1. Study on Surface Waves V. Love-Waves Propagated upon Heterogeneous Medium.

By Yasuo SATÔ,

Earthquake Research Institute.

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1. Introduction.

The existence of long waves with large horizontal transverse component of displacement in actual seismograms led A.E.H. Love[1] to the discovery of the possibility of waves with such characteristics when there is a layered structure. An alternative condition which may give rise to the waves of the same characteristics was found by E. Meissner[2] and also independently by K. Sezawa[3], and the lack of Meissner's theory was supplemented by K. Aichi[4]. The Progress in our knowledge concerning the crustal structure of the earth resulted in the refinement of the theory in various points[5].

The effect of the double superficial layer on the velocity of propagation of Love-waves was examined independently by R. Stoneley[6] and T. Matuzawa[7] in 1927, and the effect of the heterogeneity in the lower layer was also examined independently by H. Jeffreys[8] and T. Matuzawa[9].

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1) A.E.H. LOVE, *Some Problems of Geodynamics.* (1911), 176.

2) E. MEISSNER, *Vieljahr. Nat. Forsch. Ges., Zurich*, **67** (1921), 181. Verh. 2. Int. Kongr. f. Tech. Mech., Zurich, (1926), 137.

3) K. SEZAWA, "A Kind of Waves transmitted over a Semi-infinite Solid Body of Varying Elasticity." *Bull. Earthq. Res. Inst.*, **9** (1931), 310.

4) K. AICHI, "On the Transversal Seismic Waves travelling upon the Surface of Heterogeneous Medium." *Proc. Phys.-Math. Soc. Japan*, [iii], **4** (1922), 137.

5) S. HOMMA also published a paper concerning this problem. *Journ. Meteor. Soc. Jap.*, [ii], **18** (1940), 8. (In Japanese).

6) R. STONELEY and E. TILOTSON, "The Effect of a Double Surface Layer on Love Waves." *M.N.R.A.S. Geo. Sup.*, **1** (1927), 521. "The Dispersion of Waves in a Double Superficial Layer." do., 527.

7) T. MATUZAWA, "Propagation of Love Waves along a doubly Stratified Layer." *Proc. Phys.-Math. Soc. Japan*, [iii], **10** (1927), 25.

8) H. JEFFREYS, "The Effect on Love Waves of Heterogeneity in the Lower Layer." *M.N.R.A.S. Geo. Sup.*, **2** (1928), 101.

9) T. MATUZAWA, "Observation of Some Recent Earthquakes and their Time Distance Curves" *Bull. Earthq. Res. Inst.*, **6** (1929), 211.

The propagation in two dimensions and over the surface of sphere, and the problems of the generation of Love-waves were treated by K. Sezawa[10] and H. Nakano[11] and others from various points of view. R. Yosiyama[12] in general solved rigorously the waves of Love-type over the heterogeneous sphere.

In order to elucidate the nature of surface waves we meet with the neccesity of the theory of waves propagated along double superficial layer upon the heterogeneous substratum as ascertained in various parts of the world.

The theory of waves of Love type propagated in heterogeneous medium was treated by Meissner[13] Sezawa[14], Jeffreys[15], Matuzawa[16], Yosiyama[17], Homma[18] and others. The assumed heterogeneity of the medium in these studies were roughly classified into three groups; in the first group the velocity increases linearly with the depth from the surface, and in the second group the rigidity varies linearly, while in the third the rigidity varies with the n-th power of the distance from the centre[19]. The first case was fully treated by H. Jeffreys, the second case by Sezawa and Matuzawa, and the last case rigorously by R. Yosiyama. In the present analysis, the results of these authorities could of course be utilized, but in view of the refinement in mathematics, we have treated the second case and obtained the solution in Whittaker's confluent hypergeometric function, $W_{k,m}(z)$[20].

2. Waves of Love type propagated over semi-infinite solid in which rigidity increases linearly with the depth z from the surface.

We shall take Cartesian coordinates x and z, whose origin lies on the surface, and take x-axis horizontally and z vertically upward, so

10) K. SEZAWA, "Propagation of Love-waves on a Spherical Surface and Allied Problems." *Bull. Earthq. Res. Inst.*, **7** (1929), 437. "Love waves Generated from a Source of a certain Depth." *Bull. Earthq. Res. Inst.*, **13** (1929), 1.

11) H. NAKANO, "Love-wave in Cylindrical Coordinates." *Geophys. Mag.*, **8** (1929).

