41. *Distribution of Surface Stress Generating No Rayleigh-Waves.*

By Yasuo SATÔ,

Earthquake Research Institute.

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1. Introduction.

In a previous paper entitled "Seismic Focus without Rayleigh-waves,"[1] we have investigated the nature of origin situated in the semi-infinite elastic body. We have found examples of focus which radiate no Rayleigh-waves in spite of the generation of bodily waves. We began this study mainly from theoretical interest, however in this paper, we will treat a similar problem such as the distribution of surface stress which generates no Rayleigh-waves rather from the practical point of view.

Recently Don Leet[2] published a paper concerning "delay blasting" which seems to us an effort to find a technique of blasting which generates no surface waves. This is of practical importance in seismic prospecting, because the surface waves often obliterate later useful phases of P group.

In the following, we will show the way of applying surface stress without generating Rayleigh-waves.

2. Fundamental solution.

First we will start from the results obtained by Lamb[3] in his famous work "On the Propagation of Tremors over the Surface of Elastic Solid".

In three-dimensional semi-infinite elastic solid, displacement components u, v, and w which have axial symmetry around the vertical axis are expressed by using the potentials ϕ and χ as follows

$$u=\frac{\partial\phi}{\partial x}+u', \qquad v=\frac{\partial\phi}{\partial y}+v', \qquad w=\frac{\partial\phi}{\partial z}+w' \quad \ldots\ldots\ldots(2.1)^{4)}$$

1) Y. SATÔ, *Bull. Earthq. Res. Inst.*, **29** (1951), 13.
2) DON LEET, "Vibration from Delay Blasting." *Bull. Seism. Soc. Amer.*, **39** (1949), 9.
3) H. Lamb, *Phil. Trans.*, **A 203** (1904), 1–42.
4) *ibid.*

in which

$$\begin{cases} (\nabla^2+h^2)\phi=0\,, \qquad \nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2} \\ h^2=p^2/V_p^2 \\ V_p^2=(\lambda+2\mu)/\rho \end{cases} \qquad \text{............(2.2)}$$

and p means circular frequency, while λ, μ and ρ are Lamé's constants and density respectively.

On the other hand u', v' and w' are

$$u'=\frac{\partial^2\chi}{\partial x\partial z}\,, \qquad v'=\frac{\partial^2\chi}{\partial y\partial z}\,, \qquad w'=\frac{\partial^2\chi}{\partial z^2}+k^2\chi \qquad \text{..........(2.3)}$$

in which

$$\begin{cases} (\nabla^2+k^2)\chi=0 \\ k^2=p^2/V_s^2\,, \qquad V_s^2=\mu/\rho \end{cases} \qquad \text{..................(2.4)}$$

From the assumption of axial symmetry

$$\nabla^2=\frac{\partial^2}{\partial\tilde{\omega}^2}+\frac{1}{\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}}+\frac{\partial^2}{\partial z^2}$$

where $\tilde{\omega}$ is the distance from z-axis.

(2.1) and (2.3) are identical with the following expression.

$$q=\frac{\partial\phi}{\partial\tilde{\omega}}+\frac{\partial^2\chi}{\partial\tilde{\omega}\partial z}\,, \qquad w=\frac{\partial\phi}{\partial z}+\frac{\partial^2\chi}{\partial z^2}+k^2\chi \qquad \text{............(2.5)}$$

here q implies the displacement component in $\tilde{\omega}$-direction.

Adopting the fundamental solution of the type

$$\phi=A\exp(-\alpha z)\,J_0(\xi\tilde{\omega}), \qquad \chi=B\exp(-\beta z)\,J_0(\xi\tilde{\omega}) \qquad \text{......(2.6)}$$

where

$$\alpha=\sqrt{(\xi^2-h^2)}, \qquad \beta=\sqrt{(\xi^2-k^2)}$$

$$\begin{cases} q=\{-\xi A\exp(-\alpha z)+\xi\beta B\exp(-\beta z)\}\,J_1(\xi\tilde{\omega}) \\ w=\{-\alpha A\exp(-\alpha z)+\xi^2 B\exp(-\beta z)\}\,J_0(\xi\tilde{\omega}) \end{cases} \qquad \text{........(2.7)}$$

and the surface stress $[\widehat{\tilde{\omega}z}]_0$ and $[\widehat{zz}]_0$ are, at $z=0$

