7. *Experimental Studies on the Mechanism of Generation of Elastic Waves II.*

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1. Introduction.

In the previous paper[1], the author gave a report of a noticeable phenomenon which can be seen in the vicinity of a wave source. When an impulsive vertical force acts downwards on the surface of an elastic medium, a part of the surface within a certain region begins in upward motion. The disturbance which occurs in such a way is propagated as time passes on, forming gradually a perfect type of the surface wave and other kind of waves. Such phenomenon seems rather curious from the standpoint of the theory of the statical deformation of elastic bodies. Because, as Boussinesq has studied[2], every part of the surface of a semi-infinite elastic body subjected to an external vertical force acting downwards at a point, is displaced downwards and no part showing upward displacements can exist on the surface. In fact, a statical experiment carried out on the same medium produces such results.

These facts can be considered to indicate that such phenomenon as mentioned above occurs only in the transient case of elastic deformation of the surface. Inouye has formerly pointed out that the surface of the ground subject to the shock of a falling body showed a peculiar distribution of initial motion[3].

Wave phenomena which would take place on the surface of a semi-infinite elastic body subjected to an external impulsive force have been investigated theoretically by H. Lamb[4]. His consideration, however, has been limited mainly to the problem on the phenomena which can be observed at a place on the surface distant enough from the origin. We have, to the present, scarcely any theoretical basis, to which we may refer when we undertake experimental investigations on the phe-

1) K. KASAHARA, *Bull. Earthq. Res. Inst.*, **30** (1952), 267.
2) A. E. H. LOVE, The Mathematical Theory of Elasticity, (1927), 185.
3) W. INOUYE, *Bull. Earthq. Res. Inst.*, **13** (1935), 194.
4) H. LAMB, *Phil. Trans. Roy. Soc.*, **A, 203** (1904), 1.

nomena in the vicinity of the origin. These circumstances led the author into the present study, which is considered to be of use in checking the reliability of the model experiment as well as in obtaining a practical way to study the theoretical basis of related phenomena.

2. Fundamental Considerations.

The two-dimensional problem is treated in the present paper, because it is rather easy to study and may be readily compared with the results obtained in the previous experiments. Wave equations in two-dimensions are

$$\left.\begin{aligned} \partial^2\phi/\partial t^2 &= \{(\lambda+2\mu)/\rho\}\nabla^2\phi, \\ \partial^2\psi/\partial t^2 &= (\mu/\rho)\nabla^2\psi, \end{aligned}\right\} \tag{1}$$

where ρ, λ and μ represent density and Lamé's constants, respectively, and ϕ and ψ are the wave functions which have the following relations with two components of displacements u and v.

$$\left.\begin{aligned} u &= (\partial\phi/\partial x)+(\partial\psi/\partial y), \\ v &= (\partial\phi/\partial y)-(\partial\psi/\partial x). \end{aligned}\right\} \tag{2}$$

Stress components become, then,

$$\left.\begin{aligned} P_{xy} &= \mu\{(\partial u/\partial x)+(\partial v/\partial y)\} = \text{func.}\ (\phi,\ \psi), \\ P_{yy} &= \lambda\Delta+2\mu(\partial v/\partial y) \qquad\quad = \text{func.}\ (\phi,\ \psi). \end{aligned}\right\} \tag{3}$$

Let us consider, now, the case of a semi-infinite elastic body, bounded by the plane $y=0$ (Fig. 1), which is subjected to a vertical force as follows,

$$\left.\begin{aligned} [P_{xy}]_{y=0} &= 0, \\ [P_{yy}]_{y=0} &= Y\exp{(i\xi x)}\cdot\mathbf{1}, \end{aligned}\right\} \tag{4}$$

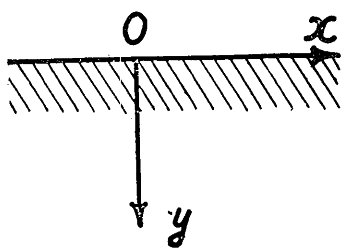

Fig. 1. Two-dimensional coordinates employed in the present study.