12) R. YOSIYAMA, *Zisin*, **10** (1938), 272 (in Japanese).

13) *loc. cit.*, 2).

14) *loc. cit.*, 3).

15) *loc. cit.*, 8).

16) *loc. cit.*, 9).

17) *loc. cit.*, 12), and *Zisin* **11** (1939), 57, 145, 247.

18) *loc. cit.*, 5), and also see *Kensin-Ziho*, **10** (1938), 459 and **12** (1942), 37.

19) S. Homma, investigated a problem assuming some other kind of heterogeneity, but has not yet published it.

20) WHITTAKER and WATSON, *Modern Analysis*, *p*. 337.

that $\mu=\mu_0-\beta z$. We shall confine our study, for the present, to the case of plane waves propagated in the direction of x-axis. However this is by no means an essential limitation, for the nature of the general cylindrical wave is exactly the same so far as the variation of the amplitude with z and the velocity of propagation are concerned.

Now in our problem the displacement has only transverse horizontal component, so $u=w=0$.

Then the equation of motion in such medium is, as usual,

$$\rho\frac{\partial^2 v}{\partial t^2}=\mu\nabla^2 v+\frac{\partial\mu}{\partial z}\frac{\partial v}{\partial z},\qquad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\quad\ldots\ldots\ldots(2.1)$$

where ρ and μ are the density and rigidity of the medium, and μ is assumed to be the function of z alone.

Putting V for

$$V\equiv\sqrt{\mu}\,v$$

we have

$$\rho\frac{\partial^2 V}{\partial t^2}=\mu\nabla^2 V-\frac{1}{2}V\frac{d^2\mu}{dz^2}+\frac{1}{4\mu}V\left(\frac{d\mu}{dz}\right)^2\quad\ldots\ldots\ldots(2.2)$$

Assuming now $V=Z\exp(ipt-ifx)$
and changing the independent variable z into μ, we have

$$\frac{d^2Z}{d\mu^2}+\left\{\frac{\rho p^2}{\mu}-f^2-\frac{1}{2\mu}\frac{d^2\mu}{dz^2}+\frac{1}{4\mu^2}\left(\frac{d\mu}{dz}\right)^2\right\}Z=0$$

which now reduces itself in our problem to

$$\frac{d^2Z}{d\mu^2}+\left\{\frac{1/4}{\mu^2}+\frac{\rho p^2/\beta^2}{\mu}-\frac{f^2}{\beta^2}\right\}Z=0\quad\ldots\ldots\ldots(2.3)$$

because $\frac{d\mu}{dz}=-\beta$, $\frac{d^2\mu}{dz^2}=0$ from the assumption.

Substituting again

$$\kappa=\rho p^2/2f\beta,\qquad \zeta=2f\mu/\beta\quad\ldots\ldots\ldots(2.4)$$

and $W=\frac{4f^2}{\beta^2}Z$

we get

$$\frac{d^2W}{d\zeta^2}+\left\{\frac{1/4}{\zeta^2}+\frac{\kappa}{\zeta}-\frac{1}{4}\right\}W=0.\quad\ldots\ldots\ldots(2.5)$$

The solution of the last equation is the Whittaker's[21] confluent hyper-

21) *loc. cit.*, 20).

geometric function $W_{\kappa,0}(\zeta)$

$$W = W_{\kappa,0}(\zeta) = e^{-\zeta/2}\zeta^{\kappa} S$$

in which $$S = 1 + \sum_{n=1}\left\{(-)^n \prod_{m=1}^{n}\left(\frac{(\kappa - m + 1/2)^2}{m\zeta}\right)\right\}$$

or $$W = M_{\kappa,0}(\zeta) = \zeta^{1/2} e^{-\zeta/2}\left\{1 + \sum_{n=1}^{\infty}\left(\prod_{m=1}^{n}\frac{(1/2 - \kappa + m - 1)}{m^2}\right)\zeta^n\right\}$$

$$= \zeta^{1/2} e^{-\zeta/2} {}_1F_1\left(\frac{1}{2} - \kappa, 1, \zeta\right) \qquad (2.6)$$

of which the former is utilized in this problem.

