40. *Study on Surface Waves III.* *Love-Waves with Double Superficial Layer.*

By Yasuo SATÔ,

Earthquake Research Institute.

(Read Jan. 18, 1949.—Received June 20, 1951.)

1. Introduction.

In the previous paper[1] we have calculated the velocity of Love-waves with a single layer. However, from the observational facts our earth seems to have double superficial layer, and calculations for such cases have already been performed by some authors[2]. But the general theory of the waves in such cases has not been fully discussed, and still there even remain doubts about the condition of existence and on the nature of dispersion, so we have good reason to believe that our new calculation has some value.

2. Characteristic equation.

First we will obtain the characteristic equation.

Density ρ_k, rigidity μ_k, velocity of S-waves V_k and the thickness of layers H_k are taken as are illustrated in Fig. 1.

We will further assume the following notations which are not the same with those adopted in the other works of us in accordance with those used by H. Menzel[3].

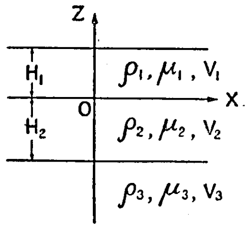

Fig. 1.

$$
\begin{aligned}
s_k &\equiv f\sqrt{\{(V^2/V_k^2)-1\}} \quad \text{when} \quad V>V_k \\
&\equiv i\sigma_k \qquad\qquad\qquad\qquad\quad V<V_k
\end{aligned}
\qquad (k=1,2,3) \qquad \ldots\ldots(2.1)
$$

in which V is the velocity of Love-waves.

---

1) Y. SATÔ, "Study on Surface Waves I." *Bull. Earthq. Res. Inst.*, **29** (1951), 1.

2) T. MATUZAWA, "Propagation of Love-Waves along a Doubly Stratified Layer." *Proc. Phys.-Math. Soc. Jap.* [iii] **10** (1928), 25. R. STONELEY and E. TILOTSON, *M.N.R.A.S. Geo. Sup.*, **1** (1927), 521. H. MENZEL, "Dispersion von seismischen Oberflächenwellen nach Registrierungen in Kopenhagen und Groß-Raum." *Beitr. z. Geophys.*, **54** (1939), 348.

3) *loc. cit.* 2)

Displacements in three media are assumed to be

$$\begin{cases} V_1=\{A_1\exp(is_1z)+B_1\exp(-is_1z)\}\exp(ifx-ipt) \\ V_2=\{A_2\exp(is_2z\}+B_2\exp(-is_2z)\}\exp(ifx-ipt) \\ V_3=\{ \qquad\qquad\qquad +B_3\exp(-is_3z)\}\exp(ifx-ipt) \end{cases} \qquad \ldots\ldots(2.2)$$

Boundary conditions are

$$\begin{array}{lllll} \text{at} \quad z=H_1\,; & \widehat{xz}_1=0\;, & \widehat{yz}_1=0 & & \\ \qquad\; z=0\;\;; & \widehat{xz}_1=\widehat{xz}_2\,, & \widehat{yz}_1=\widehat{yz}_2\,, & v_1=v_2 & \ldots\ldots.(2.3) \\ \qquad\; z=-H_2\,; & \widehat{xz}_2=\widehat{xz}_3\,, & \widehat{yz}_2=\widehat{yz}_3\,; & v_2=v_3 & \end{array}$$