$$\begin{cases} [\widehat{\tilde{\omega}z}]_0=\mu\left[\dfrac{\partial q}{\partial z}+\dfrac{\partial w}{\partial\tilde{\omega}}\right]_0=\mu\{2\xi\alpha A-(2\xi^2-k^2)\xi B\}\,J_1(\xi\tilde{\omega}) \\ [\widehat{zz}]_0=\mu\left[\lambda\Delta+2\mu\dfrac{\partial w}{\partial z}\right]_0=\mu\{(2\xi^2-k^2)A-2\xi^2\beta B\}\,J_0(\xi\tilde{\omega}) \end{cases} \qquad \text{........(2.8)}$$

3. Surface stress generating no Rayleigh-waves.

Solutions shown in the equations (2.6), (2.7) and (2.8) satisfy fundamental equation of motion regardless of ξ. Therefore we may take any functional form for A and B.

If the surface stress such as

$$\widehat{\varpi z}=\mu X(\xi)\, J_1(\xi\varpi)\exp(ipt), \qquad \widehat{zz}=\mu Z(\xi)\, J_0(\xi\varpi)\exp(ipt) \quad \ldots\ldots(3.1)$$

are applied on $z=0$, we must take A and B, so that they may satisfy

$$\left\{\begin{array}{l} 2\xi\alpha A-(2\xi^2-k^2)\xi B=X(\xi) \\ (2\xi^2-k^2)A-2\xi^2\beta B=Z(\xi) \end{array}\right. \quad \ldots\ldots(3.2)$$

From this

$$\left\{\begin{array}{l} A=\{-2\xi\beta X(\xi)+(2\xi^2-k^2)Z(\xi)\}\,/\,F(\xi) \\ B=\{-(2\xi^2-k^2)X(\xi)+2\xi\alpha Z(\xi)\}\,/\,\xi F(\xi) \end{array}\right. \quad \ldots\ldots(3.3)$$

$$F(\xi)=(2\xi^2-k^2)^2-4\xi^2\alpha\beta$$

Surface displacement q_0 and w_0 are, putting $z=0$ in (2.7), and introducing the above relations

$$\left\{\begin{array}{l} q_0=[k^2\beta X(\xi)+\xi\{2\alpha\beta-(2\xi^2-k^2\}Z(\xi)]\, J_1(\xi\varpi)\,/\,F(\xi) \\ w_0=[\xi\{2\alpha\beta-(2\xi^2-k^2)\}X(\xi)+k^2\alpha Z(\xi)]\, J_0(\xi\varpi)\,/\,F(\xi) \end{array}\right. \quad \ldots\ldots(3.4)$$

Even if we integrate the above expressions, with respect to ξ new functions $\int q_0 d\xi$ and $\int w_0 d\xi$ also represent displacement components upon the free surface of the same medium. Denoting them by Q and W,

$$\left\{\begin{array}{l} Q=\int_0^\infty [k^2\beta X+\xi\{2\alpha\beta-(2\xi^2-k^2)Z]\, J_1(\xi\varpi)\,/\,F(\xi)\cdot d\xi \\ W=\int_0^\infty [\xi\{2\alpha\beta-(2\xi^2-k^2)\}X+k^2\alpha Z]\, J_0(\xi\varpi)\,/\,F(\xi)\cdot d\xi \end{array}\right. \quad \ldots\ldots(3.5)$$

If

$$X(-\xi)=X(\xi) \text{ and } Z(-\xi)=-Z(\xi) \quad \ldots\ldots(3.6)$$

$$Q=\int_0^\infty [\qquad]\,\tfrac{1}{2}\{H_1^{(1)}(\xi\varpi)+H_1^{(2)}(\xi\varpi)\}\,/\,F(\xi)\cdot d\xi$$

$$=\tfrac{1}{2}\int_{-\infty}^\infty [\qquad]\, H_1^{(1)}(\xi\varpi)\,/\,F(\xi)\cdot d\xi \quad \ldots\ldots(3.8)$$

Similarly

$$W=\tfrac{1}{2}\int_{-\infty}^\infty [\qquad]\, H_0^{(1)}(\xi\varpi)\,/\,F(\xi)\cdot d\xi \quad \ldots\ldots(3.8)$$

