

16. Velocity of Elastic Waves Propagated in Media with Small Holes.

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1. Introduction.

Today, the theory of elastic waves is generally discussed under the assumption of isotropy and homogeneity. However, the actual material which we ordinarily observe or treat at the field or observatory has rarely such simple properties. For example, it commonly contains small obstacles or holes which are sometimes empty, but more often filled with water. These obstacles or holes exist scattered randomly throughout the material and after all it shows apparent isotropic and homogeneous property. Thus the theory treating such heterogeneous medium becomes necessary, and one attempt was skillfully performed by J. K. Mackenzie¹⁾ concerning a solid containing spherical empty holes. In this paper, we will introduce his theory briefly, and after that adapt it to the case when the holes are filled up with liquid.

As our purpose exists in the application to the theory of elastic waves, mere calculations of apparent statical elastic properties would be insufficient. Fortunately, however, the waves we observe have far longer wave-lengths compared with the radii of holes or the intervals of them. Consequently we may neglect the effect of reflection or diffraction and apply the quasi-static theory in calculating the propagation velocity of waves by means of bulk modulus and rigidity deduced from statical treatment. In fact the observed velocity of seismic waves is not that of real material with neither crevice nor obstacle, but that of a medium porous and heterogeneous.

2. Notations used in this paper.

In this section we will give some notations and fundamental quantities used in the following articles.

ρ Relative density of actual material (when holes are empty).

1) J. K. MACKENZIE, "The Elastic Constants of a Solid Containing Spherical Holes." *Proc. Phys. Soc.*, B 63 (1950), 2.

- The porosity is expressed by $1-\rho$.
- a Radius of a hole.
- r_0 Every hole of radius a is surrounded by a spherical shell of real material out to the radius r_0 .
- Therefore $n \cdot \left(\frac{4}{3}\pi r_0^3\right) = \frac{1}{\rho}$ (n is defined below.)
- $\delta, \delta_0, \delta_1$ Density of actual material, real material and of the liquid fulfilling holes respectively.
- $\lambda, \mu; \lambda_0, \mu_0$ Lamé's constants of actual material which are to be calculated, and those of real material.
- k, k_0, k_1 Bulk modulus of actual material, real material and of the liquid filling holes respectively.
- σ, σ_0 Poisson's ratio of actual and real material respectively.
- V_S, V_{S0} Velocity of S-waves propagated in actual and real material respectively.
- V_P, V_{P0} Velocity of P-waves propagated in actual and real material respectively.
- n Number of holes per unit volume of real material.
- νn Number of holes filled up with liquid per unit volume of real material.
- K Ratio of the bulk modulus of the liquid k_1 to that of the real material k_0 . ($=k_1/k_0$)
- D Ratio of the density of the liquid δ_1 to that of the real material δ_0 . ($=\delta_1/\delta_0$)

The next relations are easily obtained from the above definition.
when holes are empty $\delta = \rho \delta_0$ (2.1)

$$\frac{1}{\rho} = 1 + (1 - \rho) \quad \text{.....(2.2)}$$

In the following calculations we will always neglect the square term of $(1-\rho)^2$.

3. Mackenzie's theory concerning a medium with spherical holes.²⁾

J. K. Mackenzie discusses briefly the problem of finding the elastic constants of a solid containing spherical holes as follows. He assumes that a hole of radius a is surrounded by a spherical shell of real material out to radius r , and these in turn are surrounded by a

2) This method seems to be first indicated by FRÖHLICH and SACK (*Proc. Roy. Soc., A.* **185** (1946), 415), but unfortunately the author cannot read this paper in his country.

spherical shell of equivalent homogeneous material out to some large radius R . A particular stress is applied to this outer spherical boundary and the displacements of the outer surface are calculated. Finally, the resulting displacement is equated to the displacement which would have resulted had the sphere of radius R been completely filled with equivalent homogeneous material. (cf. Fig. 1)³⁾

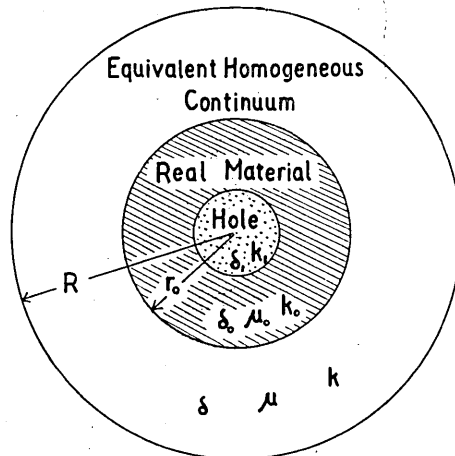


Fig. 1.