From (2.3), we have

$$D\equiv\begin{vmatrix} \exp(is_1H_1) & -\exp(-is_1H_1) & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ \mu_1s_1 & -\mu_1s_1 & -\mu_2s_2 & \mu_2s_2 & 0 \\ 0 & 0 & \exp(is_2H_2) & \exp(is_2H_2) & -\exp(is_3H_2) \\ 0 & 0 & i\mu_2s_2\cdot\exp(is_2H_2) & -i\mu_2s_2\cdot\exp(is_2H_2) & \mu_3s_3\cdot\exp(is_3H_2) \end{vmatrix}=0 \qquad .(2.4)$$

which may be reduced to

$$D\equiv D^c+D^s=0$$

$$\begin{cases} D^c=\mu_2s_2\cos s_1H_1\,(+\mu_3's_3\cos s_2H_2+\mu_2s_2\sin s_2H_2) \\ D^s=\mu_1s_1\sin s_1H_1\,(-\mu_3's_3\sin s_2H_2+\mu_2s_2\cos s_2H_2) \end{cases} \qquad \ldots\ldots(2.4a)$$

$$\mu_3'\equiv i\mu_3$$

Further, modifying the above expressions, we have

$$\begin{cases} D^c=C_{21}\,(C_{32}+S_{22}) \\ D^s=S_{11}\,(-S_{32}+C_{22}) \end{cases}$$

in which

$$\begin{cases} C_{jk}\equiv\mu_js_j\cos s_kH_k \\ S_{jk}\equiv\mu_js_j\sin s_kH_k \qquad (j,\;k=1,2,3) \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.4b)$$

$$(\text{when } j=3,\;\; \mu_j \text{ implies } \;\mu_3')$$

3. Condition of existence.

Now we will discuss the condition of existence[4].

Since we assume double superficial layer and the lower semi-infinite body, we have three kinds of media, which we call A, B, C according to the ascend-

4) This problem was already discussed by T. Matuzawa. *loc. cit.* 2)

ing order of velocity of S-waves in respective media. Let us further show, for example, by the notation $\begin{pmatrix} B \\ A \\ C \end{pmatrix}$ that the intermediate layer is the medium of lowest velocity, the upper is that of middle value, while, the semi-infinite part gives the highest velocity. Of the 27 possible permutations of the three notations A, B and C we exclude trivial (on physically identical) cases and have the following 13 ones; viz.

$$[1]\begin{pmatrix} A \\ A \\ A \end{pmatrix},\ [2]\begin{pmatrix} A \\ A \\ B \end{pmatrix},\ [3]\begin{pmatrix} A \\ B \\ A \end{pmatrix},\ [4]\begin{pmatrix} A \\ B \\ B \end{pmatrix},\ [5]\begin{pmatrix} A \\ B \\ C \end{pmatrix},\ [6]\begin{pmatrix} A \\ C \\ B \end{pmatrix}$$

$$[7]\begin{pmatrix} B \\ A \\ A \end{pmatrix},\ [8]\begin{pmatrix} B \\ A \\ B \end{pmatrix},\ [9]\begin{pmatrix} B \\ A \\ C \end{pmatrix},\ [10]\begin{pmatrix} B \\ B \\ A \end{pmatrix},\ [11]\begin{pmatrix} B \\ C \\ A \end{pmatrix} \quad \text{..........(3.1)}$$

$$[12]\begin{pmatrix} C \\ A \\ B \end{pmatrix},\ [13]\begin{pmatrix} C \\ B \\ A \end{pmatrix}.$$

Since v_3 must vanish at $z=-\infty$, s_3 must have positive imaginary part Remembering that

$$s_3/i=\sigma_3=f\sqrt{\{1-(V^2/V_3^2)\}},$$

the above condition is replaced by

$$V<V_3. \quad \text{........................(3.2)}$$

If V_3 takes the smallest or one of the smallest value among V_k ($k=1, 2, 3$), s_1 and s_2 also become imaginary from the relation (3.2). Therefore, putting $s_k=i\sigma_k$ into (2.4a), we have

$$\begin{aligned} iD=\mu_2\sigma_2 \cosh \sigma_2H_2\,(\mu_3\sigma_3 \cosh \sigma_2H_2+\mu_2\sigma_2 \sinh \sigma_2H_2) \\ +\mu_1\sigma_1 \sinh \sigma_1H_1\,(\mu_3\sigma_3 \sinh \sigma_2H_2+\sigma_2\sigma_2 \cosh \sigma_2H_2)=0 \quad \text{.....(3.3)} \end{aligned}$$

Since all the terms involved in this equation are positive, this cannot be satisfied by the real value of V.