Displacement due to Rayleigh phase is

$$\left\{\begin{array}{l} Q[R]=i\pi[k^2\beta X(\xi)+\xi\{2\alpha\beta-(2\xi^2-k^2)\}\, Z(\xi)]\cdot H_1^{(1)}(\xi\varpi)\,/\,F'(\xi)_{\xi=-K} \\ W[R]=i\pi[\xi\{2\alpha\beta-(2\xi^2-k^2)\}\, X(\xi)+k^2\alpha Z(\xi)]\cdot H_0^{(1)}(\xi\varpi)\,/\,F'(\xi)_{\xi=-K} \end{array}\right. \quad \ldots(3.9)$$

in which

$$F'(\xi)=\frac{d}{d\xi}F(\xi) \text{ and } F(K)=0$$

If the two factors in [ ] vanish; i.e. if

$$\begin{cases} [k^2\beta X(\xi)+\xi\{2\alpha\beta-(2\xi^2-k^2)\}Z(\xi)]_{\xi=-K}=0 \\ [\xi\{2\alpha\beta-(2\xi^2-k^2)\}X(\xi)+k^2\alpha Z(\xi)]_{\xi=-K}=0 \end{cases} \quad \text{.......(3.10)}$$

hold, $Q[R]$ and $W[R]$ both vanish at any point of epicentral distance. Although (3.9) are simultaneous equations, the determinant of coefficients fortunately vanishes: i.e.,

$$\begin{vmatrix} k^2\beta & \xi\{2\alpha\beta-(2\xi^2-k^2)\} \\ \xi\{2\alpha\beta-(2\xi^2-k^2)\} & k^2\alpha \end{vmatrix}_{\xi=-K}=0$$

Therefore, the ratio of $X(-K)$ and $Z(-K)$ is uniquely determined.

$$X(-K)/Z(-K)=-X(K)/Z(K)=[-\xi\{2\alpha\beta-(2\xi^2-k^2)\}/k^2\beta]_{\xi=-K} \quad \text{...(3.12)}$$

which is a constant depending neither on ξ nor p. Since it is the function of Poisson's ratio, we will denote it as $-C(\sigma)$. Then (3.12) becomes as

$$X(K)/Z(K)=C(\sigma) \quad \text{..................(3.13)}$$

4. Numerical calculations.

4.1 $C(\sigma)$.

When $\lambda:\mu=1$, or $\sigma=1/4$; $C(1/4)=1.4679$
$\lambda:\mu=2$, or $\sigma=1/3$; $C(1/3)=1.5652$ ....(4.1)
$\lambda:\mu=\infty$, or $\sigma=1/2$; $C(1/2)=1.8413$

4.2 Surface stress.

Assume the distribution of surface stress as

$$[\widehat{\varpi z}]_{z=0}=\Psi(\varpi), \qquad [\widehat{zz}]_{z=0}=\Phi(\varpi) \quad \text{................(4.2)}$$

then

$$\xi\Psi^*(\xi)\equiv\xi\int_0^\infty \Psi(\varpi)\,J_1(\xi\varpi)\varpi d\varpi \quad \text{is an even function of } \xi$$

and $$\xi\Phi^*(\xi)\equiv\xi\int_0^\infty \Phi(\varpi)\,J_0(\xi\varpi)\varpi d\varpi \quad \text{,, odd ,,}$$

Therefore we may adopt the above two functions as our $X(\xi)$ and $Z(\xi)$. (cf. (3.6))

If we choose the magnitude of both functions appropriately so that $X(K)$ and $Z(K)$ may satisfy the condition (3.13), then we shall have no Rayleigh phase from such an origin.

5. Examples.

5.1 Simple harmonic origin.

We will show an example of simple harmonic origin in the following calculations.

First we assume the functional form of $\Psi(\tilde{\omega})$ and $\Phi(\tilde{\omega})$ as follows (cf. Fig. 1);

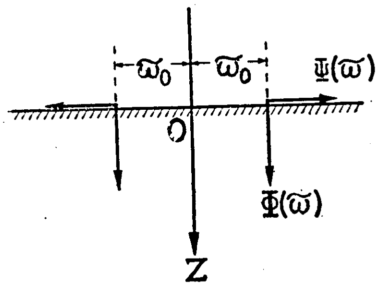

Fig. 1.

$$\Psi(\tilde{\omega})=\begin{cases}-\dfrac{\Psi_0}{\varepsilon}, \\ 0,\end{cases} \qquad \Phi(\tilde{\omega})=\begin{cases}-\dfrac{\Phi_0}{\varepsilon} & \ldots\ldots\tilde{\omega}_0<\tilde{\omega}<\tilde{\omega}_0+\varepsilon \\ 0 & \ldots\ldots\tilde{\omega}<\tilde{\omega}_0 \text{ or } \tilde{\omega}_0+\varepsilon<\tilde{\omega}\end{cases} \qquad \ldots(5.1)$$

in which ε is very small.

