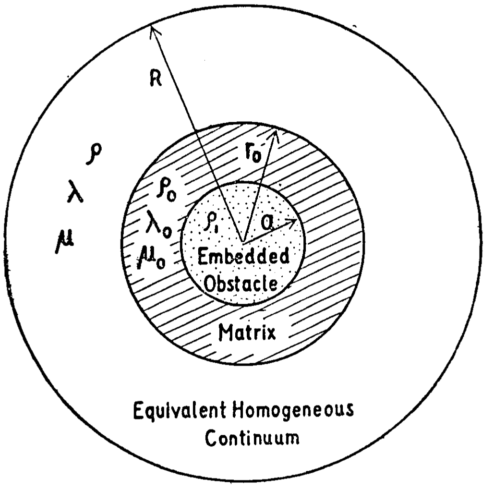

Fig. 1.

Similarly, we have

4) *cf.*, LOVE, *Elastisty.* p. 252.

$$\begin{aligned}\mathfrak{d}[\omega_n]&=E_n\mathfrak{d}_{\phi n+}+F_n\mathfrak{d}_{\phi n-}+G_n\mathfrak{d}_{\omega n+}+H_n\mathfrak{d}_{\omega n-}\\ \mathfrak{d}[\chi_n]&=I_n\mathfrak{d}_{\chi n+}\ \ +J_n\mathfrak{d}_{\chi n-}\end{aligned}\qquad\cdots\cdots\cdots(2.8)$$

In the part of matrix, we have corresponding expressions ;

$$\left\{\begin{aligned}\mathfrak{d}_0[\phi_n]&=A_{n0}\mathfrak{d}_{\phi n+}+B_{n0}\mathfrak{d}_{\phi n-}+C_{n0}\mathfrak{d}_{\omega n+}+D_{n0}\mathfrak{d}_{\chi n-}\\ \mathfrak{d}_0[\omega_n]&=E_{n0}\mathfrak{d}_{\phi n+}+F_n\mathfrak{d}_{\phi n-}\ \ +G_{n0}\mathfrak{d}_{\omega n+}+H_{n0}\mathfrak{d}_{\omega n-}\\ \mathfrak{d}_0[\chi_n]&=I_{n0}\mathfrak{d}_{\chi n+}\ \ +J_{n0}\mathfrak{d}_{\chi n-}\end{aligned}\right.\qquad\cdots\cdots\cdots(2.9)$$

and in the part of obstacle,

$$\left\{\begin{aligned}\mathfrak{d}_1[\phi_n]&=A_{n1}\mathfrak{d}_{\phi n+}+C_{n1}\mathfrak{d}_{\omega n+}\\ \mathfrak{d}_1[\omega_n]&=E_{n1}\mathfrak{d}_{\phi n+}+G_{n1}\mathfrak{d}_{\omega n+}\\ \mathfrak{d}_1[\chi_n]&=I_{n0}\mathfrak{d}_{\chi n+}\end{aligned}\right.\qquad\cdots\cdots\cdots(2.10)$$

If an arbitrary stress represented by the expression

$$\sum_n[K_n\mathfrak{T}_{\phi n+}+L_n\mathfrak{T}_{\omega n+}+M_n\mathfrak{T}_{\chi n+}]_{r=R}\qquad\cdots\cdots\cdots(2.11)$$

is applied to the spherical surface $r=R$, then the displacements in the above model are as follows:

in the equivalent homogeneous continuum ;

$$\mathfrak{d}=\sum_n K_n\mathfrak{d}[\phi_n]+\sum_n L_n\mathfrak{d}[\omega_n]+\sum_n M_n\mathfrak{d}[\chi_n]$$

in the matrix ;

$$\mathfrak{d}_0=\sum_n K_n\mathfrak{d}_0[\phi_n]+\sum_n L_n\mathfrak{d}_0[\omega_n]+\sum_n M_n\mathfrak{d}_0[\chi_n]\qquad\cdots\cdots\cdots(2.12)$$

in the obstacle ;

$$\mathfrak{d}_1=\sum_n K_n\mathfrak{d}_1[\phi_n]+\sum_n L_n\mathfrak{d}_1[\omega_n]+\sum_n M_n\mathfrak{d}_1[\chi_n]$$

3. $\mathfrak{d}[\phi_n]$. Relevalent solution of A_n, B_n, C_n and D_n.

In this article we will determine the quantities such as A_n, B_n C_n D_n which are involved in the expressions (2.7), (2.8), (2.9) and (2.10), and also in (2.11), (2.12) and (2.13) implicitly.

Boundary conditions which must hold at the boundary surfaces are

$$\begin{aligned}r=R\ ;&\quad \mathfrak{T}=\text{external stress}\\ r=r_0\ ;&\quad \mathfrak{d}=\mathfrak{d}_0\\ &\quad \mathfrak{T}=\mathfrak{T}_0\\ r=a\ ;&\quad \mathfrak{d}_0=\mathfrak{d}_1\\ &\quad \mathfrak{T}_0=\mathfrak{T}_1\end{aligned}\qquad\cdots\cdots\cdots(3.1)$$

3.1 Determination of A_n, B_n, C_n, D_n.

At first, we will take up the case when the external stress

$$\mathfrak{T}_{\phi n+}=\mu 2(n-1)\{n,1\}_n\qquad\cdots\cdots\cdots(3.2)$$

is applied at the spherical surface

$$r=R,$$

A_n, B_n, C_n, D_n ; A_{n0}, B_{n0}, C_{n0}, D_{n0} ; A_{n1}, C_{n1} are determined by the next equations which are obtained from the boundary conditions.