where $\mathbf{1}$ is the unit function, employed in operational calculus. Then the equation (1) can be written in the operational form, viz.:

$$\begin{aligned} a^2p^2\phi-\nabla^2\phi=0, \\ b^2p^2\psi-\nabla^2\psi=0, \end{aligned} \tag{5}$$

where
$$a^2=\rho/(\lambda+2\mu), \quad b^2=\rho/\mu, \tag{6}$$

and we may consider as the solutions for (5)

$$\left.\begin{aligned} \phi=A\exp(-\alpha y)\exp(i\xi x)\cdot\mathbf{1}, \\ \psi=B\exp(-\beta y)\exp(i\xi x)\cdot\mathbf{1}. \end{aligned}\right\} \tag{7}$$

In this case, we obtain from (5) the following relations,

$$\left.\begin{aligned} \{\alpha^2-(\xi^2+h^2)\}\cdot\mathbf{1}=0, \\ \{\beta^2-(\xi^2+k^2)\}\cdot\mathbf{1}=0, \end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} h^2=a^2p^2, \\ k^2=b^2p^2. \end{aligned}\right\} \tag{9}$$

Surface displacements become, then,

$$\left.\begin{aligned} u_0=(i\xi A-\beta B)\cdot\exp(i\xi x)\cdot\mathbf{1}, \\ v_0=(-\alpha A-i\xi B)\cdot\exp(i\xi x)\cdot\mathbf{1}, \end{aligned}\right\} \tag{10}$$

and stress components on the surface are

$$\left.\begin{aligned} [P_{xy}]_0=\mu\{-2i\xi\alpha A+(\xi^2+\beta^2)B\}\exp(i\xi x)\cdot\mathbf{1}, \\ [P_{yy}]_0=\mu\{(\xi^2+\beta^2)A+2i\xi\beta B\}\exp(i\xi x)\cdot\mathbf{1}. \end{aligned}\right\} \tag{11}$$

To satisfy the initial conditions (4) given for surface stresses, constants A and B in the above relations should be

$$\begin{aligned} A=\{(2\xi^2+k^2)/F(\xi)\}(Y/\mu), \\ B=\{2i\xi\alpha/F(\xi)\}(Y/\mu), \end{aligned} \tag{12}$$

where
$$F(\xi)=(2\xi^2+k^2)^2-4\xi^2\alpha\beta. \tag{13}$$

Therefore, both components for surface displacements can be written as follows,

$$\left.\begin{aligned} u_0=[i\xi\{(2\xi^2+k^2)-2\alpha\beta\}/F(\xi)]\,(Y/\mu)\exp(i\xi x)\cdot\mathbf{1}, \\ v_0=[\qquad -\alpha k^2/F(\xi) \qquad]\,(Y/\mu)\exp(i\xi x)\cdot\mathbf{1}. \end{aligned}\right\} \tag{14}$$

3. Numerical Considerations.

Let us extend our numerical considerations to the case of external force acting in the following form,

$$\left.\begin{array}{ll} 0 & \text{when } t<0 \\ Q\,H_{2m}(x)\exp(-x^2/2) & \text{when } t>0\,, \end{array}\right\} \tag{15}$$

where $H_{2m}(x)$ represents Hermite's functions of $2m$ mode. The above form may be written as

$$Q(-1)^m(2/\pi)^{1/2}\int_0^\infty H_{2m}(\xi)\cdot\exp(-\xi^2/2)\cos\xi x d\xi, \tag{16}$$

then equation (14) becomes

$$\left.\begin{aligned} u_0 &= \boldsymbol{R}(Q/\mu)(-1)^m(2/\pi)^{1/2}\int_0^\infty\{i\xi(2\xi^2+k^2-\alpha\beta)/F(\xi)\}H_{2m}(\xi) \\ &\qquad\times\exp\{(-\xi^2/2)+i\xi x\}d\xi\cdot\mathbf{1}, \\ v_0 &= \boldsymbol{R}(Q/\mu)(1-1)^m(2/\pi)^{1/2}\int_0^\infty\{-k^2\alpha/F(\xi)\}H_{2m}(\xi) \\ &\qquad\times\exp\{(-\xi^2/2)+i\xi x\}d\xi\cdot\mathbf{1}, \end{aligned}\right\} \tag{17}$$

where $\boldsymbol{R}$ represents the real part of the complex.
When $m=0$,

$$\begin{aligned} v_0 &= \boldsymbol{R}(Q/\mu)(2/\pi)^{1/2}\int_0^\infty\{-k^2\alpha/F(\xi)\}\exp\{(-\xi^2/2)+i\xi x\}d\xi\cdot\mathbf{1} \\ &= -(Q/\mu)(2/\pi)^{1/2}\int_0^\infty\{k^2\alpha/F(\xi)\}\exp(-\xi^2/2)\cos\xi x d\xi\cdot\mathbf{1}. \end{aligned} \tag{18}$$