Although his method is neat he assumes that the holes are all empty. This assumption is not sufficient for us who want to apply the results to the problem of propagation of waves in rock or soil which is often moist or contains large number of small obstacles.

His results calculated under the said assumptions are⁴⁾

$$\begin{cases} \frac{1}{k} = \frac{1}{k_0 \rho} + \frac{3(1-\rho)}{4\mu_0 \rho} + 0[(1-\rho)^3], \\ \frac{\mu_0 - \mu}{\mu_0} = 5(1-\rho) \frac{3k_0 + 4\mu_0}{9k_0 + 8\mu_0} + 0[(1-\rho)^2] \end{cases} \dots\dots\dots(3.1)$$

from which we can deduce other elastic constants and the velocity of waves propagated in this medium.

Introducing a non-dimensional quantity $\sigma_0 \equiv \lambda_0/2(\mu_0 + \mu)$, which is known as Poisson's ratio, we have easily obtained the following formulae.

$$\begin{cases} k = k_0 \left\{ 1 - (1-\rho) \frac{3}{2} \frac{1-\sigma_0}{1-2\sigma_0} \right\} \\ \mu = \mu_0 \left\{ 1 - (1-\rho) \frac{15}{7-5\sigma_0} \frac{1-\sigma_0}{1-2\sigma_0} \right\} \\ \lambda = \lambda_0 \left\{ 1 - (1-\rho) \frac{1-\sigma_0}{\sigma_0} \left[\frac{1}{2} \frac{1+\sigma_0}{1-2\sigma_0} - 5 \frac{1-2\sigma_0}{7-5\sigma_0} \right] \right\} \\ \sigma = \sigma_0 \left\{ 1 - (1-\rho) \frac{(1-2\sigma_0)(1-\sigma_0^2)}{\sigma_0} \left[\frac{1}{2(1-2\sigma_0)} - \frac{5}{7-5\sigma_0} \right] \right\} \end{cases} \dots\dots(3.2)$$

3) This figure is same with that of Mackenzie.

4) *loc. cit.*, 1). Expressions (7) and (19).

Substituting (2.2),

$$\begin{cases} V_s = V_{s0} \left\{ 1 - (1-\rho) \frac{4-5\sigma_0}{7-5\sigma_0} \right\} \\ V_P = V_{P0} \left\{ 1 - (1-\rho) \left[\frac{1}{6} \frac{-1+8\sigma_0}{1-2\sigma_0} + 5 \frac{1-2\sigma_0}{7-5\sigma_0} \right] \right\} \end{cases} \dots\dots\dots (3.3)$$

If the Poisson's ratio $\sigma_0=1/4$, or $\lambda_0=\mu_0$

$$\begin{cases} k/k_0 = 1 - 2.25(1-\rho) \\ \mu/\mu_0 = 1 - 1.957(1-\rho) \\ \lambda/\lambda_0 = 1 - 2.445(1-\rho) \\ \sigma \cdot 4 = 1 - 0.270(1-\rho) \\ V_s/V_{s0} = 1 - 0.478(1-\rho) \\ V_P/V_{P0} = 1 - 0.768(1-\rho) \end{cases} \dots\dots\dots (3.4)$$

and if $\sigma_0=1/3$, or $\lambda_0=2\mu_0$

$$\begin{cases} k/k_0 = 1 - 3(1-\rho) \\ \mu/\mu_0 = 1 - 1.875(1-\rho) \\ \lambda/\lambda_0 = 1 - 3.375(1-\rho) \\ \sigma \cdot 3 = 1 - 0.333(1-\rho) \\ V_s/V_{s0} = 1 - 0.4375(1-\rho) \\ V_P/V_{P0} = 1 - 0.865(1-\rho) \end{cases} \dots\dots\dots (3.5)$$

When the real material is incompressible, or $\sigma_0=1/2$, the process of deducing the expressions (3.4) and (3.5) from (3.2) fails to be correct and some of the results do not hold, so we will start from the expression (3.1).