$$\mathfrak{M}\mathfrak{A}=\mathfrak{S}_{\phi} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3.3)$$

or

$$\begin{pmatrix} f_1R^{n-2} & f_2R^{-n-3} & f_3R^n & f_4R^{-n-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ g_1R^{n-2} & g_2R^{-n-3} & g_3R^n & g_4R^{-n-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1r_0^{n-1} & d_2r_0^{-n-2} & d_3r_0^{n+1} & d_4r_0^{-n} & -d_{10}r_0^{n-1} & -d_{20}r_0^{-n-2} & -d_{30}r_0^{n+1} & -d_{40}r_0^{-n} & 0 & 0 \\ r_0^{n-1} & r_0^{-n-2} & r_0^{n+1} & r_0^{-n} & -r_0^{n-1} & -r_0^{-n-2} & -r_0^{n+1} & -r_0^{-n} & 0 & 0 \\ \chi f_1r_0^{-n-2} & \chi f_2r_0^{-n-3} & f_3r_0^{n} & f_4r_0^{-n-1} & f_{10}r_0^{n-2} & -f_{20}r_0^{-n-3} & -f_{30}r_0^{n} & -f_{40}r_0^{-n-1} & 0 & 0 \\ \chi g_1r_0^{n} & \chi g_2r_0^{-n-3} & g_3r_0^{n} & g_4r_0^{-n-1} & g_{10}r_0^{n-2} & -g_{20}r_0^{-n-3} & -g_{30}r_0^{n} & -g_{40}r_0^{-n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{10}r_0^{n-1} & d_{20}a^{-n-2} & d_{30}a^{n-1} & d_{40}a^{-n} & -d_{11}a^{n-1} & -d_{11}a^{n-1} \\ 0 & 0 & 0 & 0 & a^{n-1} & a^{-n-2} & a^{n+1} & a^{-n} & -a^{n-1} & -a^{n+1} \\ 0 & 0 & 0 & 0 & f_{10}a^{n-2} & f_{20}a^{-n-3} & f_{30}a^{n} & f_{40}a^{-n-1} & -\chi_1f_{11}a^{n-2} & -\chi_1f_{31}a^{n} \\ 0 & 0 & 0 & 0 & g_{10}a^{n-2} & g_{20}a^{-n-3} & g_{30}a^{n} & g_{40}a^{-n-1} & -\chi_1g_{11}a^{n-2} & -\chi_1g_{31}a^{n} \end{pmatrix} \cdot \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \\ A_{n0} \\ B_{n0} \\ C_{n0} \\ D_{n0} \\ A_{n1} \\ C_{n1} \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\cdots\cdots\cdots\cdots(3.4) \qquad (n\neq 0)$$

in which

$$\left\{\begin{array}{ll} \chi\equiv\mu/\mu_0, \quad \chi_1\equiv\mu_1/\mu_0 & \\ \mathfrak{d}_{\phi n+}=\{d_1,1\}_n\, r^{n-1}, & \mathfrak{F}_{\phi n+}/\mu=\{f_1,g_1\}_n\, r^{n-2} \\ \mathfrak{d}_{\phi n-}=\{d_2,1\}_n\, r^{-n-2}, & \mathfrak{F}_{\phi n-}/\mu=\{f_2,g_2\}_n\, r^{-n-3} \\ \mathfrak{d}_{\omega n+}=\{d_3,1\}_n\, r^{n+1}, & \mathfrak{F}_{\omega n+}/\mu=\{f_3,g_3\}_n\, r^{n} \\ \mathfrak{d}_{\omega n-}=\{d_4,1\}_n\, r^{-n}, & \mathfrak{F}_{\omega n-}/\mu=\{f_4,g_4\}_n\, r^{-n-1} \quad \text{etc.} \end{array}\right. \quad \cdots\cdots(3.5)$$

Solving these equations we have

$$\left\{\begin{array}{l} A_n=\Delta_{An}/\Delta_n \\ B_n=\Delta_{Bn}/\Delta_n \\ C_n=\Delta_{Cn}/\Delta_n \\ D_n=\Delta_{Dn}/\Delta_n \end{array}\right. \quad (n\neq 0) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3.6)$$

$$\Delta_n=\det(\mathfrak{M})=-R^{2n-2}(f_1g_3-g_1f_3)\Delta_n\binom{1,2}{1,3}$$
$$\cdot\left\{1-R^{-2n+1}\frac{f_3g_4-g_3f_4}{f_1g_3-g_1f_3}\Delta_n\binom{1,2}{3,4}\Big/\Delta_n\binom{1,2}{1,3}\right\}$$

$$\left\{\begin{array}{l}\Delta_{An}=-R^n(f_1g_3-g_1f_3)\Delta_n\binom{1,2}{1,3}+O(R^{-n-1})\\ \Delta_{Bn}=\quad R^n(f_1g_3-g_1f_3)\Delta_n\binom{1,2}{2,3}+O(R^{-n-1})\\ \Delta_{Cn}=\quad R^{-n-1}(f_1g_4-g_1f_4)\Delta_n\binom{1,2}{3,4}+O(R^{-n-3})\\ \Delta_{Dn}=-R^n(f_1g_3-g_1f_3)\Delta_n\binom{1,2}{3,4}+O(R^{-n-3})\qquad (n\neq 0)\end{array}\right. \qquad\qquad (3.7)$$

where $\Delta_n\binom{i,j}{p,q}$ means a minor determinant of the $(n-2)$-th degree which is obtained from the original Δ_n erasing the i-th and j-th rows and the p-th and q-th columns.

Introducing these expressions into (2.7), we have

$$\mathfrak{d}[\phi_n]=R\{n,1\}_n+R^{-2n+2}\Delta_n\binom{1,2}{3,4}\Big/\Delta_n\binom{1,2}{1,3}$$
$$\cdot\left[\{n,1\}_n\frac{f_3g_4-g_3f_4}{f_1g_3-g_1f_3}-\{n+\alpha_n,1\}_n\frac{f_1g_4-g_1f_4}{f_1g_3-g_1f_3}\right.$$
$$\left.+\{-n-1+\alpha_{-n-1},1\}_n+O(R^{-2})\right](n\neq 0)\qquad (3.8)$$

On the other hand, from (2.3a) and (2.4a)

$$\mathfrak{d}[\phi_0]=0 \qquad\qquad (3.8a)$$

4. $\mathfrak{d}[\omega_n]$. Relevant solution of E_n, F_n, G_n, H_n.