For the numerical evaluation of the surface displacement, the above expression is expanded in series of p^{-n}, viz.,

$$\begin{aligned} v_0 = -(Q/\mu)(2/\pi)^{1/2}(a/b^2)[p^{-1}&\int_0^\infty \exp(-\xi^2/2)\cos\xi x d\xi \\ +A'p^{-3}&\int_0^\infty \xi^2\exp(-\xi^2/2)\cos\xi x d\xi \\ -B'p^{-5}&\int_0^\infty \xi^4\exp(-\xi^2/2)\cos\xi x d\xi \\ +C'p^{-7}&\int_0^\infty \xi^6\exp(-\xi^2/2)\cos\xi x d\xi \\ -D'p^{-9}&\int \xi^8\exp(-\xi^2/2)\cos\xi x d\xi+\ldots]\cdot\mathbf{1}. \end{aligned} \tag{19}$$

Then we can replace operators p^{-1}, p^{-3},... by t, t^3,... in the following way,

$$p^{-n}\cdot\mathbf{1}=t^n/n!\,, \tag{20}$$

and finally, the surface displacement can be expressed in finite series of t when t does not assume a large value, viz.,

$$\begin{aligned} v_0 = &-(Q/\mu)(c_2^2/c_1)[t\phi_0(x) - (1/3!\cdot 2)\{A + (c_1^2/2)\}t^3\phi_2(x) \\ &+ (1/5!\cdot 4)\{B + (c_1^2A/2) - (c_1^4/8)\}t^5\phi_4(x) \\ &- (1/7!\cdot 8)\{C + (c_1^2B/2) - (c_1^4A/8) + (c_1^6/16)\}t^7\phi_6(x) \\ &+ (1/9!\cdot 16)\{D + (c_1^2C/2) - (c_1^4B/8) + (c_1^6A/16) - (5c_1^8/128)\}t^9\phi_8(x) + \ldots] \end{aligned} \tag{21}$$

where each constant in the above expression means,

$$\left.\begin{aligned} &A = -4c_2^2\{1 + (c_2/c_1)\}, \\ &B = 2c_2^3\{1 - (c_2/c_1)\}^2(-c_1 + 8c_2), \\ &C = (c_2^3/2)\{1 - (c_2/c_1)\}^2[-c_1^3\{1 + (c_2/c_1)\}^2 + 32c_1c_2^2\{1 - (c_2/c_1)\} \\ &\qquad - 128c_2^3\{1 - (c_2/c_1)\}], \\ &D = (c_2^3/4)\{1 - (c_2/c_1)\}^2[c_1^5\{1 + (c_2^2/c_1^2)\}\{1 + (c_2/c_1)\}^2 \\ &\qquad + 16c_1^2c_2^3\{1 + (c_2/c_1)\}^4\{1 - (c_2/c_1)\}^2 \\ &\qquad - 384c_1c_2^4\{1 + (c_2/c_1)\}^4\{1 - (c_2/c_1)\}^2 \\ &\qquad + 1024c_2^5\{1 + (c_2/c_1)\}^4\{1 - (c_2/c_1)\}^2], \\ &c_1 = 1/a, \quad c_2 = 1/b, \\ &\text{and} \\ &\phi_n(x) = H_n(x/\sqrt{2})\cdot\exp(-x^2/2).^{5)} \end{aligned}\right\} \tag{22}$$

Equation (21) is one of the fundamental results which clarifies the transient deformation of the surface subjected to the external force expressed by (4) and (15). When we expand (18) in series of p^n and take $t=\infty$, we can obtain the surface deformation in statical case. Equation (23) is the result which shows no upward displacement of the surface when the external force acts downwards,

$$v_0 = (-Q/2\mu)\{(\lambda + 2\mu)/(\lambda + \mu)\}\cdot\exp(-x^2/2). \tag{23}$$