Substituting $\sigma_0=1/2$, or $k_0=\infty$

$$\begin{cases} \frac{1}{k} = \frac{3(1-\rho)}{4\mu_0\rho} \\ \frac{\mu_0-\mu}{\mu_0} = \frac{5}{3}(1-\rho) \end{cases} \dots\dots\dots (3.6)$$

from which we have

$$\left\{ \begin{array}{l} k = \frac{4}{3} \mu_0 \frac{\rho}{1-\rho} \\ \mu = \mu_0 \left[1 - \frac{5}{3}(1-\rho) \right] \\ \sigma = \frac{1}{2} \left[1 - \frac{3}{4}(1-\rho) \right] \\ V_s = V_{s0} \left[1 - \frac{1}{3}(1-\rho) \right] \\ V_p = V_{p0} \sqrt{\frac{4}{3(1-\rho)}} \end{array} \right. \dots\dots\dots(3.7)$$

In every case, the propagation velocity becomes smaller from the effect of holes.

4. Theory on the media with holes filled with liquid.

Now, we will calculate the elastic constants and the velocity of waves in the substance with holes filled with liquid.

The structure of the model used for the calculation of this case is same with that of Mackenzie's except the fact that the inner hole is filled with a liquid of density δ_1 and bulk modulus k_1 . (cf. Fig. 1)

Thus the method by which the formulae (3.1) were obtained is applicable to this case also. Moreover, the rigidity of the medium is not affected by the existence of liquid in the holes. Therefore, it is sufficient for us to obtain the apparent decrease of the volume of the actual substance when the hydrostatic pressure is applied to the outer spherical boundary.

Conditions which must hold at the boundary surfaces are:

$$\left\{ \begin{array}{l} r=R; \quad [\widehat{rr}]_R = -P \\ r=r_0; \quad [\widehat{rr}]_{r_0+0} = [\widehat{rr}]_{r_0-0}, \quad [u]_{r_0+0} = [u]_{r_0-0} \dots\dots\dots(4.1) \\ r=a; \quad [\widehat{rr}]_{a+0} = [\widehat{rr}]_{a-0}, \quad [u]_{a+0} = [u]_{a-0} \end{array} \right.$$

On the other hand, the general solutions of the three parts are—in equivalent homogeneous continuum;

$$\left\{ \begin{array}{l} u = Ar + Ar^{-2} \\ \widehat{rr} = 3kA - 4\mu Br^{-3} \end{array} \right. \dots\dots\dots(4.2)$$

$$\text{in real material; } \left\{ \begin{array}{l} u = A_0 r + B_0 r^{-2} \\ \widehat{rr} = 3k_0 A_0 - 4\mu_0 B_0 r^{-3} \end{array} \right. \dots\dots\dots(4.3)$$

$$\text{in liquid; } \begin{cases} u = A_1 r \\ \widehat{r r} = 3k_1 A_1 \end{cases} \dots\dots\dots (4.4)$$

Therefore from (4.1) we have

$$\begin{pmatrix} 3k & -4\mu R^{-3} & 0 & 0 & 0 \\ 3k & -4\mu r_0^{-3} & -3k_0 & 4\mu_0 r_0^{-3} & 0 \\ r_0 & r_0^{-2} & -r_0 & -r_0^{-2} & 0 \\ 0 & 0 & 3k_0 & -4\mu_0 a^{-3} & -3k_1 \\ 0 & 0 & a & a^{-2} & -a \end{pmatrix} \begin{pmatrix} A \\ B \\ A_0 \\ B_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} -P \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \dots\dots (4.5)$$

Solving these equations with respect to A , B , A_0 , B_0 and A_1 and putting them into the expression (4.2), we can easily calculate the amount of displacement at the surface $r=R$, namely

$$[u]_R = (-P/3k) \cdot R + \{ -(k-k_0)(3k_1+4\mu_0) - (k_0-k_1)(3k+4\mu_0)(1-\rho) \} \dots\dots (4.6)$$

which must be equal to the displacement in case the same pressure had been applied to the sphere of radius R filled with equivalent homogeneous substance with a bulk modulus k .