Then $X(\xi)$ and $Z(\xi)$ are

$$\begin{cases} X(\xi)\equiv\xi\displaystyle\int_0^\infty \Psi(\tilde{\omega})\,J_1(\xi\tilde{\omega})\tilde{\omega}d\tilde{\omega} \\ \qquad \fallingdotseq -\xi\Psi_0 J_1(\xi\tilde{\omega}_0)\tilde{\omega}_0 \\ Z(\xi)=\xi\displaystyle\int_0^\infty \Phi(\tilde{\omega})\,J_0(\xi\tilde{\omega})d\tilde{\omega} \\ \qquad \fallingdotseq -\xi\Phi_0 J_0(\xi\tilde{\omega}_0)\tilde{\omega}_0 \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5.2)$$

Therefore the condition (3.13) becomes

$$[-K\Psi_0 J_1(K\tilde{\omega}_0)\tilde{\omega}_0\,/\,-K\Phi_0 J_0(K\tilde{\omega}_0)\tilde{\omega}_0]=C(\sigma)$$

or

$$\frac{J_1(K\tilde{\omega}_0)}{J_0(K\tilde{\omega}_0)}=C(\sigma)\frac{\Phi_0}{\Psi_0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.3)$$

This equation has an infinite number of roots, of which we will show the smallest two values of $K\tilde{\omega}_0$ as the functions of Φ_0/Ψ_0 in Fig. 2 and Tab. I.

$$\left(K\tilde{\omega}_0 = \frac{2\pi}{T}\frac{\tilde{\omega}_0}{V\,(\text{Rayleigh})} = 2\pi\frac{\tilde{\omega}_0}{\lambda\,(\text{Rayleigh})}\right)$$

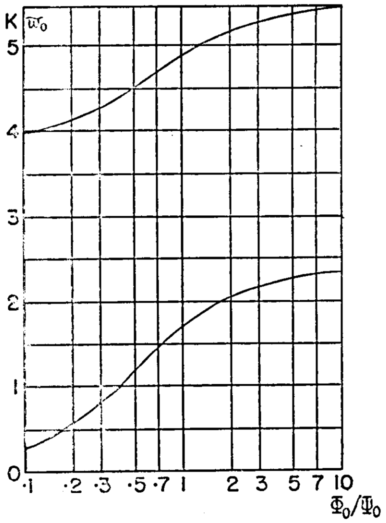

Fig. 2.

Table I.

Φ_0/Ψ_0	$K\tilde{\omega}_0$	
0.10	0.2905	3.9802
0.15	0.4302	4.0545
0.2	0.5640	4.1277
0.3	0.8070	4.2679
0.5	1.1908	4.5092
0.7	1.4552	4.6952
1.0	1.7072	4.8886
1.5	1.9315	5.0765
2	2.0499	5.1814
3	2.1698	5.2920
5	2.2650	5.3727
7	2.3053	5.4222
10	2.3356	5.4546

5.2 Aperiodic type of origin.

Usually, in prospecting or explosion a simple harmonic origin is not employed, and that of the aperiodic type is commonly met with. Therefore, in this article, we will discuss such problems.

The condition (3.13) involves the frequency p, so that even if this relation holds for some value of p, it does not hold for the other values. Thus we are not allowed simply to integrate the solution of (3.9) and get a seismic origin without Rayleigh-waves.

However, there is a way to get rid of this inconvenience. If we assume the two functions $X(\xi)$ and $Z(\xi)$ to be identical (excluding the constant factor) and make the ratio $X(\xi)/Z(\xi)$ equal to $C(\sigma)$, then the condition (3.13) holds regardless of p. Of course we cannot choose such functions strictly satisfying the above relation, for $X(\xi)$ is an even function of ξ and $Z(\xi)$ is an odd function. So we choose two functions, of which one is even and the other is odd, nearly equal on the positive half of real axis. Then the condition (3.13) can be approximately satisfied throughout the range of p, and we can make the Rayleigh-waves extremely small compared with the ordinary seismic origins.