We have thus obtained the solution v of the equation (2.1) which is now written

$$v = A(\beta/2f)^{3/2}\Phi(\zeta)e^{i(pt - fx)} \qquad (2.7)$$

where $$\Phi(\zeta) = \zeta^{\kappa - 1/2} e^{-\zeta/2} S.$$

Although we have hitherto confined ourselves to the solution of the two dimensional problem, the restriction is a matter of no essential necessity, because the solution of the three dimensional problem is obtained following the line skilfully opened by K. Sezawa provided the present $\Phi(\zeta)$ function is substituted for his $\Phi(z)$ into his solution by replacing ζ above by $2kz$ of Sezawa's notation[22].

3. Love waves propagated along stratified surface.

As first proved by Meissner[23], the boundary condition of zero surface traction can be satisfied without surface layers provided the medium is heterogeneous.

Thus $$\frac{dv}{dz} = 0$$

or $$\frac{d\Phi}{d\zeta} = 0 \qquad (3.1)$$

If there are surface layers, the free surface wave can exist as discovered by Love and Stoneley and Matuzawa.

We shall, for simplicity's sake, also assume that the medium is homogeneous and isotropic in each crustal layer while in the substratum the rigidity increases linearly with the depth from the boundary surface.

22) *loc. cit.*, 3).
23) *loc. cit.*, 2).

It is well-known that the relevant solution of the waves in the crustal layers is

$$v_i = \{A_i e^{is_i z} + B_i e^{-is_i z}\} e^{i(pt-fx)}$$

where $\qquad s_i^2 = -f^2 + k_i^2, \qquad k_i^2 = \rho_i p^2/\mu_i, \quad i=1, 2 \ldots\ldots (3.2)$

while the solution in the substratum is given by (2.7) in the previous section.

The boundary conditions to be satisfied by these waves are the vanishing of surface traction at the uppermost surface and the continuity of displacement and traction on the boundary surfaces of two media; i.e.,

$$\begin{cases} \text{at the free surface} \quad \dfrac{\partial v_1}{\partial z} = 0 \\ \text{at the boundary surfaces } v_i = v_{i+1} \text{ and } \mu_i \dfrac{\partial v_i}{\partial z} = \mu_{i+1} \dfrac{\partial v_{i+1}}{\partial z} \end{cases} \qquad (3.3)$$

When there is a superficial layer with thickness H_1, (the origin of coordinate being taken on the lower boundary of this layer) and the intermediate layer with thickness H_2, the boundary conditions gives the relations (cf. Fig. 1)

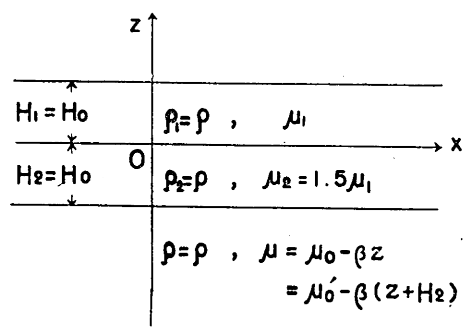

Fig. 1.

$$A_1 \exp(is_1 H_1) - B_1 \exp(-is_1 H_1) = 0$$

$$A_1 + B_1 = A_2 + B_2$$

$$\mu_1 s_1 A_1 - \mu_1 s_1 B_1 = \mu_2 s_2 A_2 - \mu_2 s_2 B_2 \qquad \ldots\ldots\ldots\ldots (3.4)$$

$$A_2 \exp(-is_2 H_2) + B_2 \exp(is_2 H_2) = A_3 \left(\frac{\beta}{2f}\right)^{3/2} \Phi(\zeta_0)$$

$$i\mu_2 s_2 A_2 \exp(-is_2 H_2) - i\mu_2 s_2 B_2 \exp(is_2 H_2) = (-2f\mu_0') A_3 \left(\frac{\beta}{2f}\right)^{3/2} \Phi'(\zeta_0)$$

which give

$$\begin{vmatrix} \exp(is_1H_1) & -\exp(-is_1H_1) & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ \mu s_1 s_1 & -\mu_1 s_1 & -\mu_2 s_2 & \mu_2 s_2 & 0 \\ 0 & 0 & \exp(-is_2H_2) & \exp(is_2H_2) & -\left(\frac{\beta}{2f}\right)^{3/2} \Phi(\zeta_0) \\ 0 & 0 & i\mu_2 s_2 \exp(-is_2H_2) & -i\mu_2 s_2 \exp(is_2H_2) & 2f\mu_0'\left(\frac{\beta}{2f}\right)^{3/2} \Phi'(\zeta_0) \end{vmatrix} = 0 \qquad (3.5)$$

where μ_0' implies the value of rigidity of heterogeneous medium at the boundary surface $z=-H_2$, i.e., $\mu=\mu_0-\beta z=\mu_0'-\beta(z+H_2)$.