Therefore

$$[u]_R = -(P/3k)R \dots\dots\dots (4.7)$$

Combining (4.6) and (4.7) we get

$$k = k_0 - (k_0 - k_1) \frac{3k_0 + 4\mu_0}{3k_1 + 4\mu_0} (1 - \rho) \dots\dots\dots (4.8)$$

This is the very formula which we have intended to obtain in this calculation.

(4.8) can be expressed in the following form using Poisson's ratio σ_0 of the real material and $K = k_1/k_0$.

$$k = k_0 \left[1 - (1 - \rho) \frac{3(1 - K)(1 - \sigma_0)}{2(1 - 2\sigma_0) + K(1 + \sigma_0)} \right] \dots\dots\dots (4.9)$$

Since the rigidity suffers no effect by the existence of the liquid (3.2) holds in this case, too. Combining these two formulae we have

$$\left\{ \begin{aligned} \lambda &= \lambda_0 \left\{ 1 - (1-\rho) \frac{1-\sigma_0}{\sigma_0} \left[\frac{(1+\sigma_0)(1-K)}{2(1-2\sigma_0)+K(1+\sigma_0)} - 5 \frac{1-2\sigma_0}{7-5\sigma_0} \right] \right\} \\ \sigma &= \sigma_0 \left\{ 1 - (1-\rho) \frac{(1-2\sigma_0)(1-\sigma_0^2)}{\sigma_0} \left[\frac{1-K}{2(1-2\sigma_0)+K(1+\sigma_0)} - \frac{5}{7-5\sigma_0} \right] \right\} \\ V_s &= V_{s0} \left\{ 1 - (1-\rho) \frac{1}{2} \left[15 \frac{1-\sigma_0}{7-5\sigma_0} - (1-D) \right] \right\} \dots\dots\dots(4.10) \\ V_p &= V_{p0} \left\{ 1 - (1-\rho) \frac{1}{2} \left[\frac{(1-K)(1+\sigma_0)}{2(1-2\sigma_0)+K(1+\sigma_0)} + 10 \frac{1-2\sigma_0}{7-5\sigma_0} - (1-D) \right] \right\} \\ \delta &= \delta_0 \{ 1 - (1-\rho)(1-D) \} \end{aligned} \right.$$

If we put $K=0$ and $D=0$ in the above formulae, we obtain expressions equivalent to (3.2) and (3.3) which are the case of empty holes.

5. General considerations concerning the velocity in heterogeneous medium.

First let us assume a medium composed of two kinds of substances denoted by A and B . If these two substances are mixed by the ratio $\rho_A:\rho_B$ in a volume, an arbitrary straight line taken in this medium will be occupied by A and B at the same ratio $\rho_A:\rho_B$. For, in an

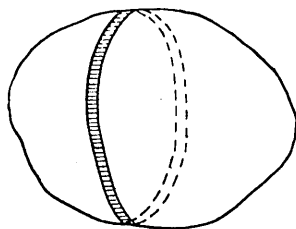


Fig. 2a.

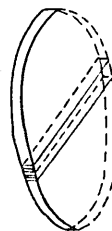


Fig. 2b.



Fig. 2c.

arbitrary sheet cut out of this mixture (cf. Figs. 2a and 2b) A and B are involved by the ratio $\rho_A:\rho_B$. Further, if a thin rod is again cut out of the said sheet, in this rod also A and B are mixed by the ratio $\rho_A:\rho_B$. If we denote the propagation velocity in A and B by V_A and V_B , the next doubt naturally arises. Can the mean velocity \bar{V} in this mixed substance be expressed by the following formula?

$$\frac{1}{\bar{V}} = \frac{\rho_A}{\rho_A + \rho_B} \frac{1}{V_A} + \frac{\rho_B}{\rho_A + \rho_B} \frac{1}{V_B} \dots\dots\dots(5.1)$$

This relation may hold if the wave-length is far smaller compared with the dimension of the mass of A or B . However, it is not certain whether the same relation holds or not when the wave-length is extremely large. Now that we have solved the problem under the latter assumption in the previous section, we can easily examine the validity of the above formula in this case.