The present problem may be advanced by replacing the time factor of the vertical force (15) with the impulsive one, viz.,

$$\left.\begin{aligned} &0 &&\text{when } t<0 \text{ and } t>(\pi/a), \\ &QH_{2m}(x)\exp(-x^2/2)\sin at &&\text{when } 0\leqq t\leqq(\pi/a). \end{aligned}\right\} \tag{24}$$

It this case, the fundamental relation (25) in the operational calculus can be applied to obtain the expression for the vertical displacement of the surface

$$g_2(p)f_1(t) = \frac{d}{dt}\int_0^t f_2(t-\tau)f_1(\tau)d\tau, \tag{25}$$

and the following expression is obtained as the result:

5) T. RIKITAKE, *Bull. Earthq. Res. Inst.*, **30** (1952), 293.

$$
\begin{aligned}
v_0 = & (-Q/\mu)(c_2^2/c_1)[(1/a^2)\{2-\cos at-\cos(at-\pi)\}\phi_0(x) \\
& -(1/3!\cdot 2)\{A+(c_1^2/2)\}(1/a^4)\{2(3a^2t^2-6)+6\cos at+6\cos(at-\pi)\}\phi_2(x) \\
& +(1/5!\cdot 4)\{B+(c_1^2A/2)-(c_1^4/8)\}(1/a^6)\{2(5a^4t^4-60a^2t^2+120) \\
& \qquad -120\cos at-120\cos(at-\pi)\}\phi_4(x) \\
& -(1/7!\cdot 8)\{C+(c_1^2B/2)-(c_1^4A/8)+(c_1^6/16)\}(1/a^8) \\
& \quad \times\{2(7a^6t^6-210a^4t^4+2250a^2t^2-5040)-5040\cos at+5040\cos(at-\pi)\}\phi_6(x) \\
& +(1/9!\cdot 16)\{D+(c_1^2C/2)-(c_1^4B/8)+(c_1^6A/16)-(5c_1^8/128)\}(1/a^{10}) \\
& \quad \times\{2(9a^8t^8-504a^6t^6+15120a^4t^4-181440a^2t^2+362880) \\
& \qquad -362880\cos at-362880\cos(at-\pi)\}\phi_8(x),
\end{aligned}
\tag{26}
$$

where $A, B, C, \ldots$ have the identical meaning with those employed in the former case.

When we undertake numerical calculation for (21) and (26), it is desirable to adopt suitable values for constants and parameters in the expression so that they may fit the conditions, under which the previous experiment had been undertaken. Values thus determined are listed in Table I, in comparison with those employed in the experiment[6].

Table I.

	Calculation	Experiment
Velocity of longitudinal wave	$3c$	610 cm/sec
Velocity of transversal wave	c	240 cm/sec
Poisson's ratio	0.48	0.41
Unit distance	x	5 cm
Unit time	$\tau=(x/c)$	20 m sec
Time interval between each observation	$\frac{1}{8}$ or $\frac{1}{4}\tau$	5 m sec
Duration time of the impulsive force	$\frac{1}{2}\tau$[7]	10 m sec
Width of the impulsive force	$5x$[8]	0.6 cm

To keep the accuracy of the calculation as high as possible the value of t in (21) and (26) should not exceed a certain limit and the final results thus obtained are illustrated in Figs. 2 and 3.

Fig. 4 is the result of the previous experiment drawn for the purpose of comparison.

6) *loc. cit.*

7) This indicates the duration time of force acting like (24).

8) This indicates the dimension of the region, in which the force can be considered acting effectively.

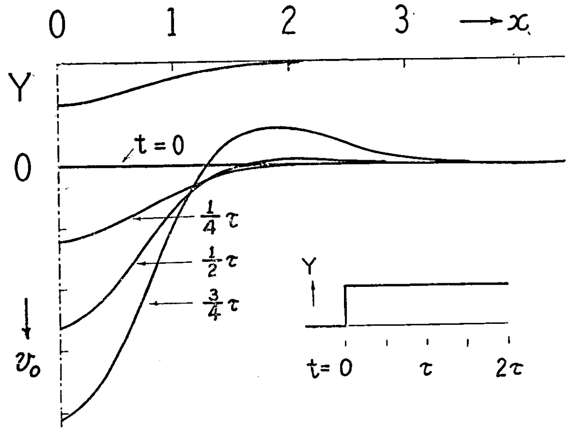

Fig. 2. Results of the numerical calculations. (1).
From top: spatial distribution of the external force, displacements of the surface at each instant, and time characteristics of the force.