Developing the above determinant we have

$$\Delta_0 \equiv \left\{1-\frac{\mu_1}{\mu_2}\frac{s_1}{s_2}\tan s_1H_1 \tan s_2H_2\right\}\left\{2f\frac{1}{s_2}\frac{\Phi'(\zeta_0)}{\Phi(\zeta_0)}\right\} + \left\{\frac{\mu_2}{\mu_0'}\tan s_2H_2+\frac{\mu_1}{\mu_0'}\frac{s_1}{s_2}\tan s_1H_1\right\} = 0 \quad \cdots\cdots(3.6)$$

If we put

$$\begin{aligned} &k^2\equiv \rho p^2/\mu_0', \quad k_j^2\equiv \rho_j p^2/\mu_j && (j=1,\ 2) \\ &s_j^2=k_j^2-f^2 && (\ \ ,,\ \ ) \\ &\frac{k_j^2}{k^2}=\frac{\rho_j}{\rho}\frac{\mu_0'}{\mu_j}\equiv r_j'^2 && (\ \ ,,\ \ ) \quad \cdots\cdots\cdots(3.7) \\ &\omega_j\equiv kH_j && (\ \ ,,\ \ ) \\ &\mathfrak{v}\equiv k/f \end{aligned}$$

then

$$s_jH_j=\omega_j\sqrt{(r_j'^2-\mathfrak{v}^{-2})} \qquad (j=1,\ 2)$$

Here, $\mathfrak{v}$ is the phase velocity of surface waves, and ω is the frequency of the waves. The expression (3.6) becomes

$$\Delta_0=\left\{1-\frac{\mu_1}{\mu_2}\frac{\sqrt{r_1'^2-\mathfrak{v}^{-2}}}{\sqrt{r_2'^2-\mathfrak{v}^{-2}}}\tan\omega_1\sqrt{r_1'^2-\mathfrak{v}^{-2}}\cdot\tan\omega_2\sqrt{r_2'^2-\mathfrak{v}^{-2}}\right\}\cdot\left\{2\frac{\mathfrak{v}^{-1}}{\sqrt{r_2'^2-\mathfrak{v}^{-2}}}\frac{\Phi'(\zeta_0)}{\Phi(\zeta_0)}\right\}$$

$$+\left\{\frac{\mu_2}{\mu_0'}\tan\omega_2\sqrt{r_2'^2-\mathfrak{v}^{-2}}+\frac{\mu_1}{\mu_0'}\frac{\sqrt{r_1'^2-\mathfrak{v}^{-2}}}{\sqrt{r_2'^2-\mathfrak{v}^{-2}}}\tan\omega_1\sqrt{r_1'^2-\mathfrak{v}^{-2}}\right\}=0$$

$$\frac{\Phi'(\zeta_0)}{\Phi(\zeta_0)}=\frac{\kappa-1/2}{\zeta}-\frac{1}{2}+\frac{S'}{S},$$

$$S=1+\sum_{n=1}\left\{(-)^n\prod_{r=1}^{n}\frac{(\kappa-r+1/2)^2}{r\zeta}\right\}, \quad S'=\sum_{n=1}\left[\left(-\frac{n}{\zeta}\right)\left\{(-)^n\prod_{r}\frac{(\kappa-r+1/2)^2}{r\zeta}\right\}\right] \quad \cdots\cdots\cdots(3.8)$$

4. Numerical calculations.

4.1 Determination of velocity.

In this article we will give the results of numerical calculations performed under the assumptions shown in the following lines (cf. Fig. 1).

$$\begin{aligned} &H_1 = H_2 \equiv H_0 \\ &\rho_1 = \rho_2 \equiv \rho \\ &\frac{\mu_1}{\mu_2} = \frac{\mu_2}{\mu_0'} = \frac{1}{1.50} \end{aligned} \qquad \ldots\ldots\ldots\ldots(4.1)$$

from which we obtain

$$r_1'^2 = 2.25, \qquad r_2'^2 = 1.50$$
$$\omega_1 = \omega_2 \equiv \omega.$$

Further we assume

$$\theta_0 \equiv D_0/H_0 = 40 \qquad \ldots\ldots\ldots\ldots(4.2)$$

where $D_0 \equiv \mu_0'/\beta$ implies the depth where the rigidity is twice that of the boundary surface.