We assume the substance A to be an elastic medium with Lamé's constants λ_0 and μ_0 , and B a liquid with the bulk modulus k_1 .

Further we put

$$\begin{cases} V_A = V_{P0}, & V_B = V_{P1}, \\ \rho_A = \rho, & \rho_B = 1 - \rho, \end{cases} \dots\dots\dots (5.2)$$

then the right hand member of (5.1) becomes

$$\frac{\rho}{V_{P0}} + \frac{1-\rho}{V_{P1}} = \frac{1}{V_{P0}} \left\{ 1 + (1-\rho) \left[-1 + 1/V \left(\frac{K}{D} \frac{1}{3} \frac{\sigma_0}{1-\sigma_0} \right) \right] \right\} \dots (5.3)$$

which differs clearly from the value expected from the expression of V_P in (4.10), namely

$$\frac{1}{V_{P0}} \left\{ 1 + (1-\rho) \left[\frac{1}{6} \frac{-1+8\sigma_0}{1-2\sigma_0} + 5 \frac{1-2\sigma_0}{7-5\sigma_0} \right] \right\} \dots\dots\dots (5.4)$$

Therefore the above doubt has been negatively cleared up.

6. When the holes are partly filled by the liquid.

When the holes are not entirely filled with the liquid, we can suppose two cases. One case is when, as illustrated in Fig. 3a, all the holes are partly filled with the liquid with some empty parts remaining,

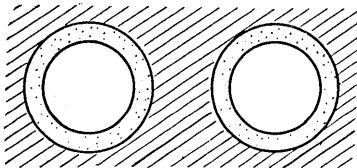


Fig. 3a.

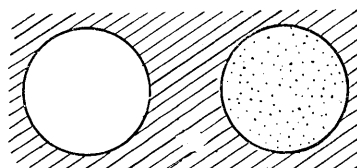


Fig. 3b.

and the other when some of the holes are entirely filled up with the liquid, with the rest remaining empty (cf. Fig. 3b). Of course, we cannot decide which of these two cases really exists in natural conditions.

If there remains some empty part, as in the first model, the existence of the liquid does not affect the amount of the bulk modulus, the only effect being to increase the density.

In the second model, the effect due to empty holes (—the theory of this case was developed by Mackenzie)—and that due to holes filled up with some liquid, must be summed up.

Fortunately all the equations in this calculation are linear, hence we can simply superpose both the effects and obtain the final results. Of the n holes existing in an unit volume of real material, we assume that νn holes are filled up with liquid and the others are empty. Thus any material constant such as the density or the bulk modulus may be expressed as the weighted mean of the values corresponding to perfectly empty and perfectly wet cases. Therefore

$$\begin{aligned} k & \text{ (partly filled with the liquid)} \\ &= (1-\nu) \cdot k \text{ (empty)} \quad \dots\dots\dots (6.1) \\ &+ \nu \cdot k \text{ (entirely filled up with the liquid)} \end{aligned}$$

etc.

7. Numerical examples.

To illustrate the above theory we will assume some adequate numerical values for the material constants and will perform calculations.

Our assumptions are:

$$\left\{ \begin{array}{ll} \delta_0 = 2.778 \text{ gr/cm}^3, & \delta_1 = 1.000 \text{ gr/cm}^3 \\ \lambda_0 = \mu_0 = 3.90 \times 10^{11} \text{ c.g.s.}, & k_1 = 2.10 \times 10^{10} \text{ c.g.s.} \quad \dots\dots\dots (7.1) \\ \sigma_0 = 1/4 \end{array} \right.$$

from which we have

$$\begin{aligned} k_0 &= 6.50 \times 10^{11} \text{ c.g.s.} \\ V_{s0} &= 3.75 \quad \text{km/sec} \\ V_{P0} &= 6.49 \quad \text{km/sec.} \end{aligned}$$

and

$$\begin{aligned} K &= k_1/k_0 = 0.0323. \quad \dots\dots\dots (7.2) \\ D &= \delta_1/\delta_0 = 0.36. \end{aligned}$$