By a similar treatment, we can obtain E_n, F_n, G_n, H_n ; E_{n0},...... although, in this case, the external stress takes the form

$$\begin{aligned}\mathfrak{F}_{\omega n+}&=\mu\{(n-3)\alpha_n+2(n^2-2n-1),\,2n+\alpha_n\}_n\\ &\equiv\mu\{f_3,g_3\}\end{aligned}\qquad\qquad (4.1)$$

Simultaneous equations to determine E_n, F_n, and so forth are expressed by the following form ;

$$\mathfrak{M}\cdot\mathfrak{E}=\mathfrak{S}_\omega \qquad\qquad (4.2)$$

in which

$$\mathfrak{E}=\begin{pmatrix} E_n \\ F_n \\ G_n \\ H_n \\ E_{n0} \\ F_{n0} \\ G_{n0} \\ H_{n0} \\ E_{n1} \\ G_{n1} \end{pmatrix}, \qquad \mathfrak{S}_\omega=\begin{pmatrix} f_3 \\ g_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and $\mathfrak{M}$ is same with the previous one. ....(4.3)

Solving these equations we obtain

$$\begin{cases} E_n=\Delta_{En}/\Delta_n \\ F_n=\Delta_{Fn}/\Delta_n \\ G_n=\Delta_{Gn}/\Delta_n \\ H_n=\Delta_{Hn}/\Delta_n \qquad (n\neq 0) \end{cases} \qquad \text{......................(4.4)}$$

in which

$$\begin{cases} \Delta_{En}= R^{-n-1}(f_3g_4-g_3f_4)\Delta_n\begin{pmatrix}1,2\\1,4\end{pmatrix}+O(R^{-n-3}) \\ \Delta_{Fn}= R^{n-2}(f_1g_3-g_1f_3)\Delta_n\begin{pmatrix}1,2\\1,2\end{pmatrix}O(R^{-n-1}) \\ \Delta_{Gn}=-R^{n-2}(f_1g_3-g_1f_3)\Delta_n\begin{pmatrix}1,2\\1,3\end{pmatrix} \\ \qquad\qquad +(f_3g_4-g_3f_4)R^{-n-1}\Delta_n\begin{pmatrix}1,2\\3,4\end{pmatrix}+O(R^{-n-3}) \\ \Delta_{Hn}= R^{n-2}(f_1g_3-g_1f_3)\Delta_n\begin{pmatrix}1,2\\1,4\end{pmatrix}+O(R^{-n-3}) \end{cases}$$

and Δ_n has already appeared in the previous section.

Introducing these expressions into (2.8), we have

$$\begin{aligned} \mathfrak{d}[\omega_n]=&R\{n+\alpha_n,1\} \\ &-R^{-2n}\frac{\Delta_n\begin{pmatrix}1,2\\1,4\end{pmatrix}}{\Delta_n\begin{pmatrix}1,2\\1,3\end{pmatrix}}\left[\frac{f_3g_4-g_3f_4}{f_1g_3-g_1f_3}\{n,1\}+\{-n-1+\alpha_{-n-1},1\}\right] \\ &+O(R^{-2n-2}) \qquad (n\neq 0) \qquad \text{................(4.5)} \end{aligned}$$

When $n=0$, the solution must be obtained in another way. Introducing the expressions (2.3a) and (2.4a) into the boundary conditions, we have

$$\begin{pmatrix} 4\mu R^{-3} & 3\alpha_0 k & 0 & 0 & 0 \\ -r_0^{2} & \alpha_0 r_0 & r_0^{-2} & -\alpha_{00} r_0 & 0 \\ 4\mu r_0^{-3} & 3\alpha_0 k & -4\mu_0 r_0^{-3} & -3\alpha_{00} k_0 & 0 \\ 0 & 0 & -a^{-2} & \alpha_{00} a & -\alpha_{01} a \\ 0 & 0 & 4\mu_0 a^{-3} & 3\alpha_{00} k_0 & -3\alpha_{01} k_1 \end{pmatrix} \begin{pmatrix} F \\ G \\ F_0 \\ G_0 \\ G_1 \end{pmatrix} = \begin{pmatrix} 3\alpha_0 k \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots\ldots (4.6)$$

from which we arrive at the following solution

$$\mathfrak{d}[\omega_0] = \alpha_0 R + \left\{1 + \frac{4\mu}{3k}\right\} R^{-2} \frac{\Delta_0\binom{1}{1}}{\Delta_0\binom{1}{2}} \quad \ldots\ldots\ldots\ldots (4.7)$$

in which $\Delta_0\binom{i}{p}$ is a minor of the (i, p)-element of the determinant of degree 5 obtained from (4.6).

5. $\mathfrak{d}[\chi_n]$. Relevant solution of I_n, J_n.

In the former sections we have obtained A_n, B_n, C_n, D_n and E_n, F_n, G_n, H_n, which give the relevalent solution of the cases of type ϕ_n and ω_n. Now here, we will get the solution of I_n and J_n which are necessary to express the displacement of the type χ_n.

External stress in this case, applied to the outer spherical surface is

$$\mathfrak{F}_{\chi n+} = -\mu(n-1)S_n'\mathfrak{s} \quad \ldots\ldots\ldots\ldots (5.1)$$

Boundary conditions are

$$\begin{aligned} r=R: \quad & \mathfrak{F}[\chi_n] = -\mu(n-1)S_n'\mathfrak{s} \\ r=r_0; \quad & \mathfrak{d}[\chi_n] = \mathfrak{d}_0[\chi_n] \\ & \mathfrak{F}[\chi_n] = \mathfrak{F}_0[\chi_n] \qquad \ldots\ldots\ldots\ldots (5.2) \\ r=a; \quad & \mathfrak{d}_0[\chi_n] = \mathfrak{d}_1[\chi_n] \qquad (n \neq 0) \\ & \mathfrak{F}_0[\chi_n] = \mathfrak{F}_1[\chi_n] \end{aligned}$$

in which $\mathfrak{d}[\chi_n]$, $\mathfrak{d}_0[\chi_n]$ etc. are given in (2.8) and so on.