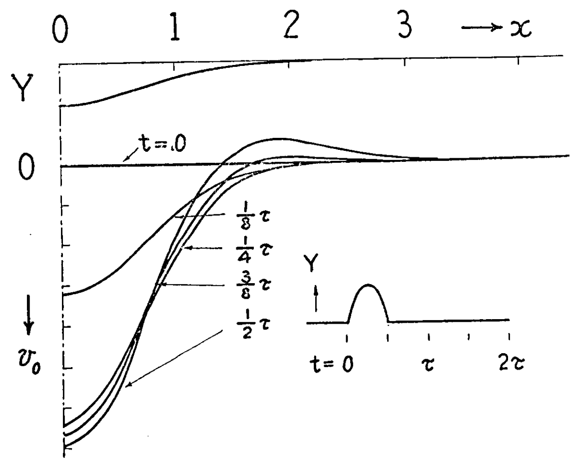

Fig. 3. Results of the numerical calculations (2).
Curves are drawn similarly with Fig. 2.

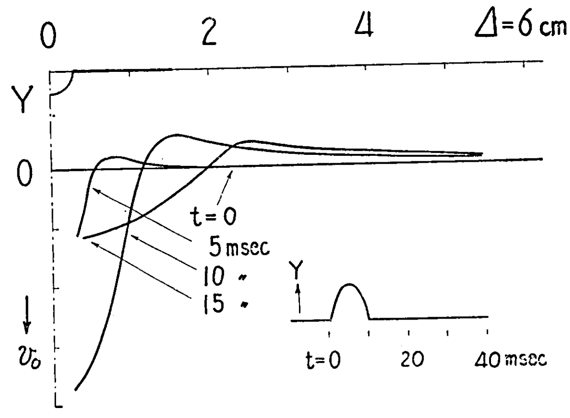

Fig. 4. Results of the previous experiments.
Curves are drawn similarly with Fig. 2.

4. Discussions and Conclusions.

In Figs. 2 and 3 we observe remarkable swelling up occurring in a certain region near to the origin on which the external force is acting downwards. As time passes on, the swelling becomes higher and wider within the period in which the present calculation is carried on. These features are quite similar in their characters with those observed through experiments. We cannot, however, overlook a characteristic discrepancy seen between results of experiments and calculations. In spite of the extinguished travelling away of the peak of the swelling in Fig. 4, the former two do not show such features. The discrepancy results possibly from the wide distribution of the external force expressed by (15), that is, as can be seen from Table I, the effective width of the force amounts to more than several ten times of that employed in the experiments and this condition seems to have caused the above-stated discrepancy. The speculation is expected to be checked by the progressed calculation employing more sharply concentrated force in place of (15), but the study will be postponed to some future occassion.

We have, in the present paper, studied mathematically the possibility of swelling up of the surface subjected to the impulsive vertical force. The method and the result may be considered to be useful to study related problems in future.

These kinds of phenomena are caused probably by the concentration and releasing of local stresses produced by the external force, but their physical mechanism is left unknown, in the present step.

5. Acknowledgements.

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7. 弾性波の発生機構に関する実験 (第2報)

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半無限弾性体の表面に垂直衝撃力を下向きに作用させると，ある範囲内の表面は上向きに動き始めることが，第1報の模型実験において観察された．半無限体の波動現象のうち，原点附近の問題は理論的にも従来殆んど取扱われていないが，近似的にせよ之を取扱う方法を見出す事は今後の研究をすゝめる上にも無意味であるまい．

本報においては H. LAMB の展開した方法に沿いつゝ，之を演算子法的に取扱っている．二次元の半無限体において原点を中心に垂直衝撃力が作用した時の表面の垂直変位が主な対象であり，実例として，ごく簡単な條件の下で解いた数値計算の結果を図示してある．それによると原点近傍ある範囲の表面は明らかに盛上りの傾向を示す．時と共に，この範囲は遠方に及ぶが，模型実験の際観られたような“ピーク”の移動は明瞭でない．この差異は，計算の際仮定した外力の空間的分布が実験條件に比べて余りに広すぎた為ではないかと思われる．