Introducing these values into the formula (4.10), we have

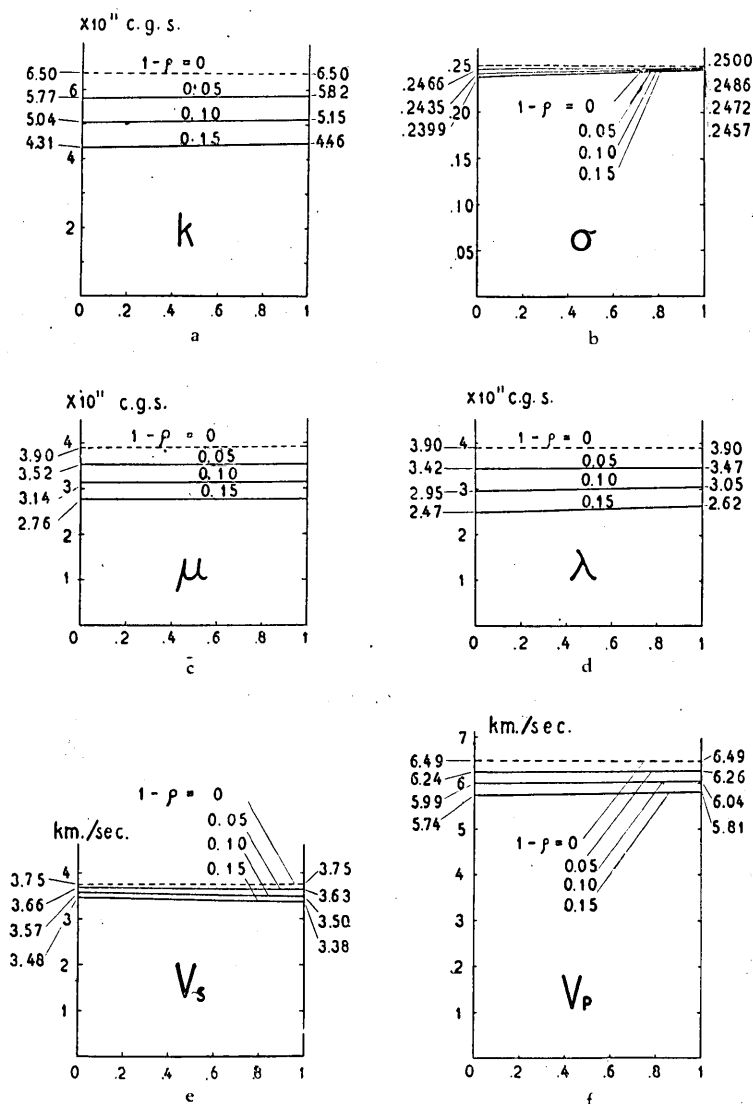


Fig. 4. Numerical values of the elastic constants k , σ , λ , μ and the propagation velocity V_s and V_p , assuming the material constants

$$\begin{aligned} \delta_0 &= 2.778 \text{ gr/cm}^3, & \delta_1 &= 1.000 \text{ gr/cm}^3 \\ \lambda_0 &= \mu_0 = 3.90 \times 10^{11} \text{ c.g.s.}, & k_1 &= 2.10 \times 10^{10} \text{ c.g.s.} \\ \sigma_0 &= 1/4 \end{aligned}$$

Parameter is the porosity $1-\rho$, and the abscissa is ν , or the proportion of the number of holes filled with the liquid.

$$\left\{ \begin{array}{ll} k = 6.50 \times 10^{11} \cdot \{1 - (1 - \rho) 2.092\} & \text{c.g.s.} \\ \mu = 3.90 \times 10^{11} \cdot \{1 - (1 - \rho) 1.957\} & \text{c.g.s.} \\ \lambda = 3.90 \times 10^{11} \cdot \{1 - (1 - \rho) 2.184\} & \text{c.g.s.} \\ \sigma = 0.25 & \cdot \{1 - (1 - \rho) 0.1134\} \dots\dots\dots (7.3) \\ \delta = 2.778 & \cdot \{1 - (1 - \rho) 0.64\} \text{ gr/cm}^3 \\ V_s = 3.75 & \cdot \{1 - (1 - \rho) 0.6582\} \text{ km/sec.} \\ V_p = 6.49 & \cdot \{1 - (1 - \rho) 0.6960\} \text{ km/sec.} \end{array} \right.$$

Combining these expressions with (3.4) we can arrive at the final results, which are illustrated in

- Fig. 4a.—Bulk modulus k ,
 Fig. 4b.—Poisson's ratio σ ,
 Fig. 4c.—Rigidity μ ,
 Fig. 4d.—Lamé's constant λ ,
 Fig. 4e.—Velocity of S-waves V_s ,
 and Fig. 4f.—Velocity of P-waves V_p ,

where the abscissa is ν and the parameter is the porosity $(1 - \rho)$. When the material is perfectly dense, or the porosity is zero, the variation of ν is meaningless. However, for comparison with other cases, we have written it in a broken line.