Simultaneous equations to determine I_n, J_n; I_{n0}, J_{n0}; and I_{n1} which are involved in (5.2) take the next form by the matrix expression.

$$\mathfrak{M}\mathfrak{k} \cdot \mathfrak{J} = \mathfrak{S}_\chi \quad \ldots\ldots\ldots\ldots (5.3)$$

or

$$\mathfrak{M}\mathfrak{k} = \begin{pmatrix} (n-1)R^{n-1} & -(n+2)R^{-n-2} & 0 & 0 & 0 \\ -r_0^{n} & -r_0^{-n-1} & r_0^{n} & r_0^{-n-1} & 0 \\ -\chi(n-1)r_0^{n-1} & \chi(n+2)r_0^{-n-2} & (n-1)r_0^{-n-2} & -(n+2)r_0^{-n-2} & 0 \\ 0 & 0 & a^{n} & a^{n-1} & -a^{n} \\ 0 & 0 & -(n-1)a^{n-1} & (n+2)a^{-n-2} & \chi_1(n-1)a^{n-1} \end{pmatrix}$$

and

$$\mathfrak{J}=\begin{pmatrix} I_n \\ J_n \\ I_{n0} \\ J_{n0} \\ I_{n1} \end{pmatrix}, \qquad \mathfrak{S}_\chi=\begin{pmatrix} n-1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{......................(5.4)}$$

The solution is easily obtained in the following form

$$\begin{aligned} I_n &= \Delta_n^\dagger\binom{1}{1}\Big/\Delta_n^\dagger \\ J_n &= -\Delta_n^\dagger\binom{1}{2}\Big/\Delta^\dagger \qquad \text{......................(5.5)} \\ \Delta_n^\dagger &= \det(\mathfrak{M}\dagger) \end{aligned}$$

$\Delta_n^\dagger\binom{p}{q}$ is a minor determinant of the (p, q)-element of $\Delta_n^\dagger$.

Hence, at $r=R$

$$\begin{aligned} \mathfrak{d}[\chi_n] &= I_n(-R^n S_n'\mathfrak{s}) + J_n(-R^{-n-1}S_n'\mathfrak{s}) \\ &= \left[-R+R^{-2n}\frac{\Delta_n^\dagger\binom{1}{2}}{\Delta_n^\dagger\binom{1}{1}}\left\{1+\frac{n+2}{n-1}\right\}\right]\cdot(S_n'\mathfrak{s}) \quad (n\neq 0) \quad \text{...(5.6)} \end{aligned}$$

When $n=0$

$$\mathfrak{d}[\chi_0]=0 \qquad \text{..................................(5.6a)}$$

6. General form of displacement in the equivalent homogeneous shell corresponding to the general type of stress.

When a general type of stress

$$\mathfrak{F}=\sum_n [K_n\mathfrak{F}_{\phi n+} + L_n\mathfrak{F}_{\omega n+} + M_n\mathfrak{F}_{\chi n+}]_{r=R} \qquad \text{.............(6.1)}$$

is given at the outer spherical surface $r=R$, the corresponding displacement is

$$\mathfrak{d}=\sum_n [K_n\mathfrak{d}[\phi_n] + L_n\mathfrak{d}[\omega_n] + M_n\mathfrak{d}[\chi_n]] \qquad \text{................(6.2)}$$

which is now explicity written down by means of the above preparations.

Introducing $\mathfrak{d}[\phi_n]$, $\mathfrak{d}[\omega_n]$ and $\mathfrak{d}[\chi_n]$ given in (3.8), (3.8a), (4.5), (4.7) (5.6) and (5.6a) we have

$$\begin{aligned} \mathfrak{d} &= L_0\cdot R\alpha_0 \\ &\quad + \sum_{n=1}[K_n\cdot R\{n, 1\}_n + L_n\cdot R\{n+\alpha_n, 1\}_n + M_n\cdot(-)R] \\ &\quad + L_0\cdot R^{-2}\left\{1+\frac{4\mu}{3k}\right\}\Delta\binom{1}{1}\Big/\Delta\binom{1}{2} \end{aligned}$$

$$+\sum_{n=1}\Big[K_n\cdot R^{-2n+2}\Big\{\Delta_n\binom{1,\ 2}{3,\ 4}\Big/\Delta_n\binom{1,\ 2}{1,\ 3}\Big\}\cdot\Big[\{n,1\}_n\frac{f_3g_4-g_3f_4}{f_1g_3-g_1f_3}-\dots\Big]$$

$$+L_n\cdot R^{-2n}(-)\Big\{\Delta_n\binom{1,\ 2}{1,\ 4}\Big/\Delta_n\binom{1,\ 2}{1,\ 3}\Big\}\cdot\Big[\{n,1\}_n\frac{f_3g_4-g_3f_4}{{}_1fg_3-g_1f_3}-\dots\Big]$$

$$+M_n\cdot R^{-2n}\Big\{\Delta_n^\dagger\binom{1}{2}\Big/\Delta_n^\dagger\binom{1}{1}\Big\}\cdot\Big[\Big\{1+\frac{n+2}{n-1}\Big\}\cdot S_n{}'\mathfrak{s}\Big]\Big] \quad \dots(6.3)$$