Thus of the thirteen cases in (3.1), [1], [3], [7], [10], [11] and [13] must be omitted. The remaining cases are

$$[2]\begin{pmatrix} A \\ A \\ B \end{pmatrix},\ [4]\begin{pmatrix} A \\ B \\ B \end{pmatrix},\ [5]\begin{pmatrix} A \\ B \\ C \end{pmatrix},\ [6]\begin{pmatrix} A \\ C \\ B \end{pmatrix},\ [8]\begin{pmatrix} B \\ A \\ B \end{pmatrix},\ [6]\begin{pmatrix} B \\ A \\ C \end{pmatrix},\ [12]\begin{pmatrix} C \\ A \\ B \end{pmatrix} \ldots \text{(3.4)}$$

Of these seven cases, the existence of the solution of first two cases ([2] and [4]) is self-evident, and the third one ([5]) has been already treated by T.

Matuzawa and R. Stoneley[5]. So we will here discuss the four remaining cases separately.

3.1 Examples.

In order to prove the existence of Love-waves it is sufficient for us to show a single example of solutions satisfying the conditions (2.3) for a given succession of the media. And indeed, we can easily give an example in each case.

I. Example of case [8].... $\begin{pmatrix} B \\ A \\ B \end{pmatrix}$.

We assume:

$$H_1 = H_2 \equiv H_0$$
$$\rho_1 = \rho_2 = \rho_3 \equiv \rho \qquad \text{(3.5)}$$
$$\mu_1 = 10\mu_2 = \mu_3 \equiv \mu$$

then we can get the solution of (2.4), which we will give in Fig. 2 and Tab. I. For comparison we also give the dispersion curve of simple Love-waves. (Thickness of the layer; $2H_0$. Density; ρ. Rigidity of upper layer is one tenth of that of lowest medium.) A remarkable point which we notice in the figure is that the phase velocity of simple Love-waves is larger than that of the above case when the wave-length is fairly large.

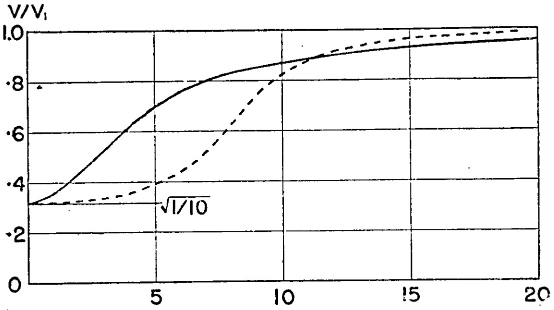

Fig. 2. Dispersion curve of example I (Case [8]).
Abscissa: Wave length (unit: H_0)
Broken line shows the dispersion curve of simple Love-waves. (Unit of wave length is same with above.)

Table I.

$(V/V_1)^2$	fH_0
0	0
0.99	0.10935
0.96	0.21405
0.9039	0.3322
0.84	0.4525
0.75	0.6471
0.64	0.9078
0.51	1.2169
0.4375	1.4149
0.36	1.6844
0.2775	2.1182
0.19	3.089
0.1536	4.08
0.10	∞

II. Example of case [6].... $\begin{pmatrix} A \\ C \\ B \end{pmatrix}$

We assume:

5) *loc. cit.*, 2)

$$\begin{aligned} &H_1 = H_2 \equiv H_0 \\ &\rho_1 = \rho_2 = \rho_3 \equiv \rho \qquad \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.6) \\ &10\mu_1 = \mu_2 = 3\mu_3 \equiv \mu \end{aligned}$$

The results of calculations are given in Fig. 3 and Tab. II. In this case the upper limit of wave-length of period exists, and we will call this value as the "cut-off period", which gives a phase velocity equal to that of S-wave in the lowest medium[6]. The proof is as follows.