8. Summary and conclusion.

Under the assumptions cited in § 1, we have calculated the elastic constants and propagation velocity in media with small holes, following the way of investigation of J.K. Mackenzie.

When the holes are empty, Mackenzie's results are used unconditionally, and the convenient expressions for practical use are shown in § 3.

When the holes are filled with some liquid we cannot apply the above formula. Hence we calculated anew the volume change of the equivalent homogeneous sphere with a single hole applying hydrostatic pressure to the outer boundary. In this way we got the bulk modulus k of such a medium. Rigidity remains unaltered even if a hole is filled up with liquid, therefore combining it with the former formula we have the elastic constants λ , σ and the velocity of P and S waves.

Next we discussed whether the next relation can hold in this medium.

$$\frac{1}{V_P} = \frac{\rho}{V_{P0}} + \frac{1-\rho}{V_{P1}}$$

The answer is negative; we cannot deduce the propagation velocity in heterogeneous medium from the weighted harmonic mean of the velocity of two component substances.

When the holes are partly filled with liquid we can use a simple formula of weighted mean such as (6.1).

Numerical examples are shown in (7.3) and Figs. 4a-4f.

16. 小さな穴のあいた媒質内を傳はる弾性波の速さ

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J.K. Mackenzie は、小さな穴のあいた弾性體の、體積弾性率及び剛性率を、equivalent homogeneous continuum の考へを用ゐて、極めてたくみに導き出した。彼のとつた方法は静力學的なものであるから、その結果をそのまま應用して、弾性波が傳はる場合の速さを求めることは、厳密には正しくない。しかし、波長が十分に長い場合には、準靜的な取扱ひが許されるであらうから、第一近似としては、上にえられた k と μ とから、波の傳播速度を出してもさしつかへないであらう。

このような考へに従つて、空隙率 $(1-\rho)$ の二乗以上を省略して得られた公式が (3.2)~(3.5) および (3.7) である。

次に我々は、Mackenzie と全く同様な考へに従つて、穴を液體が満してゐる場合を調べた。この時剛性率には變化のないこと勿論である。結果は (4.8) または (4.9) であるが、これと前の (3.2) の第 2 式を組み合わせ、 λ その他を求めたものが (4.10) にあげてある。

二種類の物質 A, B が體積で ρ_A と ρ_B の割合で混合してゐる時、この中を傳はる弾性波の速さ \bar{V} が、weight をつけた簡單な調和平均の式

$$\frac{1}{\bar{V}} = \frac{\rho_A}{\rho_A + \rho_B} \frac{1}{V_A} + \frac{\rho_B}{\rho_A + \rho_B} \frac{1}{V_B}$$

と與へられるかもしれない、といふ事は誰にも考へられる事であるが、§4 にえられた公式を適用して、その当否をしらべて見た。結果は否定的である。

液體がすべてのすき間を満してゐない時にも、Fig. 3a のやうな状態ではなく、Fig. 3b のやうになつてゐるとするならば、穴全體の中で、液體が充満してゐるものが占める割合 ν を用ゐて、簡單に弾性常數や速度を求めることができる。すべての式は $(1-\rho)$ の一次式であるから、(6.1) の形の公式が成立つわけである。

最後に適当な數値 (7.1) を代入して計算を行つた。結果は (7.3) と Figs. 4a~4f に示してある。普通の岩石と、水との組合せでは、全然水がない場合とあまり大きなちがいはない。この事は圖から一目で知ることができる。圖の横軸は ν であり、 $(1-\rho)$ が 0 の時には、 ν をかへて考へるといふ事は意味を持たないが、他との比較のために破線で示した。