On the other hand, the displacement at the same surface $r=R$ of the equivalent homogeneous sphere under the same external stress is

$$\sum_{n=0}[K_n\mathfrak{d}_{\phi n+}+L_n\mathfrak{d}_{\omega n+}+M_n\mathfrak{d}_{\chi n+}]_{r=R}$$

$$=L_0R\alpha_0+\sum[K_n\cdot R\{n,1\}_n+L_n\cdot R\{n+\alpha_n,1\}_n+M_n\cdot(-)R] \quad \dots(6.4)$$

which is exactly equal to the first and second lines of (6.3). Therefore the right hand member of the expression (6.3), excluding these two lines, must vanish. Thus

$$L_0\mathrm{R}^{-2}\Big\{1+\frac{4\mu}{3k}\Big\}\Delta\binom{1}{1}\Big/\Delta\binom{1}{2}$$

$$+\sum_{n=1}\Big[K_n\cdot R^{-2n+2}\Big\{\Delta_n\binom{1,\ 2}{3,\ 4}\Big/\Delta_n\binom{1,\ 2}{1,\ 3}\Big\}\cdot\Big[\dots\Big]$$

$$+L_nR^{-2n}(-)\Big\{\Delta_n\binom{1,\ 2}{1,\ 4}\Big/\Delta_n\binom{1,\ 2}{1,\ 3}\cdot\Big[\dots\Big]$$

$$+M_n\cdot R^{-2n}\Big\{\Delta_n^\dagger\binom{1}{2}\Big/\Delta_n^\dagger\binom{1}{1}\Big\}\cdot\Big[\dots\Big]\Big]=0 \quad \dots\dots(6.5)$$

or

$$K_1\cdot\Big\{\Delta_1\binom{1,\ 2}{3,\ 4}\Big/\Delta_1\binom{1,\ 2}{1,\ 3}\Big\}\cdot[\{1,1\}_1\ \dots]$$

$$+R^{-2}\cdot\Big[L_0\Big\{1+\frac{4\mu}{3k}\Big\}\Delta\binom{1}{1}\Big/\Delta\binom{1}{2}$$

$$+K_2\Big\{\Delta_2\binom{1,\ 2}{3,\ 4}\Big/\Delta_2\binom{1,\ 2}{1.\ 3}\Big\}\cdot[\{2,1\}_2\ \dots]$$

$$+L_1(-)\Big\{\Delta_1\binom{1,\ 2}{1,\ 4}\Big/\Delta_1\binom{1,\ 2}{1,\ 3}\Big\}\cdot[\{1,1\}_1\ \dots]$$

$$+M_1\Big\{\Delta_1^\dagger\binom{1}{2}\Big/\Delta_1^\dagger\binom{1}{1}\Big\}\cdot[\ \dots\dots]\Big]$$

$$+O(R^{-4}) \qquad =0 \qquad \dots\dots\dots\dots\dots(6.5)$$

Since this expression is arranged according to the descending power of R, an upper term is larger than the lower one. Therefore we will equate every term to zero from above one by one.

Fortunately the first line involving K_1 vanishes, for the determinant

$\Delta_1\begin{pmatrix}1, 2\\3, 4\end{pmatrix}$ is identically zero. (This terms denotes a mere translation of a sphere to the direction of z-axis as a rigid body.)

Among the remaining terms, the fourth line involving L_1 also vanishes, for

$$\Delta_1\begin{pmatrix}1, 2\\1, 4\end{pmatrix}=0$$

The fifth line involving M_1 is a term derived from χ_1. Now, this solution implies the mere rotation of a sphere as a rigid body. Therefore it has nothing to do with the stress distribution.

Now, we must equate the remaining terms to zero. Since L_0 and K_2 take independent values both terms must vanish independently.

$$\Delta\begin{pmatrix}1\\1\end{pmatrix}=0 \qquad \qquad \text{......(6.7)}$$

$$\Delta_2\begin{pmatrix}1, 2\\3, 4\end{pmatrix}=0 \qquad \qquad \text{......(6.8)}$$

7. Bulk modulus.

Solving the equation (6.7), we have the following solution;

$$k=k_0-(k_0-k_1)\frac{3k_0+4\mu_0}{3k_1+4\mu_0}(1-\rho) \qquad \qquad \text{......(7.1)}$$ [5]

which is equivalent to the formula of bulk modulus obtained in our former paper. ($1-\rho$ is the proportion of the volume occupied by the obstacle.)

7.1 Weighted mean of the compressibility.

Some thirty years ago, Adams[6] made a comprehensive investigation upon the compressibility of heterogeneous medium. He concludes that at a high pressure (above about 3000 bars) the compressibility of a rock is the average (according to volumes) of those of constituent minerals.

At a sight this conclusion seems very natural, but the weighted mean (according to volumes) of the compressibilities $1/k_0$ and $1/k_1$ of two materials is (σ_0 implies Poisson's ratio.)

$$\rho\frac{1}{k_0}+(1-\rho)\frac{1}{k_1}=\frac{1}{k_0}\left\{1+(1-\rho)\frac{1-K}{K}\right\} \qquad \qquad \text{......(7.2)}$$

5) *loc. cit.*, 1) Expressions (4.8), (4.9).

6) L. H. Adams, *Beitr. z. Geophys.*, **31** (1931), 315. Afterwards he modified his theory a little. *cf. Journ. Franclin Inst.*, **208** (1929).

while the corresponding value calculated by means of (7.1) is

$$\frac{1}{k}=\frac{1}{k_0}\left[1+(1-\rho)\frac{3(1-K)(1-\sigma_0)}{2(1-2\sigma_0)+K(1+\sigma_0)}\right],\ (K=k_1/k_0) \qquad \dots\dots(7.3)$$

and these two expressions are generally not equal.

Equating the right hand members of (7.2) and (7.3) we can easily find the condition when Adams' simple theory strictly holds. This gives

$$(1-2\sigma_0)(1-K)^2=0 \qquad \dots\dots\dots\dots(7.4)$$

which is statisfied by

$$\sigma_0=1/2 \quad \text{or} \quad K=1 \qquad \dots\dots\dots\dots(7.5)$$

The former condition implies that the material of the matrix is incompressible, while the latter gives $k_0=k_1$, or the condition that the compressibilities of the pebble and the matrix are altogether equal.

8. Rigidity.

The equation $\Delta_2\begin{pmatrix}1, 2\\3, 4\end{pmatrix}=0$ gives the rigidity.