Then

$$\begin{aligned} \kappa &\equiv \rho p^2/2f\beta \\ &= \theta_0\, \omega\mathfrak{v}/2 = 20\omega\mathfrak{v} \\ \zeta_0 &= 2f\mu_0'\beta \\ &= 2\theta_0\omega/\mathfrak{v} = 80\,\omega/\mathfrak{v} \end{aligned} \qquad \ldots\ldots\ldots\ldots(4.3)$$

Although the process of computation is very much complicated, its principle is simple.

At first we assume a pair of the values of ω and $\mathfrak{v}$; then we calculate Δ_0 in (3.8) by means of the expressions (2.6) and (2.7). If it does not vanish, we assume another value of ω ($\mathfrak{v}$ remains unaltered) and again calculate the value Δ_0. We repeat such processes and by interpolation determine the pair of the values of ω and $\mathfrak{v}$ which satisfy $\Delta_0 = 0$. Thus we obtain the following results. (Table I and Fig. 2)

Table I.

$1/\mathfrak{v}$	ω	$\mathfrak{v}$ (phase vel.)	$1/\omega$	group vel.
0.90	0.161	1.111	6.21	1.012
0.95	0.2300	1.053	4.348	0.932
0.97	0.272	1.031	3.68	—
1.00	0.3295	1	3.035	0.871
1.05	0.4360	0.952	2.294	0.814
1.10	0.5510	0.909	1.815	0.750
1.15	0.6726	0.870	1.485	0.707
1.20	0.8094	0.833	1.236	0.679
1.25	0.9746	0.800	1.026	0.659
1.30	1.1870	0.769	0.8425	0.652
1.35	1.5098	0.741	0.6623	0.648
1.40	2.0475	0.714	0.4884	0.646
1.45	3.2635	0.690	0.3064	0.653
1.475	—	—	—	0.658
1.50	∞	0.667	0	0.667

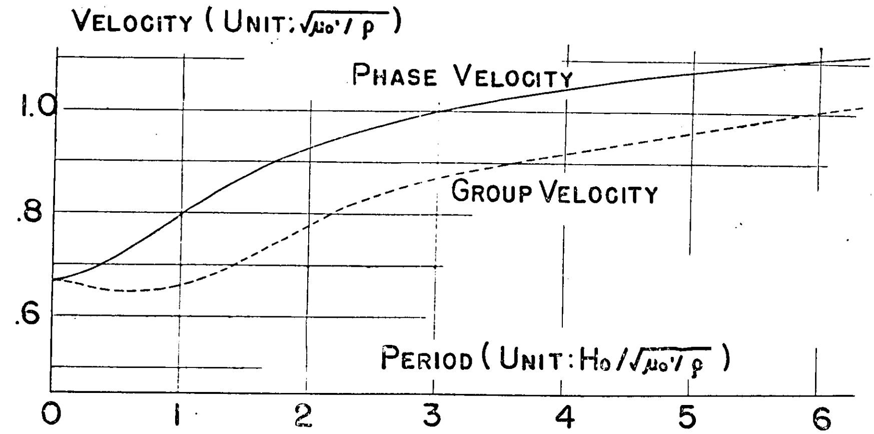

Fig. 3.

The last column of the same table gives the value of group velocity, which is determined by using the method of numerical differentiation (we adopted Stirling's formula), and graphical differentiation (in this case Askania's apparatus was used).

In this calculation, the determination of ω, when the wave length is not so small, is extremely troublesome. For, in this case, κ and ζ_0 are both large, and the convergency of the series S and S' are not good. Thus the asymptotic expansion in such a case is desirable, which, however, is not obtained yet.

4.2. Vertical distribution of amplitude.

Now that the velocity has been determined in the previous chapter, we will obtain the vertical distribution of amplitude here.