Pairs of two functions having the said property are not few, but we will here show an example most simple and common.

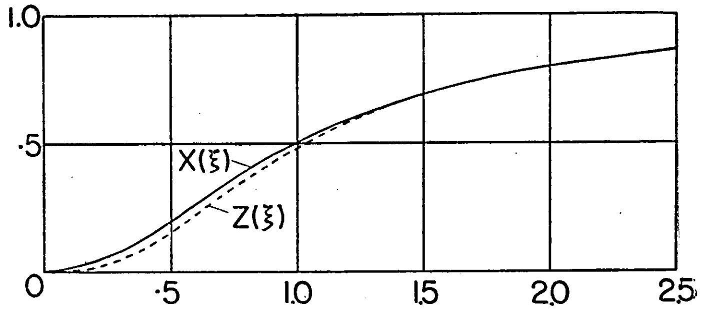

Fig. 3.

We take

$$X(\xi)=\frac{\xi^2}{\xi^2+1}$$

and

$$Z(\xi)=\frac{\xi^2}{\xi^2+1}\cdot\tanh 2\xi \quad \dots\dots\dots\dots\dots\dots(5.4)$$

then the stress distribution at the surface is

$$\begin{cases} [\widehat{\varpi z}]_{z=0}=\Psi(\varpi)=\int_0^\infty X(\xi)\,J_1(\xi\varpi)d\xi \\ [\widehat{zz}]_{z=0}=\Phi(\varpi)=\int_0^\infty Z(\xi)\,J_0(\xi\varpi)d\xi \end{cases} \quad \dots\dots\dots\dots(5.5)$$

Introducing (5.4), we have[5]

$$\begin{cases} \Psi(\varpi)=\int_0^\infty \frac{\xi^2}{\xi^2+1}\,J_1(\xi\varpi)d\xi=K_1(\varpi) \\ \Phi(\varpi)=\int_0^\infty \frac{\xi^2}{\xi^2+1}\cdot\tanh 2\xi\cdot J_0(\xi\varpi)d\xi \\ \qquad =\int_0^\infty J_0(\xi\varpi)d\xi-\int_0^\infty\frac{1}{\xi^2+1}J_0(\xi\varpi)\,d\xi+\int_0^\infty(\tanh 2\xi-1)\frac{\xi^2}{\xi^2+1}\,J_0(\xi\varpi)d\xi \\ \qquad =\frac{1}{\varpi}-\frac{\pi}{2}\{I_0(\varpi)-\boldsymbol{L}_0(\varpi)\}+\int_0^\infty(\tanh 2\xi-1)\frac{\xi^2}{\xi^2+1}J_0(\xi\varpi)d\xi\,. \end{cases}$$

$$\dots\dots(5.6)$$

The evaluation of the above expressions are performed in the following way.

Table of $e^x K_1(x)$ is found in Watson's famous book[6].

5) WATSON, *Theory of Bessel Functions.* p. 425.

Table of $\boldsymbol{L}_0(x)$ cannot be found in any book, so we calculated the value of $I_0(x)-\boldsymbol{L}_0(x)$ by the formula[7]

$$I_0(x)-\boldsymbol{L}_0(x)=\frac{2}{\pi}\int_0^{\pi/2}\exp(-x\cos\theta)\,d\theta \quad \text{............}(5.7)$$

using Simpson's $\frac{3}{8}$-rule (taking the interval as 10°).
The last term of $\Phi(\tilde{\omega})$ is modified into

$$\int_0^a (\tanh 2\xi-1)\frac{\xi^2}{\xi^2+1}J_0(\xi\tilde{\omega})\,d\xi+\int_a^\infty \cdots d\xi \quad \text{..........}(5.8)$$

and

$$\begin{aligned}\left|\int_a^\infty\right| &< \int_a^\infty |\tanh 2\xi-1|\cdot\frac{\xi^2}{\xi^2+1}\cdot|J_0(\xi\tilde{\omega})|\,d\xi \\ &< \int_a^\infty(1-\tanh 2\xi)\,d\xi \\ &< 2\int_a^\infty \exp(-4\xi)\,d\xi \\ &= \exp(-4a)/2 \quad \text{........................}(5.9)\end{aligned}$$