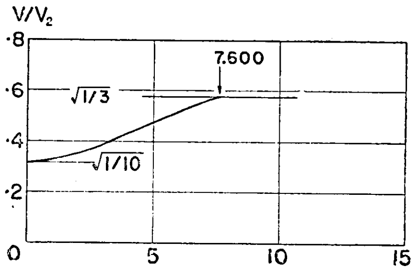

Fig. 3. Dispersion curve of example II (Case [6])

Table II.

$(V/V_2)^2$	fH_0
1/3	0.8268
0.325	0.8645
0.319375	0.8840
0.3111	0.9120
0.2775	1.0295
0.2256	1.272
0.19	1.536
0.1536	2.034
0.1351	2.5429
0.125775	2.986
0.1	∞

When $V^2 = V_3^2(1-\varepsilon^2), \qquad (\varepsilon \ll 1)$

$$s_1 = f\sqrt{(r_1^2-1)} + 0(\varepsilon^2), \quad s_2 = f\sqrt{(r_2^2-1)} + 0(\varepsilon^2), \quad (r_j = V_3/V_j) \quad \ldots\ldots(3.7)$$
$$s_3 = i\varepsilon f$$

Therefore, on neglecting small quantities, velocity equation takes the following form,

$$\begin{aligned} D = &\mu_2\sqrt{(r_2^2-1)}\cdot f^2 \cos\{\sqrt{(r_1^2-1)}\cdot\xi\}\cdot\mu_2\sqrt{(r_2^2-1)}\cdot\sin\{\sqrt{(r_2^2-1)}\cdot\xi\} \\ &+\mu_1\sqrt{(r_1^2-1)}\cdot f^2 \sin\{\sqrt{(r_1^2-1)}\cdot\xi\}\cdot\mu_2\sqrt{(r_2^2-1)}\cdot\cos\{\sqrt{(r_2^2-1)}\cdot\xi\}\ldots(3.8) \\ = &0 \qquad\qquad (\xi \equiv fH_0) \end{aligned}$$

Since r_2 is smaller than unity in this case, the above expression is reduced to

$$\begin{aligned} &-\mu_2\sqrt{(1-r_2^2)}\cdot\tanh\{\sqrt{(1-r_2^2)}\cdot\xi\} \\ &+\mu_1\sqrt{(r_1^2-1)}\cdot\tan\{\sqrt{(r_1^2-1)}\cdot\xi\} = 0 \qquad \ldots\ldots\ldots\ldots(3.9) \end{aligned}$$

which is satisfied by some finite value of ξ. ($\xi = 0$ is an isolated point.)

---

6) If we judge from the Fig. 3a. of the paper of T. Matuzawa, $V = b_3$ (b_3 corresponds to our V_3) seems to be an asymptote, which however, does not apply to this case. (cf. Fig. 3)

III. Example of case [9]....$\begin{pmatrix} B \\ A \\ C \end{pmatrix}$.

This case is fully discussed by T. Matuzawa, so we will only give the example of calculations in this paper.

The assumption of this case is

$$\begin{aligned} &3\mu_1 = 10\mu_2 = \mu_3 \equiv \mu \\ &H_1 = H_2 \equiv H_0 \\ &\rho_1 = \rho_2 = \rho_3 \equiv \rho \end{aligned} \qquad \text{.....................(3.10)}$$

Numerical results will be found in Fig. 4 and Tab. III.

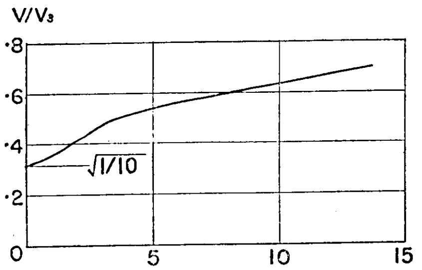

Fig. 4. Dispersion curve of example III (Case [9]).