The necessary formulae are obtained utilizing the above four expressions in (3.4) and developing the five determinants of 4th degree. The result is as follows,

$$\begin{cases} A_1: & \{(\beta/2f)^{3/2}\,\mu_2 s_2\,\Phi(\zeta_0)\cdot(\cos s_1H_1 - i\sin s_1H_1)\} \\ =B_1: & \{(\beta/2f)^{3/2}\,\mu_2 s_2\,\Phi(\zeta_0)\cdot(\cos s_1H_1 + i\sin s_1H_1)\} \\ =A_2: & \{(\beta/2f)^{3/2}\,\Phi(\zeta_0)\cdot(\mu_2 s_2\cos s_1H_1 - i\mu_1 s_1\sin s_1H_1)\} \\ =B_2: & \{(\beta/2f)^{3/2}\,\Phi(\zeta_0)\cdot(\mu_2 s_2\cos s_1H_1 + i\mu_1 s_1\sin s_1H_1)\} \\ =A_3: & \{2(\mu_2 s_2\cos s_1H_1 - \mu_1 s_1\sin s_1H_1\sin s_2H_2)\} \end{cases}$$

We introduce these expressions into (3.2) and (2.7), and get the following three expressions which are sufficient to show the vertical distribution of the amplitude of the waves in question.

$$\begin{cases} v_1 = C\,\Phi(\zeta_0)\,\{\mu_2 s_2 \cos s_1 H_1 \cos s_1 z + \mu_2 s_2 \sin s_1 H_1 \sin s_1 z\} \exp(ipt - ifx) \\ v_2 = C\,\Phi(\zeta_0)\,\{\mu_2 s_2 \cos s_1 H_1 \cos s_2 z + \mu_1 s_1 \sin s_1 H_1 \sin s_2 z\} \exp(ipt - ifx) \quad \ldots (4.5) \\ v_3 = C\,\Phi(\zeta)\,\{\mu_2 s_2 \cos s_1 H_1 \cos s_2 H_2 - \mu_1 s_1 \sin s_1 H_1 \sin s_2 H_2\} \exp(ipt - ifx) \end{cases}$$

If we introduce the numerical values which we have adopted in §4.1, and further modify them in order to obtain the expressions more convenient for numerical calculations, we arrive at the following expressions,

$$\begin{cases} v_1 = C'\left\{\cos\left(\omega\sqrt{2.25-\mathfrak{v}^{-2}}\,\frac{z}{H_0}\right) + \tan(\omega\sqrt{2.25-\mathfrak{v}^{-2}})\sin\left(\omega\sqrt{2.25-\mathfrak{v}^{-2}}\,\frac{z}{H_0}\right)\right\} \\ v_2 = C'\left\{\cos\left(\omega\sqrt{1.50-\mathfrak{v}^{-2}}\,\frac{z}{H_0}\right) + \frac{1}{1.50}\frac{\sqrt{2.25-\mathfrak{v}^{-2}}}{\sqrt{1.50-\mathfrak{v}^{-2}}} \cdot \tan(\omega\sqrt{2.25-\mathfrak{v}^{-2}})\sin\left(\omega\sqrt{1.50-\mathfrak{v}^{-2}}\,\frac{z}{H_0}\right)\right\} \quad \ldots (4.6) \\ v_3 = C'\left(\frac{\zeta}{\zeta_0}\right)^{\kappa-\frac{1}{2}} \exp\{(\zeta_0-\zeta)/2\}\,\frac{S(\zeta)}{S(\zeta_0)}\cos(\omega\sqrt{1.50-\mathfrak{v}^{-2}}) \cdot\left\{1-\frac{1}{1.50}\frac{\sqrt{2.25-\mathfrak{v}^{-2}}}{\sqrt{1.50-\mathfrak{v}^{-2}}}\tan(\omega\sqrt{2.25-\mathfrak{v}^{-2}})\tan(\omega\sqrt{1.50-\mathfrak{v}^{-2}})\right\} \end{cases}$$

Table II.

Case	A	B	C
$1/\mathfrak{v}$	0.997	1.10	1.40
ω	0.324025	0.55	2.044
κ	6.5	10.0	29.2
ζ_0	25.7645	48.40	228.93
ζ/ζ_0	$\left(\frac{\zeta}{\zeta_0}\right)^{\kappa-\frac{1}{2}}\exp\{-(\zeta-\zeta_0)/2\}\frac{S(\zeta)}{S(\zeta_0)}$		
1.00	1.000	1.000	1.000
1.02	0.918	—	—
1.04	0.836	—	—
1.05	—	0.546	0.0168
1.06	0.761	—	—
1.08	0.688	—	—
1.10	—	0.292	0.00027
1.12	0.554	—	—
1.20	0.344	0.0786	—
1.30	0.1769	0.01677	—
1.50	0.0395	0.00068	—