In Mackenzie's paper[7], when he calculates the bulk modulus, he arbitrarily applies the hydrostatic pressure, and when he aims at the rigidity, he uses the spherical solid harmonics of degree two, but without sufficient proof. Therefore one fears that another way of calculation using other types of stress distribution may give another result.

However, in this paper, we have employd a general type of solution, and arrived at the same result with the former paper in regard to the bulk modulus. Also with respect to the rigidity, we arrived at a similar conclusion, which means the method of applying the stress given by the spherical solid harmonics of degree two was justified.

We will now procced to the solution of (6.8).

$$\Delta_2\begin{pmatrix}1, 2\\3, 4\end{pmatrix}=0 \qquad \dots\dots\dots\dots(6.8 \text{ bis})$$

After some complicated calculations the above equation assumes the following form :

7) J. K. MACKENZIE, "The Elastic Constants of a Solid Containing spherical Holes," *Proc. Phys. Soc.*, **B63** (1950), 2.

$$\Delta_2\binom{1,\ 2}{3,\ 4}=\mu_0^4\cdot 50a^{-4}r_0^{-1}(-\alpha_{20}+2)\{-8\alpha_{21}+(3\alpha_{21}+10\chi_1)\}$$

$$\cdot\Big[-(\chi-1)\{(2\chi_1+3)(\alpha_{-30}-5)+5(2\chi_1+1)\}$$

$$+(\chi_1-1)\frac{a^3}{r_0^3}5(\alpha_{-30}-2)$$

$$+0(a^5/r_0^5)\Big]$$

$$=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.1)$$

in which

$$\alpha_{n0}=\frac{4(2n+1)\sigma_0-2(3n+1)}{-4\sigma_0+n+5}$$

$$\chi=\mu/\mu_0$$

$$\chi_1=\mu_1/\mu_0$$

σ_0 is Poisson's ratio of the matrix.

Since the first line of the expression (8.1) does not vanish, terms in [ ] must be equal to zero.

Therefore we have, as the first approximation, the following formula

$$(\chi-1)\{(2\chi_1+3)(\alpha_{-30}-5)+5(2\chi_1+1)\}=(\chi_1-1)\frac{a^3}{r_0^3}5(\alpha_{-30}-2)$$

or

$$\mu=\mu_0\left[1+(1-\rho)\frac{15(\chi_1-1)(1-\sigma_0)}{2\chi_1(4-5\sigma_0)+(7-5\sigma_0)}\right] \qquad \ldots\ldots\ldots\ldots(8.2)$$

If we put $\chi_1=0$ (, or $\mu_1=0$) into the above expression, we have

$$\mu=\mu_0\left[1+(1-\rho)\frac{-15(1-\sigma_0)}{7-5\sigma_0}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8.3)$$

This is equivalent to the formula introduced by Mackenzie[8] and the author[9], and applicable to the case of hollow or liquid sphere.

9. Other elastic constants.

We have obtained two independent elastic constants; i.e. k (cf. (7.1) or (7.3)) and μ (cf. (8.3)), from which we are able to deduce other elastic constants and the velocity of prepagation of waves with long wave-lengths.

The principle of calculation is very simple in each case. We will only show the results here.

8) *loc. cit.*, 7). Expression (19).

9) *loc. cit.*, 1). Expression (3.2).

$$\left\{\begin{aligned}
\lambda &= \lambda_0\Big[1+(1-\rho)\Big(-\frac{1-\sigma_0}{\sigma_0}\Big) \\
&\quad \cdot\Big\{\frac{(1-K)(1+\sigma_0)}{2(1-2\sigma_0)+K(1+\sigma_0)}+\frac{5(\chi_1-1)(1-2\sigma\ )}{2\chi_1(4-5\sigma_0)+(7-5\sigma_0)}\Big\}\Big] \quad \ldots(9.1) \\
\sigma &= \sigma_0\Big[1+(1-\rho)\Big(-\frac{(1-2\sigma_0)(1-\sigma_0^2)}{\sigma_0}\Big) \\
&\quad \cdot\Big\{\frac{1-K}{2(1-2\sigma_0)+K(1+\sigma_0)}+\frac{5(\chi_1-1)}{2\chi_1(4-5\sigma_0)+(7-5\sigma_0)}\Big\}\Big] \quad \ldots(9.2)
\end{aligned}\right.$$

On the other hand, density δ is

$$\delta=\delta_0[1-(1-\rho)(1-D)] \qquad \ldots\ldots\ldots\ldots\ldots(9.3)$$

in which δ_0 and $\delta_1 \equiv D\delta_0$ are the density of the matrix and the obstacle respectively.

From (8.2), (9.1) and (9.3) we have

$$\begin{aligned}
V_p = V_{p0}\Big[1+(1-\rho)\frac{1}{2}\Big\{(1-D) \\
+\frac{-(1-K)(1+\sigma_0)}{2(1-2\sigma_0)+K(1+\sigma_0)}+\frac{10(\chi_1-1)(1-2\sigma_0)}{2\chi_1(4-5\sigma_0)+(7-5\sigma_0)}\Big\}\Big] \quad \ldots.(9.4)
\end{aligned}$$

$$V_s = V_{s0}\Big[1+(1-\rho)\frac{1}{2}\Big\{(1-D)+\frac{15(\chi_1-1)(1-\sigma_0)}{2\chi_1(4-5\sigma_0)+(7-5\sigma_0)}\Big\}\Big] \quad \ldots(9.5)$$

as the propagation velocity of longitudinal and transverse waves in this composite medium.