Table III

$(V/V_3)^2$	fH_0
1	0
0.941	0.1443
0.725	0.2802
0.584	0.375
0.500	0.459
0.389	0.668
0.325	0.962
0.244	1.881
0.200	2.556
0.149	3.984
0.125	6.556
0.1	∞

IV. Example of case [12]....$\begin{pmatrix} C \\ A \\ B \end{pmatrix}$.

Existence of solution can be easily supposed by the former three cases. We will only give an example, of which no explanation will be necessary.

The assumption of this case is:

$$\begin{aligned} &H_1 = H_2 \equiv H_0 \\ &\rho_1 = \rho_2 = \rho_3 \equiv \rho \\ &\mu_1 = 10\mu_2 = 3\mu_3 \equiv \mu \end{aligned} \qquad \text{........................(3.11)}$$

The results of calculations are given in Fig. 5 and Tab. IV. In this case, too, there exists the cut-off period. For, putting $\sqrt{(r_1^2-1)} = i\sqrt{(1-r_1^2)}$ into (3.8), we have

$$\begin{aligned} &\mu_2\sqrt{(r_2^2-1)}\cdot\tan\{\sqrt{(r_2^2-1)}\cdot\xi\} \\ &-\mu_1\sqrt{(1-r_1^2)}\cdot\tanh\{\sqrt{(1-r_1^2)}\cdot\xi\} = 0\,, \end{aligned} \qquad \text{............(3.12)}$$

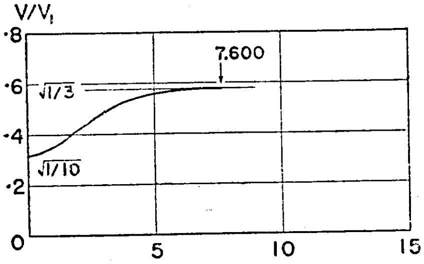

Fig. 5. Dispersion curve of example IV (Case [12]).

Table IV.

$(V/V_1)^2$	fH_0
0/3	0.8268
0.325	1.1120
0.319375	1.1940
0.311	102997
0.2775	1.6459
0.2256	2.209
0.19	2.267
0.1536	3.785
0.1351	4.808
0.125775	5.706
0.1	∞

from which we can easily obtain the above conclusion[7].

Summing up the discussion of this article we may assert that the condition of existence of Love-waves in double superficial layer is:

"The velocity of lowest medium is not the smallest of three values corresponding to three media."

4. Discussion of the paper of H. Menzel.[8]

H. Menzel discusses in his paper published some ten years ago that the group velocity of Love-waves propagated on double superficial layer has a discontinuity at the point where the phase velocity of Love-waves takes equal value with that of S-waves in the intermediate layer. He asserts that the above inference is to be confirmed theoretically as well as by observation[9]. Our doubt on this statement resulted in the following discussion.

His theoretical evidence resides in his calculation that the denominator of the expression for group velocity vanishes when the phase velocity takes the value $\sqrt{(\mu_2/\rho_2)}$. However, he does not examine whether the numerator does not vanish at the same time, so that his evidence is not sufficient.

In this article we will show that the said numerator vanishes at the same time with the denominator and the limiting value of ratio, after all, becomes finite.

7) See also the foot-note of p. 439.

8) *loc. cit.*, 2)

9) It is quite easy to percieve that apparent discontinuity in the velocity may result if the identification of a wave at two stations is mistaken by one wave length.

Denoting the group velocity as U, we have

$$U=\frac{dp}{df}$$

$$=V+\frac{p}{V}\frac{dV}{dp}U$$

$$\therefore \quad \frac{1}{U}=\frac{1}{V}\left(1-\frac{p}{V}\frac{dV}{dp}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

Putting

$$e\equiv 1/V^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4\cdot 2)$$

in (4.1), we have

$$\frac{1}{U}=\frac{1}{V}\left(1+\frac{p}{2/V^2}\frac{de}{dp}\right),$$

while e is an implicit function of p defined by (2.4), therefore

$$\frac{1}{U}=\frac{1}{V}\left(1+\frac{pD_p}{-2eD_e}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

where the suffixes p and e mean partial differentiation.