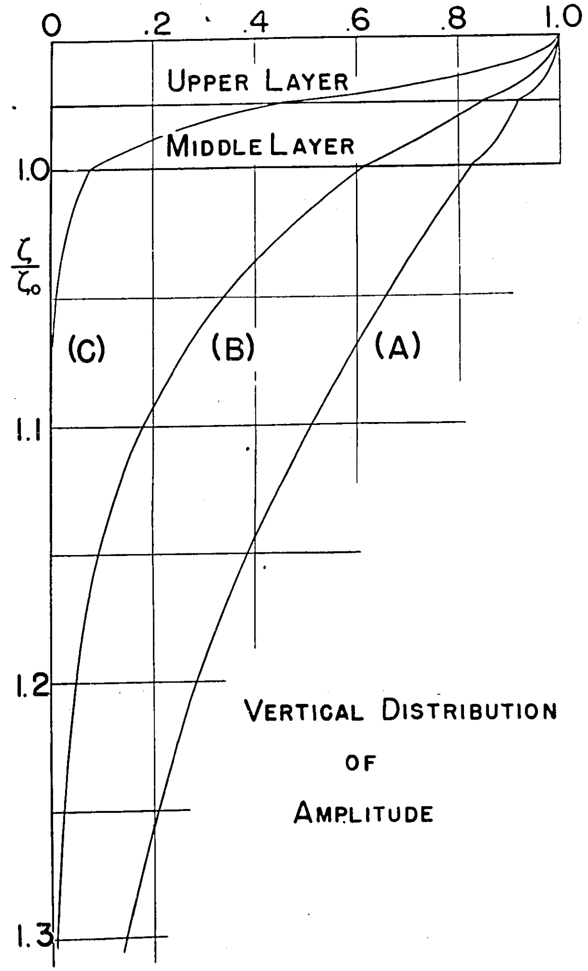

Fig. 3.

Since the waves show dispersive property in this problem, the vertical distribution of amplitude is not invariant with regard to period. Therefore we have performed numerical calculations corresponding to some typical values of velocity (v=0.997, 1.10, 1.40) or period. The results are shown in Table II and in Fig. 3.

This paper was completed under the kind guidance of Prof. H. Kawasumi. The author offers heartiest thanks to his incessant instruction.

1. 表面波の研究 V.
不均質媒質を傳はるラブ波

地震研究所 佐藤泰夫

今日までに我我に知られた所によれば，地球の表面近くの構造は，極めて複雑なものであるが，簡單なモデルとしては，一様な二つの層を表面におき，その下の彈性率を，深さの 1 次式でかはるものと假定して，大きなあやまりはないやうである．猶，密度はこの三つの部分で大したちがひはない．

そこで我々は，第 1 圖に示すやうな物質の配置を考へて，この中をつたはるラブ型の表面波についてしらべたが，これは今日の所では，實際に觀測するものに，もつともよく適合する可能性を持つた解の一つと思ふ．

問題は 2 次元的な扱ひをしたが，3 次元としても，本質的な變化があるわけではない．

彈性率が 1 次式でかはる媒質内の解としては，變數を適當に變換して confluent hypergeometric の形にみちびき，Whittaker の $W_{k,m}(z)$ 函數の形に解を求めた．

上の式を用ゐて速度方程式をかきあらはし，更に $\mu_0'=1.50\,\mu_2$, $\mu_2=1.50\,\mu_1$, $\theta_0\equiv D_0/H_0\equiv\mu_0'/H_0\beta=40$ (D_0 は μ の値が，境界面の値 μ_0' の 2 倍になる深さ) と假定してこの方程式を數値的にといた．結果は第 I 表及び第 2 圖に示す．なほ，このやうにして得られた位相速度を，數値的 (Stirling の公式によつた) もしくは圖的に (Askania の微分器を用ゐた) 微分して，群速度をも求めたが，これも同じ表と圖に示してある．

一度速度が定まるならば，媒質内の振幅の分布も決定する．もつとも，現在の問題では波が分散性であり (正分散)，速度は周期によつて異るから，振幅の分布も當然周期に依存する．我我は代表的な三つの場合をえらんで數値計算を行ひ，その結果を第 II 表と第 3 圖とに掲げた．

本研究は河角教授のねんごろな御指導によるものであり，日頃からの絶える事ない御教示にあはせて心からの感謝をささげたい．