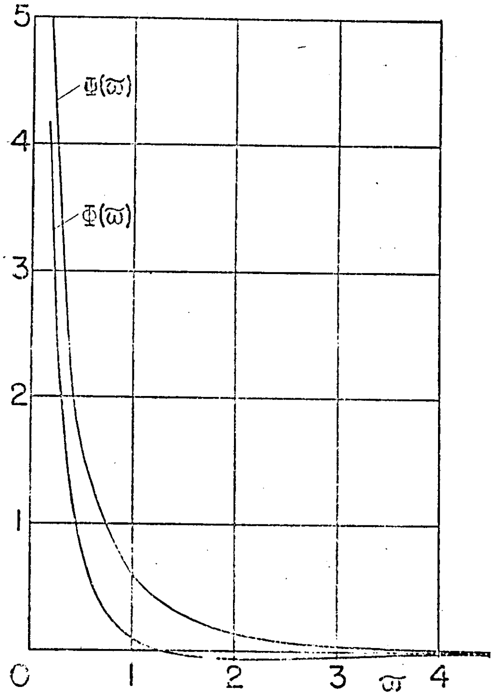

Fig. 4.

Table II.

$\tilde{\omega}$	$\Psi(\tilde{\omega})$	$I_0(\tilde{\omega})-\boldsymbol{L}_0(\tilde{\omega})$	$\int_a^\infty$	$\Phi(\tilde{\omega})$
0	∞	1	.036	∞
0.2	4.776	.882	.036	3.578
0.4	2.184	.781	.035	1.238
0.6	1.303	.695	.035	.541
0.8	.862	.620	.033	.243
1.0	.602	.559	.032	.090
1.2	.435	.501	.030	.017
1.4	.321	.452	.028	−.024
1.6	.241	.410	.026	−.046
1.8	.183	.374	.024	−.056
2.0	.140	.342	.022	−.059
2.5	.074	.279	.016	−.054
3.0	.040	.232	.011	−.043
3.5	.022	.198	.008	−.032
4.0	.012	.171	.005	−.024
4.5	.007	.151	.003	−.017

6) *loc. cit.*, 5) p. 698.
7) *do.*, p. 425.

Therefore if we assume a to be 1.8, the error due to the integral $\int_a^\infty$ is smaller than 0.0004. The first term of was integrated numerically.

The results of computations are given in Tab. 2 and illustrated in Fig. 4.

6. Conclutions.

The above discussions are interpreted as follows.

If the stress distribution of simple harmonic type are applied on the free surface of the elastic body and their functional form are appropriately chosen so that the condition (3.13) is satisfied, then Rayleigh-waves is not generated from such an origin. Of course, we can find any number of examples, of which we have given only one simple case in §5.1.

If the stress distribution varies its intensity according to an arbitrary function with respect to time, the above circumstances become more complicated, and we cannot find the distribution that strictly satisfies the condition of no Rayleigh-waves. However we can find examples which nearly satisfy the conditions, and the calculation given in §5.2 is a simple case of such cases[8]. If we put $X(\xi)$ and $Z(\xi)$ given in (5.4) into (3.7) and (3.8), we must calculate the terms which appear from the poles of these two functions. However these terms decrease rapidly with $\bar{\omega}$, so that we have no need to take them into consideration.

We hope the above theory will be used effectively for practical purposes.

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41. レーリー波を發生しない表面應力の分布

地震研究所　佐藤泰夫

半無限彈性體の表面に應力を加へる時，レーリー波を發生することは，ラムの論文以來よくしられてゐる事であるが，加へる應力の分布如何によつては，レーリー波を發生しないこともあり得るのではないか。もしこのやうな場合が存在すれば，理論的にも，地震探鑛等の實用にも，興味ある事と思ふ。

本論文においては，まづ應力の時間的な變りかたが exp(ipt) に比例するものについて，上の問題の肯定的な解を示し，次に數値を代入して，周期的な場合及び非周期的場合について，それぞれ簡單な例題をといた。

8) We chose these functions not because they agree well with our theory, but because they are simple and the calculation may be performed easily; so that if we do not mind the trouble of complicated numerical calculations, we can give sets of functions which satisfy (3.13) more strictly.