We employ a new notation

$$e_k=1/V_k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.4)$$

then

$$\left\{\begin{aligned} & s_k=p\sqrt{(e_k-e)} \\ & \frac{\partial}{\partial p}C_{jk}=\frac{1}{p}(C_{jk}-s_kH_kS_{jk}) \\ & \frac{\partial}{\partial p}S_{jk}=\frac{1}{p}(S_{jk}+s_kH_kC_{jk}) \end{aligned}\right. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.5)$$

After a little calculation

$$(D^C)_p=(C_{21})_p(C_{32}+S_{22})+C_{21}\{(C_{32})_p+(S_{22})_p\}$$
$$=\frac{1}{p}\{(2C_{21}-s_1H_1S_{21}-s_2H_2S_{11})C_{32}+(-S_{32}+C_{22})(2S_{11}+s_1H_1C_{11})\} \quad \ldots(4.6a)$$

Similarly

$$(D^S)_p=\frac{1}{p}\{(2C_{21}-s_1H_1S_{21}-s_2H_2S_{11})S_{22}+(-S_{22}+C_{22})s_2H_2C_{21}\} \quad \ldots\ldots(4.6b)$$

From (4.6a) and (4.6b) we can get pD_p.

Next we will consider eD_e.

Introducing a new notation

$$e/(e_k-e)\equiv\eta_k \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.7)$$

We have

$$\frac{\partial}{\partial e}C_{jk}=-\frac{1}{2e}\{\eta_j C_{jk}-\eta_k s_k H_k S_{jk}\}$$
$$\frac{\partial}{\partial e}S_{jk}=-\frac{1}{2e}\{\eta_j S_{jk}+\eta_k s_k H_k C_{jk}\} \qquad \cdots\cdots(4.8)$$

Therefore

$$(D^C)_e=-\frac{1}{2e}[C_{21}\{(\eta_2+\eta_3)C_{32}+2\eta_2 S_{22}\}$$
$$-\{\eta_1 s_1 H_1 S_{21}(C_{32}+S_{22})+\eta_2 s_2 H_2 C_{21}(S_{32}-C_{22})\}] \qquad \cdots\cdots(4.9)$$
$$(D^S)_e=-\frac{1}{2e}[S_{11}\{-(\eta_1+\eta_3)S_{32}+(\eta_1+\eta_2)C_{22}\}$$
$$+\{\eta_1 s_1 H_1 C_{11}(-S_{32}+C_{22})-\eta_2 s_2 H_2 S_{11}(C_{32}+S_{22})\}]$$

Introducing (2.4b)

$$-2eD_e=\eta_2\{(-s_2H_2C_{32}+S_{32})S_{11}+C_{21}S_{22}+s_2H_2(C_{21}(S_{32}-C_{22})-S_{11}S_{22})\}$$
$$+\{\eta_3 C_{21}C_{32}+(-(\eta_1+\eta_3)S_{32}+\eta_1 S_{22})S_{11}+\eta_1 s_1 H_1(-S_{21}C_{32}+(-S_{32}+C_{22})C_{11})\} \qquad \cdots\cdots(4.10)$$