10. Numerical examples.

10.1 When the obstacle is rigid.

When the embedded spherical body is rigid we must take

$$K=\infty, \qquad \chi_1=\infty. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(10.1)$$

Therefore, we have from (7.3), (8.2), (9.1), (9.2) and (9.4).

$$\left\{\begin{aligned}
k &= k_0\Big[1+(1-\rho)\frac{3(1-\sigma_0)}{1+\sigma_0}\Big], \\
\mu &= \mu_0\Big[1+(1-\rho)\frac{15(1-\sigma_0)}{2(4-5\sigma_0)}\Big], \\
\lambda &= \lambda_0\Big[1+(1-\rho)\frac{3(1-\sigma_0)}{2\sigma_0(4-5\sigma_0)}\Big], \qquad \ldots(10.2) \\
\sigma &= \sigma_0\Big[1+(1-\rho)\frac{(1-\sigma_0^2)(1-2\sigma_0)}{\sigma_0}\Big\{\frac{1}{1+\sigma_0}-\frac{5}{2(4-5\sigma_0)}\Big\}\Big], \\
V_p &= V_{p0}\Big[1+(1-\rho)\frac{1}{2}\Big\{\frac{3(3-5\sigma_0)}{4-5\sigma_0}+(1-D)\Big\}\Big],
\end{aligned}\right.$$

$$V_s = V_{sc}\left[1+(1-\rho)\frac{1}{2}\left\{\frac{15(1-\sigma_0)}{2(4-5\sigma_0)}+(1-D)\right\}\right].$$

These formulae will be applied when the bulk modules and the rigidity of the embedded obstacles are far larger than those of the matrix. It will be interesting to compare them with the formulae which are applicable to the case of vacant spherical holes[10].

Figs. 2a, b, c and d show the variation of k/k_0, μ/μ_0, λ/λ_0 and σ/σ_0 as functions of $(1-\rho)$. The parameter in the figures is Poisson's ratio σ_0 of the matrix.

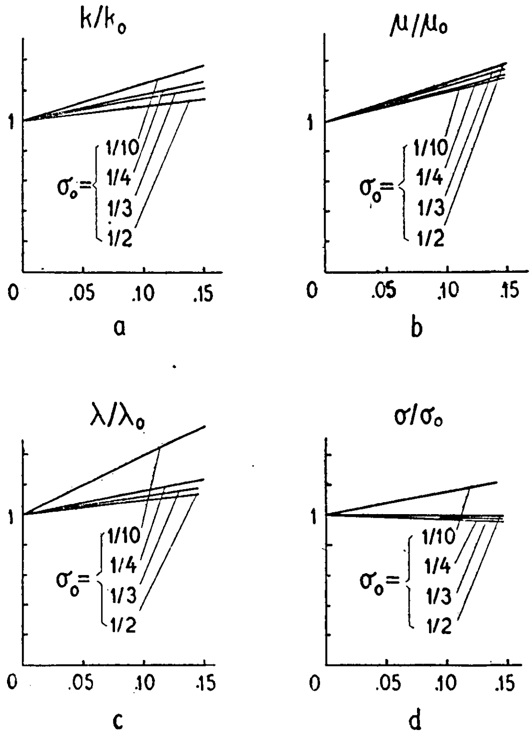

Fig. 2. Variation of k/k_0, μ/μ_0 and σ/σ_0 when the embedded obstacle is rigid. Abscissa is $1-\rho$, or the proportion of volume occupied by the obstacle.

10.2 When the holes are filled with water or ice.

Suppose a soft material with a sound velocity as small as

$$V_{p0}=500\ \mathrm{m/sec},$$

and

$$\delta_0=2.0\ \mathrm{gr/cm^3},\quad \sigma_0=1/4. \qquad \cdots\cdots(10.3)$$

Then

$$\lambda_0=\mu_0=1.667\times10^9\ \mathrm{dyne/cm^2}.$$

If water is contained in the holes of this material, we have

$$k_1=2.1\times10^{10}\ \mathrm{dyne/cm^2},\quad \delta_1=1\ \mathrm{gr/cm^3}, \qquad \cdots\cdots(10.4)$$

or combining with the above values we can easily obtain

$$K=k_1/k_0=75.6,\quad D=\delta_1/\delta_0=0.5 \qquad \cdots\cdots(10.5)$$

10) *loc. cit.*, 1). Expression (3.2).

Putting these numerical values into the expressions (7.1), (8.2), (9.1), (9.2), (9.4) and (9.5) we get

$$\left\{\begin{aligned} k&=2.778\times10^{9}\,[1+1.76(1-\rho)]\ \text{dyne/cm}^2,\\ \mu&=1.667\times10^{9}\,[1-1.96(1-\rho)]\ \text{dyne/cm}^2,\\ \lambda&=1.667\times10^{9}\,[1+4.23(1-\rho)]\ \text{dyne/cm}^2,\\ \sigma&=0.250\,[1+3.10(1-\rho)]\\ V_p&=500.0\,[1+0.303(1-\rho)]\ \text{m/sec},\\ V_s&=288.7\,[1-0.729(1-\rho)]\ \text{m/sec}. \end{aligned}\right. \qquad \cdots\cdots\cdots(10.6)$$

which are plotted by full lines in Figs. 3a, b, c, d, e and f.

If the water in holes freezes, we must introduce other values of k_1 and δ_1 instead of those given in (10,4). In this case we assume[11)]

$$\left\{\begin{aligned} V_{p1}&=3480\ \text{m/sec},\\ \delta_1&=1/1.09\ \text{gr/cm}^3,\\ \sigma_1&=1/3 \end{aligned}\right. \qquad \ldots(10.7)$$

Then we have

$$\left\{\begin{aligned} \lambda_1&=5.56\times10^{11}\ \text{dyne/cm}^2,\\ \mu_1&=2.78\times10^{11}\ \text{dyne/cm}^2. \end{aligned}\right.$$