Near the point $V=V_2$, neglecting the terms of $0\{s_2^3\}$, we have

$$\begin{cases} p(D^C)_p=(2\mu_2 s_2\cdot\cos s_1H_1-s_1H_1\cdot\mu_2 s_2\cdot\sin s_1H_1-s_2H_2\cdot S_{11})\mu_3' s_3 \\ \qquad +(-\mu_3' s_3\cdot s_2H_2+\mu_2 s_2)\cdot(2S_{11}+s_1H_1\cdot C_{11})+0\{s_2^3\} \qquad \cdots\cdots(4.11) \\ p(D^S)_p=0\{s_2^3\} \end{cases}$$

or, simplifying further,

$$p(D_p)=\mu_2 s_2 C_{11}\Big\{2\frac{\mu_3'}{\mu_1}\cdot\frac{s_3}{s_1}+\Big(-\frac{\mu_3'}{\mu_1}\cdot s_3H_1-3\frac{\mu_3'}{\mu_2}s_3H_2+2\Big)\tan s_1H_1$$
$$+\Big(-\frac{\mu_3'}{\mu_2}s_3H_2+1\Big)s_1H_1\Big\} \qquad \cdots\cdots(4.12)$$

and similarly

$$-2e(D_e)=\mu_2 s_2 C_{11}\Big\{f_2^2H_2^2\Big[\Big(\frac{1}{3}\frac{\mu_3'}{\mu_2}s_3H_2-1\Big)\tan s_1H_1-\frac{\mu_3'}{\mu_1}\frac{s_3}{s_1}+2\frac{\mu_2}{\mu_1}\frac{1}{s_1H_2}\Big]$$
$$+\Big[-\eta_1 s_1H_1\cdot\frac{\mu_3'}{\mu_1}\frac{s_3}{s_1}\cdot\tan s_1H_1+\eta_3\frac{\mu_3'}{\mu_1}\frac{s_3}{s_1}$$
$$+\eta_1(\tan s_1H_1+s_1H_1)\Big(-\frac{\mu_3'}{\mu_1}s_3H_2+1\Big)$$
$$-\eta_3\frac{\mu_3'}{\mu_2}s_3H_2\tan s_1H_1\Big]\Big\} \qquad \cdots\cdots(4.13)$$

in which $f_2=[f]_{V=V_2}$

Using the above two expressions, we can obtain the value of U.

$$\left[\frac{1}{U}\right]_{V=V_2}=\frac{1}{V_2}\{1+(pD_p)/(-2eD_e)\}_{V=V_2} \quad \ldots\ldots\ldots\ldots(4.14)$$

Employing the numerical constants of T. Matuzawa[10)]

k	1	2	3	
ρ_k	2.7	3.0	3.4	gr. cm^{-3}
V_k	3.3	4.0	4.5	km. sec^{-1}
μ_k	29.4	48.0	68.9	km. gr. sec.
H_k	20	30		km

we have

$$f_2H_2=1.47$$
$$s_1H_1=0.67, \qquad s_3H_2=i0.46$$
$$(pD_p)/(-2eD_e)=0.156,$$

and if we adopt the values due to R. Stoneley[11)], $H_1=H_2\equiv H_0$ and

k	1	2	3	
ρ_k	2.7	3.0	3.4	gr. cm^{-3}
V_k	3.15	3.90	4.40	km. sec^{-1}

we have

$$f_2H_2=1.02$$
$$s_1H_0=0.75, \qquad s_3H_0=i0.48$$
$$(pD_e)/(-2eD_e)=0.220.$$

If we introduce these numerical values, it will be soon noticed that the value of U in every case, takes finite value ($U/V_2=0.865$ (Matuzawa), 0.820 (Stoneley)). Thus we may conclude that group velocity does never become infinite at the point.

---

40. 表面波の研究 III

二つの表面層をもつラブ波

地震研究所 佐藤泰夫

表面に二層ある場合のラブ波の存在條件は，三つの媒質内のS波速度を比べた時，最下部のものが，最小の値をとらないことである。但し，二層のいづれか一方が，下部よりも大きな値を持つ時には，ラブ波の周期には上限が存在し，これより長周期の波は存在しえない。

H. Menzel は，かつてラブ波の位相速度が中間層の速度と等しくなる所では，群速度が無限に大きくなると主張したが，これはあやまりである。

10) *loc. cit.*, 2)

11) *loc. cit.*, 2)