Consequently

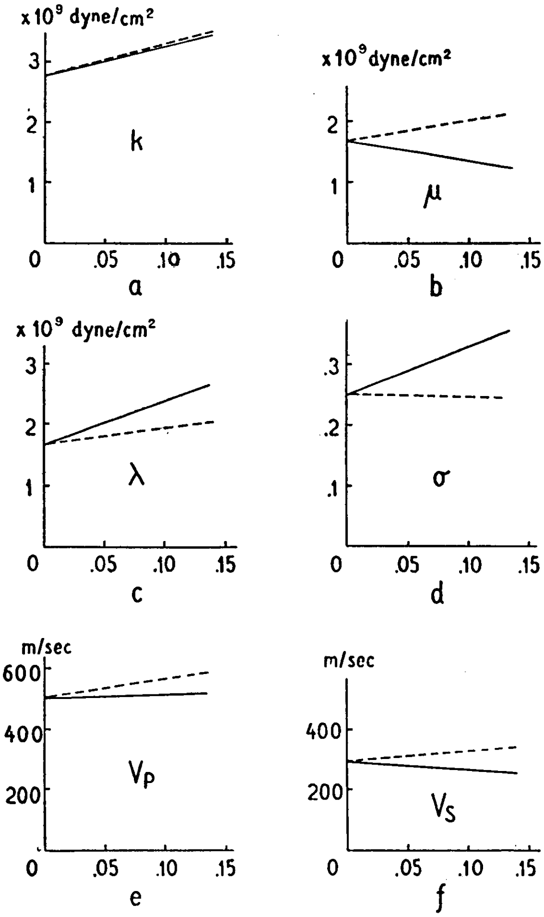

Fig. 3. Numerical values of the elastic constants k, μ, λ, σ and the propagation velocity V_p and V_s, assuming the material constants
$V_{p0}=500$ m/sec, $\delta_0=2.0$ gr/cm^3, $\sigma_0=1/4$.
($\lambda_0=\mu_0=1.667\times10^9$ dyne/cm^2)
Full line shows the case when the holes are filled with water
($k=2.1\times10^{10}$ dyne/cm^2, $\rho_1=1.000$ gr/cm^3),
while the broken line shows the case when the water was frozen.
$V_{p1}=3480$ m/sec,
$\delta_1=1/1.09$ gr/cm^3,
$\sigma_1=1/3$.
($\lambda_1=5.56\times10^{11}$ dyne/cm^2,
$\mu_1=2.78\times10^{11}$ dyne/cm^2)

11) Y. SATÔ, *Bull. Earthq. Res. Inst.*, **29** (1951), 223. See foot-note 17).

$$\left\{\begin{array}{l} \chi_1=\mu_1/\mu_0=167 \\ K=k_1/k_0=267 \\ D=\delta_1/\delta_0=0.459 \end{array}\right. \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (10.8)$$

Introducing (10.7) and (10.8) into the previous formulae, we obtain the following numerical expressions:

$$\left\{\begin{array}{l} k=k_0\ [1+1.79\ (1-\rho)] \\ \mu=\mu_0\ [1+2.02\ (1-\rho)] \\ \lambda=\lambda_0\ [1+1.60\ (1-\rho)] \\ \sigma=\sigma_0\ [1-0.218(1-\rho)] \\ V_p=V_{p0}[1+1.211(1-\rho)] \\ V_s=V_{s0}[1+1.281(1-\rho)] \end{array}\right. \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (10.9)$$

the values of which are plotted by broken lines in the same figure with the former case. The most remarkable difference is to be seen in Figs. 3b and 3d.

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1. 小さな異物を含む媒質内を伝はる彈性波の速さ

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前の論文においては,[1] 小さな球形の穴を持つ媒質の彈性常数を，Mackenzie[8] の結果を用ゐて求め，あはせて十分に長い波長の彈性波がつたはる場合の速さを近似的に出したのであった．なほ，穴の中に流体がつまってゐる場合にも公式を拡張したが，本論文では，この穴をうづめてゐる物質が，固体の時にはどのやうな式が得られるかといふ問題を扱った．Mackenzie の論文では，適当なモデルを考へた上で，体積彈性率を求める場合には外力として靜水圧を，剛性率を求める時には 2 次の球函数で与へられるやうな応力を仮定して来た．しかし，勝手にこのやうな函数形を採用してその結果えられた式が，果してどのやうな外力を加へた場合にも通用する，一般的な公式を与へ得るか否か，といふ事は必ずしも自明のことではない．

そこで，この論文では，まづ n 次の球函数で与へられる外力を仮定して，これに対する解を求め，次に，任意の外力を[2] 球函数に展開する事によって，一般の解をさきに得られた解の線型結合としてあらはした．その上で境界條件をあはせ (第 1 図参照), equivalent homogeneous continuum の考へに従って彈性常数を求めた．すると，体積彈性率を求める場合には結局靜水圧に対する項が，剛性率については 2 次の球函数の項が，ものをいってくる事がわかる．従って，先に Mackenzie が採用し，又我々が従って来た方法は正しいものであることが証明される (§ 8).

かくして得られる結果は，体積彈性率については，穴の中が流体の場合と同じである．((7.1), (7.3) 参照.) 剛性率について，2 次の球函数を用ゐて計算を行へば，途中は複雑であるが，最後には簡単な結果 (8.2) に到達する．この式で，内部の剛性を 0 とおけばさきに得られた公式と一致する[8], [9].
他の彈性常数は，k と μ とから容易に求められる．((9.1)~(9.5)参照)

混合物質の圧縮率は，それを構成する物質の圧縮率を体積の割合で平均すれば得られることの考へがあり,[7]一応もっとものやうであるが，嚴密にはなりたたないもののやうである，(§ 7.1)

数値計算の例として，異物が剛体の場合 (§ 5.1), 水の場合，それが氷で置きかへられた場合 (§ 5.2) を行った。結果は式 (10.2)，第2図 a, b, c, d, および式 (10.6), (10.9) と第3図 a, b, c, d, e, f に示してある. (§ 10)

以上の計算を行ふにあたって我我の採用した前提は前と同様で (§ 1),

1) 多数の異物を含む相当の体積をもった媒質は全体として一様な連続した媒質と同等であること，

2) 異物は同じ大きさの球であること，

3) 異物が全媒質内でしめる体積の割合は十分に小さく， その二次の項を省略してもさしつかへないこと，

4) 境界面では変位と応力が連続であること，

5) 弾性波の速さを考へる時には，波長が異物相互間の距離や直徑にくらべて十分に大きく，準靜的な取扱ひが許されること，

などである.